Kinetic Equations, Moment Closures, and Fluid Regimes

C. David Levermore
Department of Mathematics *and*
Institute for Physical Science and Technology
University of Maryland, College Park
lvrmr@math.umd.edu

Kinetic FRG Young Researchers Workshop:
Kinetic Description of Multiscale Phenomena: Modeling, Theory, and Computation
College Park, 5 March 2009

Introduction

Kinetic equations are usually solved numerically either by particle methods, such as Monte Carlo (MC) or molecular dynamics (MD) methods, or by finite difference, finite element, or finite volume methods (FXMs). All of these methods become inefficient in regimes where collisions dominate, so-called small mean-free-path regimes, because kinetic equations become stiff.

In such regimes collisions typically drive the underlying kinetic densities toward local equilibrium. This fact allows solutions of the kinetic equation to be approximated by solutions of a reduced system, typically a diffusion equation or a fluid dynamical system, that then can be solved efficiently by classical numerical methods.

Transition Regimes

The most difficult regimes to simulate are the so-called transition regimes. In these collisions are plentiful enough to make the full kinetic equation fairly stiff and expensive to solve, but are not plentiful enough to drive the underlying kinetic densities very close to local eqilibrium.

Transition regimes arise in problems that include both small mean-free-path regimes and large mean-free-path regimes. Astrophysics, radiation transport, plasma physics, and areospace are sources of such problems.

Transition regimes also arise in problems where the mean-free-path is on the scale of the problem spatial domain. Semiconductor fabrication and design and areospace are sources of such problems.

Transition Regime Models

Transition regime models must be designed to bridge the gap between small mean-free-paths regimes in which traditional reduced models are accurate, and large mean-free-paths regimes in which kinetic equations can be solved efficiently by traditional numerical methods. They should:

- be able to recover traditional reduced models derived for small mean-free-path regimes;
- be more accurate than the reduced models in transition regimes, where the reduced models typically fail badly;
- be solved more efficiently than the full kinetic equation.

Uses of Transition Regime Models

Such a model can then be used either:

- as a stand-alone model for problems that lie completely within the small mean-free-path and transition regimes,
- as a matching model between small mean-free-path regimes and large mean-free-path regimes in a hybrid simulation,
- as the basis for a preconditioner in a full kinetic simulation.

Classical Kinetic Equations

Consider a Boltzmann-like collisional kinetic equation for $F(v, x, t)$ over a D-dimensional spatial domain:

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\epsilon} \mathcal{C}(F) \,, \tag{1}$$

where ϵ is the Knudsen number, which is small in fluid dynamical regimes. (Of course, $D = 3$ is usually the physical case!)

We assume that the collision operator $\mathcal{C}$ respects Galilean symmetry, locally conserves mass, momentum and energy, locally dissipates entropy, and has local Maxwellians as equilibria.

Local Conservation Laws

The collision operator $\mathcal{C}$ satisfies

$$\langle \mathbf{e}\, \mathcal{C}(F) \rangle = 0 \quad \text{for “every” } F\,,$$

where we employ the notation

$$\langle f \rangle = \int_{\mathbb{R}^D} f(v)\, \mathrm{d}v\,, \qquad \mathbf{e} = \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix}\,.$$

This yields the local conservation laws

$$\partial_t \langle \mathbf{e}\, F \rangle + \nabla_x \cdot \langle v\, \mathbf{e}\, F \rangle = 0\,.$$

Local Entropy Dissipation Law

The collision operator $\mathcal{C}$ satisfies

$$\langle \eta'(F)\, \mathcal{C}(F) \rangle \leq 0 \quad \text{for “every” } F\,,$$

where $\eta(F) = F \log(F) - F$.

This yields the local entropy dissipation law

$$\partial_t \langle \eta(F) \rangle + \nabla_x \cdot \langle v\, \eta(F) \rangle = \frac{1}{\epsilon} \langle \eta'(F)\, \mathcal{C}(F) \rangle \leq 0\,.$$

Local Equilibra

For “every” F the following are equivalent:

(1) $\langle \eta'(F)\, \mathcal{C}(F) \rangle = 0$;

(2) $\mathcal{C}(F) = 0$;

(3) F has the Maxwellian form

$$F = \mathcal{E}(\boldsymbol{\rho}) = \frac{\rho}{(2\pi\theta)^{D/2}} \exp\left(- \frac{|v-u|^2}{2\theta} \right),$$

where $\quad \boldsymbol{\rho} = \langle \mathbf{e}\, F \rangle = \begin{pmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho|u|^2 + \frac{D}{2}\rho\theta \end{pmatrix}$.

Fluid Dynamical Closures

We decompose F into its *local equilibrium* $\mathcal{E}$ and *deviation* $\widetilde{F}$ as

$$F = \mathcal{E} + \widetilde{F}\,, \quad \text{where } \mathcal{E} = \mathcal{E}(\boldsymbol{\rho}) \text{ with } \boldsymbol{\rho} = \langle \mathbf{e}\, F \rangle.$$

One sees that $\langle \mathbf{e}\, \mathcal{E} \rangle = \boldsymbol{\rho}$ and $\langle \mathbf{e}\, \widetilde{F} \rangle = 0$.

Expressed in terms of $\boldsymbol{\rho}$ and $\widetilde{F}$, the local conservation laws are

$$\partial_t \boldsymbol{\rho} + \nabla_x \cdot \langle v\, \mathbf{e}\, \mathcal{E} \rangle + \nabla_x \cdot \langle v\, \mathbf{e}\, \widetilde{F} \rangle = 0\,.$$

A fluid dynamical closure is specified by expressing $\widetilde{F}$ above in terms of $\boldsymbol{\rho}$ and its derivatives.

Fluid Dynamical System

The resulting fluid dynamical system takes the form

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0\,, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + pI + \widetilde{P}) &= 0\,, \\ \partial_t (\rho e) + \nabla_x \cdot (\rho e u + pu + \widetilde{P} u + \tilde{q}) &= 0\,, \end{aligned} \tag{2}$$

where $p = \rho\theta$, $e = \frac{1}{2}|u|^2 + \frac{D}{2}\theta$, and

$$\widetilde{P} = \langle A\, \widetilde{F} \rangle\,, \qquad \tilde{q} = \langle B\, \widetilde{F} \rangle\,,$$

with A and B defined by

$$\begin{aligned} A &= (v-u) \otimes (v-u) - \tfrac{1}{D}|v-u|^2 I\,, \\ B &= \tfrac{1}{2}|v-u|^2 (v-u) - \tfrac{D+2}{2}\theta (v-u)\,. \end{aligned}$$

Euler Approximation

Maxwell was the first to argue that in fluid regimes F should be near $\mathcal{E}$. He therefore set $\widetilde{F} = 0$ in (2) to first derive the compressible Euler system:

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0\,, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + \rho\theta I) &= 0\,, \\ \partial_t (\rho e) + \nabla_x \cdot (\rho e u + \rho\theta u) &= 0\,. \end{aligned} \tag{3}$$

Its solutions formally dissipate the so-called Euler entropy $\langle \eta(\mathcal{E}) \rangle$ as

$$\partial_t \langle \eta(\mathcal{E}) \rangle + \nabla_x \cdot \langle v\, \eta(\mathcal{E}) \rangle \leq 0\,.$$

Deviation Equation

If one wants to improve upon the Euler approximation, one needs a better approximation for $\widetilde{F}$. The deviation $\widetilde{F}$ satisfies

$$\partial_t \widetilde{F} + \widetilde{\mathcal{P}} v \cdot \nabla_x \widetilde{F} + \widetilde{\mathcal{P}} v \cdot \nabla_x \mathcal{E} = \frac{1}{\epsilon} \mathcal{C}(\mathcal{E} + \widetilde{F}) \,, \tag{4}$$

where $\widetilde{\mathcal{P}} = \mathcal{I} - \mathcal{P}$ with $\mathcal{P}$ defined by

$$\mathcal{P}(\boldsymbol{\rho}) f = \mathcal{E}_{\boldsymbol{\rho}}(\boldsymbol{\rho}) \langle \mathrm{e}\, f \rangle \,.$$

One can show $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are orthogonal projections over $L^2(\mathcal{E}^{-1}\mathrm{d}v)$.

Navier-Stokes Balance

Maxwell argued that the deviation can be approximated by balancing the leading terms on each side of (4) — namely, that $\widetilde{F} \approx \widetilde{F}_{NS}$ where

$$\widetilde{\mathcal{P}} v \cdot \nabla_x \mathcal{E} = \frac{1}{\epsilon} D\mathcal{C}(\mathcal{E}) \widetilde{F}_{NS} \,.$$

This leads to the approximations

$$\begin{aligned} \widetilde{P} \approx \langle A\, \widetilde{F}_{NS} \rangle &= \epsilon \langle A\, D\mathcal{C}(\mathcal{E})^{-1} v \cdot \nabla_x \mathcal{E} \rangle \\ &= -\epsilon\, \mu \Big(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{D} I \nabla_x \cdot u \Big) \,, \\ \tilde{q} \approx \langle B\, \widetilde{F}_{NS} \rangle &= \epsilon \langle B\, D\mathcal{C}(\mathcal{E})^{-1} v \cdot \nabla_x \mathcal{E} \rangle = -\epsilon\, \kappa \nabla_x \theta \,, \end{aligned} \tag{5}$$

where $\mu(\rho, \theta)$ and $\kappa(\rho, \theta)$ are the coefficients of shear viscosity and thermal conductivity respectively. Most classical collision operators satisfy $\mathcal{C}(\lambda F) = \lambda^2 \mathcal{C}(F)$. In that case μ and κ will only depend on θ.

Navier-Stokes Approximation

This yields the compressible Navier-Stokes system.

$$
\begin{aligned}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0\,, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + \rho\theta I) &= \epsilon\, \nabla_x \cdot \Big[\mu\Big(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{D} I \nabla_x \cdot u\Big)\Big]\,, \\
\partial_t (\rho e) + \nabla_x \cdot (\rho e u + \rho\theta u) &= \epsilon\, \nabla_x \cdot [\kappa \nabla_x \theta]\,.
\end{aligned}
\tag{6}
$$

Its solutions formally dissipate the Euler entropy as

$$
\partial_t \langle \eta(\mathcal{E}) \rangle + \nabla_x \cdot \langle v\, \eta(\mathcal{E}) \rangle + \nabla_x \cdot \langle v\, \eta'(\mathcal{E}) \widetilde{F}_{NS} \rangle = \frac{1}{\epsilon} \langle \eta''(\mathcal{E}) \widetilde{F}_{NS}\, D\mathcal{C}(\mathcal{E}) \widetilde{F}_{NS} \rangle\,.
$$

Beyond Navier-Stokes via Expansions

Hilbert (1912) introduced a derivation of the Navier-Stokes system based on a systematic expansion in the (small) Knudsen number ϵ. This so-called Hilbert expansion yields the Euler system at leading order, and corrections that satisfy linearized Euler systems driven by lower order terms. These have to be summed through order ϵ to obtain the Navier-Stokes system.

Short afterward, Enskog (1917) introduced a slightly different expansion in ϵ, subsequently dubbed the Chapman-Enskog expansion, that led directly to the Navier-Stokes system at order ϵ.

Both these approaches fail to systematically yield corrections to the Navier-Stokes system that are formally well-posed.

Beyond Navier-Stokes via Moments

Grad (1949) proposed building approximations by taking appropriate velocity moments of (1). In particular, he developed his famous 13 moment system (for $D = 3$) which takes moments with respect to

$$1\,, \qquad v\,, \qquad v \otimes v\,, \qquad |v|^2 v\,.$$

The resulting systems classically have the form

$$\partial_t \langle \mathbf{m}\, F \rangle + \nabla_x \cdot \langle v\, \mathbf{m}\, F \rangle = \frac{1}{\epsilon} \langle \mathbf{m}\, \mathcal{C}(F) \rangle\,.$$

This system must be closed by expressing $\langle v\, \mathbf{m}\, F \rangle$ and $\langle \mathbf{m}\, \mathcal{C}(F) \rangle$ in terms of the densities $\langle \mathbf{m}\, F \rangle$. This appoach has also failed to systematically yield useful corrections to the Navier-Stokes system. Still, this is the path we will now take.

Gaussian Moment System

Taking the moments of (1) with respect to 1, v, and $v \vee v$, we obtain

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0\,, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \vee u + \rho \Theta) &= 0\,, \\ \partial_t (\rho u \vee u + \rho \Theta) + \nabla_x \cdot (\rho u \vee u \vee u + 3\rho \Theta \vee u + Q) &= R\,, \end{aligned} \tag{7}$$

where

$$\rho = \langle F \rangle\,, \quad \rho u = \langle v\, F \rangle\,, \quad \rho \Theta = \langle (v-u) \vee (v-u)\, F \rangle\,,$$
$$Q = \langle (v-u) \vee (v-u) \vee (v-u)\, F \rangle\,,$$
$$R = \langle v \vee v\, \mathcal{C}(F) = \langle (v-u) \vee (v-u)\, \mathcal{C}(F) \rangle\,.$$

Here $\vee$ denotes the symmetric tensor product.

Gaussian Closure Criteria

To close this system Q and R must be expressed in terms of ρ, u, and $\ominus$. This should be done so that the Gaussian moment system (7):

- repects Galilean symmetry,
- has an entropy structure,
- preserves the nonnegative definiteness of $\ominus$,
- recovers known fluid dynamical regimes.

Simple Gaussian Closures

A simple class of closures that formally meets these criteria has the form

$$
\begin{aligned}
R &= \frac{\rho}{\tau}\,(I\theta - \Theta)\,, \\
Q &= \tfrac{9}{D+2}\,\nu_0\, I \vee \operatorname{tr}\Big(\nabla_x \vee \Theta^{-1}\Big) \\
&\quad + 3\,\nu_1 \Big(\nabla_x \vee \Theta^{-1} - \tfrac{3}{D+2} I \vee \operatorname{tr}\Big(\nabla_x \vee \Theta^{-1}\Big)\Big)\,,
\end{aligned}
\tag{8}
$$

where $\theta = \frac{1}{D}\operatorname{tr}(\Theta)$, $\tau(\rho,\theta)$ is a relaxation time, and $\nu_0(\rho,\theta)$ and $\nu_1(\rho,\theta)$ are nonnegative transport coefficients. Moreover,

$$
\tau(\rho,\theta) = \frac{\mu(\rho,\theta)}{\rho\theta}\,, \qquad \nu_0(\rho,\theta) = \tfrac{2}{D+2}\,\theta^2 \kappa(\rho,\theta)\,,
$$

where $\mu(\rho,\theta)$ and $\kappa(\rho,\theta)$ are the coefficients of shear viscosity and thermal conductivity.

Convective Form

The Gaussian moment system (7) has the convective form

$$\begin{aligned}
\partial_t \rho + u \cdot \nabla_x \rho + \rho \nabla_x \cdot u &= 0\,, \\
\rho(\partial_t u + u \cdot \nabla_x u) + \nabla_x \cdot (\rho \Theta) &= 0\,, \\
\rho(\partial_t \Theta + u \cdot \nabla_x \Theta) + \rho\Big(\Theta \cdot \nabla_x u + (\Theta \cdot \nabla_x u)^T\Big) + \nabla_x \cdot Q &= R\,.
\end{aligned}$$

Because

$$(\partial_t + u \cdot \nabla_x)\det(\Theta) = \det(\Theta)\,\mathrm{tr}\Big(\Theta^{-1}(\partial_t + u \cdot \nabla_x)\Theta\Big)\,,$$

we see that

$$\begin{aligned}
\rho(\partial_t + u \cdot \nabla_x)\det(\Theta) + 2\rho\det(\Theta)\nabla_x \cdot u + \det(\Theta)\,\mathrm{tr}\Big(\Theta^{-1}\nabla_x \cdot Q\Big) \\
= \det(\Theta)\,\mathrm{tr}\Big(\Theta^{-1}R\Big)\,.
\end{aligned}$$

Entropy Dissipation

Let $\sigma = \log(\det(\Theta)/\rho^2)$. Then

$$\rho(\partial_t \sigma + u \cdot \nabla_x \sigma) + \operatorname{tr}\left(\Theta^{-1} \nabla_x \cdot Q\right) = \operatorname{tr}\left(\Theta^{-1} R\right),$$

which becomes

$$\partial_t(\rho\sigma) + \nabla_x \cdot (\rho u \sigma) + \nabla_x \cdot \left(\Theta^{-1} : Q\right) = \left(\nabla_x \vee \Theta^{-1}\right) \vdots Q + \Theta^{-1} : R\,.$$

For the Gaussian closures given by (8) we have

$$\begin{aligned}
\Theta^{-1} : R &= \frac{\rho}{\tau}\left(\theta \operatorname{tr}\left(\Theta^{-1}\right) - D\right) \geq 0\,, \\
\left(\nabla_x \vee \Theta^{-1}\right) \vdots Q &= 3\,\nu_0 \left|\operatorname{tr}\left(\nabla_x \vee \Theta^{-1}\right)\right|^2 \\
&\quad + 3\,\nu_1 \left|\nabla_x \vee \Theta^{-1} - \tfrac{3}{D+2} I \vee \operatorname{tr}\left(\nabla_x \vee \Theta^{-1}\right)\right|^2 \geq 0\,,
\end{aligned}$$

which is in accord with the second law of thermodynamics.

Mathematical Entropy Structure

The mapping

$$\Big(\rho\,,\ \rho u\,,\ \rho u \vee u + \rho\Theta\Big) \mapsto -\rho\sigma \quad \text{is strictly convex}\,.$$

Moreover, because

$$\Theta^{-1} : R = \frac{\rho}{\tau}\Big(\theta\,\mathrm{tr}\Big(\Theta^{-1}\Big) - D\Big) \geq 0\,,$$

with

$$\Theta^{-1} : R = 0 \quad \Longleftrightarrow \quad R = 0 \quad \Longleftrightarrow \quad \Theta = I\theta\,,$$

the quantity $-\rho\sigma$ plays the role of a mathematical entropy for our system.

Remark: If we assume that R depends linearly upon $\Theta - I\theta$ while Q depends linearly upon $\nabla_x \vee \Theta^{-1}$, and that these linear relations depend only on ρ and θ and respect Galilean symmetry, then they have the form given earlier.

Decomposition of the Gaussian Moment System

We introduce the Knudsen number ϵ into the Gaussian moment system (7) and decompose it into its conservation laws

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0\,, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \vee u + \rho \Theta) &= 0\,, \\ \partial_t (\rho e) + \nabla_x \cdot (\rho e u + \rho \Theta \cdot u + \epsilon\, q) &= 0\,, \end{aligned} \tag{9}$$

where $e = \frac{1}{2}|u|^2 + \frac{D}{2}\theta$ and $q = \frac{1}{2}\operatorname{tr}(Q)$, and its relaxation laws

$$\begin{aligned} \rho\Big(\partial_t \widetilde{\Theta} + u \cdot \nabla_x \widetilde{\Theta}\Big) + \rho\Big[\Theta \cdot \nabla_x u + (\Theta \cdot \nabla_x u)^T - \tfrac{2}{D} I \Theta : \nabla_x u\Big] \\ + \epsilon \Big[\nabla_x \cdot Q - \tfrac{2}{D} I \nabla_x \cdot q\Big] = \frac{1}{\epsilon} R\,, \end{aligned}$$

where $\widetilde{\Theta} = \Theta - I\theta$, while Q and R are given by the closure (8).

Fluid Regimes: Leading Order

In fluid dynamical regimes ϵ is small. We use this fact to obtain approximate expressions for $\widetilde{\Theta}$ in terms of ρ, u, and θ. We expand $\widetilde{\Theta}$ as

$$\widetilde{\Theta} = \widetilde{\Theta}^{(0)} + \epsilon\,\widetilde{\Theta}^{(1)} + \epsilon^2\widetilde{\Theta}^{(2)} + \cdots .$$

Because $R = -\frac{\rho}{\tau}\widetilde{\Theta}$, we see that to leading order $\widetilde{\Theta}^{(0)} = 0$. Hence, $\Theta^{(0)} = I\theta$ and the closure relation (8) implies that

$$Q^{(0)} = -\tfrac{9}{D+2}\,\frac{\nu_0}{\theta^2}\,I \vee \mathrm{tr}\Big(\nabla_x\theta \vee I\Big) = -3\,\frac{\nu_0}{\theta^2}\,I \vee \nabla_x\theta\,,$$

whereby

$$q^{(0)} = \tfrac{1}{2}\,\mathrm{tr}\Big(Q^{(0)}\Big) = -\tfrac{D+2}{2}\,\frac{\nu_0}{\theta^2}\nabla_x\theta\,,$$

$$\nabla_x\cdot Q^{(0)} - \tfrac{2}{D}I\nabla_x\cdot q^{(0)} = -2\left[\nabla_x \vee \left(\frac{\nu_0}{\theta^2}\nabla_x\theta\right) - \tfrac{1}{D}I\nabla_x\cdot\left(\frac{\nu_0}{\theta^2}\nabla_x\theta\right)\right].$$

Navier-Stokes Regimes

If we assume that $\nabla_x \rho$, $\nabla_x u$, and $\nabla_x \theta$ are of the same order then to next order the relaxation equations yield

$$-\frac{\rho}{\tau}\widetilde{\Theta}^{(1)} = \rho\theta\Big(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{D} I \nabla_x \cdot u\Big)$$

The conservation laws (9) through order ϵ then become the compressible Navier-Stokes system (6) provided that we set

$$\tau(\rho,\theta) = \frac{\mu(\rho,\theta)}{\rho\theta}\,, \qquad \nu_0(\rho,\theta) = \tfrac{2}{D+2}\,\theta^2 \kappa(\rho,\theta)\,,$$

where $\mu(\rho,\theta)$ and $\kappa(\rho,\theta)$ are the coefficients of shear viscosity and thermal conductivity. This implies that the Gaussian moment system (7) with closure (8) will formally recover the compressible Euler and incompressible Navier-Stokes limits — both Boussinesq and dominant balance.

Ghost Effect Regime Scaling

If we assume that $\nabla_x \rho$ and $\nabla_x \theta$ are large while

$$u = O(\epsilon)\,, \qquad \nabla_x \cdot (\rho \Theta) = O(\epsilon^2)\,, \qquad \partial_t = O(\epsilon)\,,$$

then, after rescaling, the conservation laws (9) can be recast as

$$\begin{aligned} \partial_t \rho + u \cdot \nabla_x \rho + \rho \nabla_x \cdot u &= 0\,, \\ \rho(\partial_t u + u \cdot \nabla_x u) + \frac{1}{\epsilon^2} \nabla_x \cdot (\rho \Theta) &= 0\,, \\ \frac{D}{2} \rho(\partial_t \theta + u \cdot \nabla_x \theta) + \Theta : \nabla_x u + \nabla_x \cdot q &= 0\,, \end{aligned}$$

while the relaxation equations become

$$\begin{aligned} \rho\Big(\partial_t \widetilde{\Theta} + u \cdot \nabla_x \widetilde{\Theta}\Big) + \rho\Big[\Theta \cdot \nabla_x u + (\Theta \cdot \nabla_x u)^T - \tfrac{2}{D} I \Theta : \nabla_x u\Big] \\ + \Big[\nabla_x \cdot Q - \tfrac{2}{D} I \nabla_x \cdot q\Big] = \frac{1}{\epsilon^2} R\,. \end{aligned}$$

Ghost Effect Regime Balance

We therefore expand $\widetilde{\Theta}$ as

$$\widetilde{\Theta} = \widetilde{\Theta}^{(0)} + \epsilon^2 \widetilde{\Theta}^{(1)} + \epsilon^4 \widetilde{\Theta}^{(2)} + \cdots .$$

Because $R = -\frac{\rho}{\tau}\widetilde{\Theta}$, we again see that to leading order $\widetilde{\Theta}^{(0)} = 0$. Hence, $\Theta^{(0)} = I\theta$ and, as before,

$$q^{(0)} = \tfrac{1}{2}\,\mathrm{tr}\Big(Q^{(0)}\Big) = -\tfrac{D+2}{2}\,\frac{\nu_0}{\theta^2}\nabla_x\theta\,,$$

$$\nabla_x\cdot Q^{(0)} - \tfrac{2}{D} I\nabla_x\cdot q^{(0)} = -2\left[\nabla_x \vee \left(\frac{\nu_0}{\theta^2}\nabla_x\theta\right) - \tfrac{1}{D} I\nabla_x\cdot\left(\frac{\nu_0}{\theta^2}\nabla_x\theta\right)\right].$$

However, this time to next order the relaxation equations yield

$$-\frac{\rho}{\tau}\widetilde{\Theta}^{(1)} = \rho\theta\Big(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{D} I\nabla_x\cdot u\Big) - 2\left[\nabla_x \vee \left(\frac{\nu_0}{\theta^2}\nabla_x\theta\right) - \tfrac{1}{D} I\nabla_x\cdot\left(\frac{\nu_0}{\theta^2}\nabla_x\theta\right)\right].$$

Ghost Effect System

We thereby arrive at the ghost effect system

$$\rho(\partial_t u + u\cdot\nabla_x u) + \nabla_x p + \nabla_x\cdot S = 0\,,$$
$$\tfrac{D}{2}\rho(\partial_t\theta + u\cdot\nabla_x\theta) - \partial_t(\rho\theta) - \nabla_x\cdot(\kappa\nabla_x\theta) = 0\,,$$

where $\rho\theta$ and u are constrained so that

$$\nabla_x(\rho\theta) = 0\,, \qquad \tfrac{D}{2}\partial_t(\rho\theta) + \tfrac{D+2}{2}\rho\theta\nabla_x\cdot u - \nabla_x\cdot(\kappa\nabla_x\theta) = 0\,,$$

and S is given by the constitutive relation

$$S = -\mu\Big(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{D}I\nabla_x\cdot u\Big) + \tfrac{4}{D+2}\frac{\mu}{\rho\theta}\Big(\nabla_x\vee(\kappa\nabla_x\theta) - \tfrac{1}{D}I\nabla_x\cdot(\kappa\nabla_x\theta)\Big)\,.$$

This limiting dynamics cannot be recovered from the compessible Navier-Stokes system, but is recovered from kinetic theory and from our Gaussian moment system.

Open Problems

- Local well-posedness for the Gaussian moment system
- Global existence of weak solutions for the Gaussian moment system
- Justification of any of the above formal fluid limits
- Any derivation of boundary conditions or any boundary layer theory (This is needed for proper use of these models in hybrid codes.)