Loss of smoothness and energy conservation in the 3D Euler equations

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joint work with

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Introduction

Use the shear flow to comment a series of issues concerning the effect of **dimension** 3 **on solutions of the Euler equation.**
The shear flow is a special example introduced in our community by Di Perna and Majda.
It does not fit in the statistical theory of turbulence because

- It is not turbulent...
- In a statistical theory it corresponds to a very "special class of events"

I use it to shows that there is no hope to prove statistical results by deterministic functional analysis.

1. In the Holder spaces C^1 is a critical space for well posedness
2. Weak limit of oscillating solution may not be solutions of the Euler equation and the corresponding "turbulent " tensor is different from what would be predicted by statistical theory of turbulence.
3. There exist solutions with vorticity containing a superficial density on a surface which may not be smooth. Conjecture that this is due to the $3d$ effect.
4. There are singular solutions that conserve the energy. Onsager conjecture : For weak solutions equation energy decay is related to some loss of regularity. 1/3 appears to be a critical value of such regularity. What is proven is that any solution more regular than the Besov space $\mathcal{B}^{\frac{1}{3}}_{3,\infty}$ conserve energy Constantin E Titi; Cheskidov Constantin Friedlander.

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The Shear Flow

Shear Flow

$$u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2)) \quad (1)$$

$$\nabla \cdot u = 0 \quad \textit{and } \partial_t u + \nabla(u \otimes u) = -\nabla p \;\; \textit{with } p \equiv 0 \quad (2)$$

$$\frac{d}{dt} \int\int\int |u(x_1, x_2, x_3)|^2 dx_1 dx_2 dx_3 = 0 \quad (3)$$

2 (in the sense of distributions) and 3 (on the torus) are true under the only hypothesis that

$$u_1 \in L^2_{x_2} \text{ and } u_3 \in L^2_{x_1}$$

Instability of Cauchy Problem, Loss of regularity

Cauchy Problem

For initial data in $C^{1,\alpha}$ the Euler equation has a unique local in time solution in $C^{1,\alpha}$

$$\partial_t \|\omega\|_{0,\alpha} \leq \|\omega \cdot \nabla u\|_{0,\alpha} + \|\omega \nabla u\|_{0,\alpha} \|u\|_{lip}$$

The original proof of Lichtenstein (1925) based on

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u$$

Same result later for initial data in $H^s, s > \frac{5}{2}$.

$W^{1,p}$ Instability

Theorem

Di Perna - Lions For every $p \geq 1$, $T > 0$ and $M > 0$ there exists a smooth shear flow solution for which $\|u(x,0)\|_{W^{1,p}} = 1$ and $\|u(x,T)\|_{W^{1,p}} > M$.

Proof (Bardos-Titi version)

$$\partial_{x_2} u_3(x_1 - tu_1(x_2)) = -t\partial_{x_2} u_1(x_2)\partial_{x_1} u_3(x_1)(x_1 - tu_1(x_2))$$
$$u(0) \in W^{1,p} \text{ does not implies } \quad u(t) \in W^{1,p}\,.$$

C^1 Criticallity

Theorem

In the Holder spaces C^1 is the critical space for local in time well posedness

1. *For $(u_1(x), u_3(x)) \in C^{1,\alpha}$, $0 \leq \alpha < 1$ the shear flow solution is always in $C^{1,\alpha}$,*
2. *For $(u_1(x), u_3(x)) \in C^{0,\alpha}$ the shear flow solution is always in C^{0,α^2} , There exists shear flow solutions which for $t = 0$ belong to $C^{0,\alpha}$ and wich for $t \neq 0$ are not in $C^{0,\beta}$ for $\beta > \alpha^2$.*

Proof

Regularity results concern only the component u_3

$$\frac{|u_3(x_1 - tu_1(x_2+h)) - u_3(x_1 - tu_1(x_2))|}{h^{\alpha^2}} =$$

$$\frac{|u_3(x_1 - tu_1(x_2+h)) - u_3(x_1 - tu_1(x_2))|}{|tu_1(x_2+h) - tu_1(x_2)|^\alpha} \left(\frac{|tu_1(x_2+h) - tu_1(x_2)|}{h^\alpha}\right)^\alpha$$

$$\leq |t|^\alpha \|u_3\|^{0,\alpha} (\|u_1\|^{0,\alpha})^\alpha .$$

Introduces two periodic functions u_1 and u_3 which near the point $x = 0$ coincide with $|x|^\alpha$. Then the for t given and x_1 and x_2 small enough $u_3(x_1 - tu_3(x_2))$ coincides with

$$|x_1 - t|x_2|^\alpha|^\alpha$$

For $(x_1, x_2, x_3) = (0, x_2, x_3)$ one has

$$u_3(x_1 - tu_3(x_2)) = |t|^\alpha |x_2|^{\alpha^2}$$

and the conclusion follows.

Other spaces and optimal spaces

The Cauchy problem is well posed in H^s , $s > \frac{n}{2} + 1$ It is not well posed in H^s , $\frac{n}{2} < s < \frac{n}{2} + 1$.

$$\alpha^2 < s - \frac{n}{2} < \alpha \,, u(0) \in C^{0,\alpha} \subset H^s \text{and } u(t) \notin C^{0,\alpha} \Rightarrow \notin H^s$$

The Besov spaces:

$$\mathcal{B}^s_{p,q} = \{f \backslash \sum_{j\in\mathbb{Z}} \|2^{js}\Delta_j f\|^p_{L^p} < \infty\}$$

$$C^{1,\alpha} = B^{1+\alpha}_{\infty,\infty} \subset B^1_{\infty,1} \subset C^1 \subset F^1_{\infty,2} \subset B^1_{\infty,\infty} \subset B^\alpha_{\infty,\infty} = C^{0,\alpha} \,.$$

Theorem

The 3d Euler equation is well posed in $B^1_{\infty,1}$(Pak and Park). It is not well posed in $B^1_{\infty,\infty}$ or in the Triebel-Lizorkin space $\subset F^1_{\infty,2}$

Proof

$B^1_{\infty,\infty}$ is the Zygmund class ie bounded functions with

$$\sup_{x\in\mathbb{R},h\in\mathbb{R}} \frac{|f(x+h)+f(x-h)-2f(x)|}{|h|} < \infty$$

They are not Lipschitz but log-Lipschitz

$$|f(x+h)-f(x)| \le C|h|\log\frac{1}{|h|}$$

Now $v(y)$ smooth outside 0 with

$$v(y) \sim y\log\frac{1}{|y|} \text{ near } 0$$

is in the Zygmund class. Then with $u_1(y)=u_3(y)=v(y)$ and $x_1=0$

$$|u_3(-tu_1(h))-u_3(-tu_1(0))| \sim th(\log h)^2!!$$

Same proof for $\subset F^1_{\infty,2}$. More delicate: Construction of a log lipschitz function in this space.

Weak Limit of Oscillating Initial Data

Original Di Perna-Majda example: Sequence of weak solutions with energy estimate:

$$\nabla \cdot \overline{u} = 0 \ \text{ in } \Omega\,, \quad \overline{u} \cdot \vec{n} = 0 \ \text{ on } \partial\Omega\,,$$
$$\partial_t \overline{u} + \nabla \cdot (\overline{u} \otimes \overline{u}) + \nabla \cdot RT(u_\epsilon) + \nabla \overline{p} = 0 \ \text{ in } \Omega\,,$$
$$RT(u_\epsilon)(x,t) = \lim_{\epsilon \longrightarrow 0} ((u_\epsilon - \overline{u}) \otimes (u_\epsilon - \overline{u}))$$

$$u^\epsilon(x,t) = (u_1(\frac{x_2}{\epsilon}), 0, u_3(x_1 - t u_1(\frac{x_2}{\epsilon})) \ \int_0^1 u_1(s) ds = 0$$

$$\lim_{\epsilon \longrightarrow 0} \mathrm{w}eak \ u^\epsilon = (0, 0, \overline{u_3})\,, \overline{u_3} = \int_0^1 u_3(x_1 - t u_1(s)) ds$$

$$\partial_t \overline{u_3} + \lim_{\epsilon \longrightarrow 0} \nabla \cdot u^\epsilon \otimes u_3^\epsilon = 0$$

$$\lim_{\epsilon \longrightarrow 0} \nabla \cdot u^\epsilon \otimes u_3^\epsilon = \partial_{x_1} \int_0^1 u_1(s) u_3(x_1 - t u_1(s)) ds \neq \partial_{x_1} \cdot \overline{u}_1 \otimes \overline{u}_3 = 0$$

Weak Limit of Oscillating Initial Data. Remarks

There is some similarity between weak convergence and statistical theory of turbulence.

$$RT = \langle u \otimes u \rangle - \langle u \rangle \otimes \langle u \rangle$$

$$\langle \hat{u}(k,\cdot) \otimes \overline{\hat{u}(k,\cdot)} \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ik\cdot r} \langle u(x+\frac{r}{2},\cdot) \otimes u(x-\frac{r}{2},\cdot) \rangle dr$$

Spectra with homogeneity isotropy and decay as $|k|^{-\frac{5}{3}}$

None of the above properties holds for the above example:

1 Because the forcing is low frequency.

2 Because it is a specific example ???

Shear flow with density on interface . Example 1

In order for the vorticity of the shear flow solution be concentrated on an interface the solution must be of the form:

$$u_1(s) = \begin{cases} \alpha_1 & \text{for} \quad s < \xi_2 \\ \beta_1 & \text{for} \quad s > \xi_2 \end{cases} \quad \text{and} \quad u_3(s) = \begin{cases} \alpha_3 & \text{for} \quad s < \xi_1 \\ \beta_3 & \text{for} \quad s > \xi_1 \end{cases} ,$$

for some fixed real parameters $\alpha_1, \alpha_3, \beta_1, \beta_3, \xi_1, \xi_2$, satisfying $\alpha_1 \geq \beta_1$ and $\alpha_3 \neq \beta_3$. Consequently, the corresponding vorticity of the above solution is concentrated on the singular surface:

Singular Vortex Sheet

$$\Sigma(t) = \{(x_1, x_2, x_3) | \, x_2 = \xi_2\} \cup$$
$$\{(x_1, x_2, x_3) | \, x_1 = \xi_1 + t\alpha_1, \, x_2 \leq \xi_2\}$$
$$\cup \{(x_1, x_2, x_3) | \, x_1 = \xi_1 + t\beta_1, \, x_2 \geq \xi_2\} \, .$$

Shear flow with density on interface . Example 2

Proposition In $3d$ with the following configuration

$$u_3(s) = \begin{cases} 1 & \text{for } x < 0\,, \\ 0 & \text{for } x > 0 \end{cases}$$

and $y = u_2(s)$ a C^1 curve:

$$u(x) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2)))$$

is a solution with vorticity, singular on the surface

$$\Gamma(t) = \{(x_1, x_2, x_3)/x_1 = tu_1(x_2), \} \ :$$

Vorticity surface density

$$\omega(x,t) =$$

$$((-t\frac{\partial_{x_2} u_1}{(|t\partial_{x_2} u_1|^2 + 1)^{\frac{1}{2}}}, \frac{1}{(|t\partial_{x_2} u_1|^2 + 1)^{\frac{1}{2}}}) \otimes \delta_{\Gamma(t)}, -\partial_{x_2} u_1(x_2))$$

Comparison with the Kelvin Helmholtz problem in 2 and 3d

Kelvin Helmholtz: Vorticity concentrated on a oriented curve (2d) or surface (3d)

The first example is a solution with a density of vorticity concentrated on a manifold with corners this seems not possible to construct same type of configuration in 2d More room in 3d
In the second example the function $x_2 \mapsto u_1(x_2)$ needs to be C^1 no more and maintain this regularity.

For the Kelvin Helmholtz such property is not possible in 2d and may be possible in 3d

$$u(x,t) =$$
$$\frac{1}{2\pi} R_{\frac{\pi}{2}} \int \frac{x - r(t,\lambda')}{|x - r(t,\lambda')|^2} \tilde{\omega}(t, r(t,\lambda')) |\partial_\lambda (r(t,\lambda'))| d\lambda' \text{ in } 2d\,,$$
$$-\frac{1}{4\pi} \int \frac{x - r(t,\lambda',\mu')}{|x - r(t,\lambda',\mu')|^3} \wedge \tilde{\omega}(t, r(t,\lambda',\mu')) |\partial_\lambda (r(t,\lambda')) \wedge \partial_\mu (r(t,\lambda')) | d\lambda$$

When x converges to $r \in \Gamma(t)$ the velocity $u(x,t)$ converges to two different values $u_\pm(r)$

$$u_+(r)\cdot\vec{n} = u_-(r)\cdot\vec{n}\ ,\quad \omega = (u_+(r) - u_-(r))\wedge\vec{n}\otimes\delta_{\Gamma(t})$$

The vorticity density $\omega = (u_+(r) - u_-(r))\wedge\vec{n}\otimes\delta_{\Gamma(t}$:

$$= \left| \begin{array}{c} 2d\ \tilde{\omega}(t, r(t,\lambda'))|\partial_\lambda(r(t,\lambda'))|d\lambda'\ , \\ 3d\ \tilde{\omega}(t, r(t,\lambda',\mu'))|\partial_\lambda(r(t,\lambda'))\wedge\partial_\mu(r(t,\lambda'))|d\lambda' d\mu'\ . \end{array} \right.$$

On the interface

$$v = \frac{u_+ + u_-}{2}$$

$$(\partial_t r - v) \cdot \vec{n} = 0\,,$$

In $2d$

$$\partial_t \tilde{\omega} + \frac{\partial}{\partial \lambda}\left(\frac{\tilde{\omega}}{|r_\lambda|^2} (v - r_\lambda) \cdot r_\lambda \right) = 0$$

In $3d$ with $N = \partial_\lambda r(t,\lambda,\mu) \wedge \partial_\mu r(t,\lambda,\mu)$

$$\partial_t \tilde{\omega} + \frac{\partial}{\partial \lambda}\left(\frac{\tilde{\omega}}{\|N\|^2} (v - \partial_t r, \partial_\mu r, N) \right) - \frac{\partial}{\partial \mu}\left(\frac{\tilde{\omega}}{\|N\|^2} (v - \partial_t r, \partial_\lambda r, N) \right)$$
$$= \frac{1}{\|N\|^2} (\partial_\mu r, N, \tilde{\omega}) \partial_\lambda v - \frac{1}{\|N\|^2} (\partial_\lambda r, N, \tilde{\omega}) \partial_\mu v = 0\,.$$

KH classical results

1. The initial value problem is locally in time well posed in $2d$ and $3d$ the class of analytic data.
2. There exist in $2d$ in $3d$?? analytic solutions which after a finite time exhibit singularities. Ex: Caflisch and O. Orellana analytic solutions for $0 < t < T$ with a cusp for $t \to T$ (with $0 < \nu < 1$) :

$$\lim_{t\to T}(\Gamma(t), \tilde{\omega}(t)) = (\Gamma(T), \tilde{\omega}(T)) \Bigg| \quad \begin{array}{l} \notin C^{1+\nu} \times C^{\nu}, \\ \in C^{1+\nu'} \times C^{\nu'} \ \ \forall\, 0 < \nu' < \nu\,. \end{array}$$

3. In $2d$ If in the neighbourhood of a point $r(t_0, \lambda_0), \tilde{\omega}(t_0, \lambda_0)$ the density of vorticity does not vanishes and if the functions $r(t, \lambda), \tilde{\omega}(t, \lambda)$ have some *limited regularity* then in fact they are analytic in this neighbourhood.

$$(r(t,\lambda),\tilde\omega(t,\lambda)) \in C^{1+\alpha}\times C^{\alpha}$$
$$|\lambda-\lambda'| \le C|r(t,\lambda)-r(t,\lambda')| \text{ with } C<\infty$$

The catch in the above $2d$ results:

$$\partial_t y - v_2 = -(v_1\partial_x y)\,, \quad (4)$$
$$\partial_t\tilde\omega + \partial_x(v_1\Omega_0) = -\epsilon\partial_x(v_1\tilde\omega)\,, \quad (5)$$
$$v_1(x,t) = -\frac{1}{2\pi}P.V.\int \frac{y(x,t)-y(x',t)}{(x-x')^2+\epsilon^2(y(x,t)-y(x',t))^2}(\Omega_0+\epsilon\tilde\omega)dx' \quad (6)$$
$$v_2(x,t) = \frac{1}{2\pi}P.V.\int \frac{x-x'}{(x-x')^2+\epsilon^2(y(x,t)-y(x',t))^2}(\Omega_0+\epsilon\tilde\omega)dx'\,. \quad (7)$$

This system describes perturbations in $\mathbb{R}^2$ of the stationary solution

$$y(x,0)=0\,, u_- = \frac{\Omega_0}{2}\,, u_+ = -\frac{\Omega_0}{2}\,.$$

The expansion

$$\frac{1}{\pi}P.V.\int \frac{f(x)-f(x')}{(x-x')^2+\epsilon^2(y(x,t)-y(x',t))^2}dx' =$$
$$\frac{1}{\pi}P.V.\int \frac{f(x)-f(x')}{(x-x')^2+\epsilon^2(y(x,t)-y(x',t))^2}dx' =$$
$$\frac{1}{\pi}P.V.\int \frac{f(x)-f(x')}{(x-x')^2}\left(1+\sum_{n\geq 1}(-1)^n\epsilon^{2n}\left(\frac{y(x)-y(x'))}{x-x'}\right)^2\right)dx'$$

leads to the introduction of the operators (Hilbert transform):

$$Hf(x)=\frac{1}{\pi}\int \frac{1}{x-x'}f(x')dx' = F^{-1}(-i\mathrm{sgn}(\xi)\hat{f}(\xi)) \tag{8}$$
$$|D|f(x)=\frac{1}{\pi}P.V.\int \frac{f(x)-f(x')}{(x-x')^2} = \partial_x(Hf(x)) = F^{-1}(|\xi|)\hat{f}(\xi))\,. \tag{9}$$

This gives, with the formula (**??**), (**??**) , for the perturbation of the stationary solution:

$$\partial_t y_x - \Omega_0 |D| \omega = \epsilon F(y_x, \omega)_x$$
$$\partial_t \omega - |D| y_x = \epsilon G(y_x, \omega)_x$$

In the right hand side F and G are first order operators. Eventually with the introduction of the "Laplacian" one has:

$$\partial_{tt}(y_x) + \Omega_0^2 \partial_{xx}(y_x) = \epsilon(\partial_t(F(y_x, \omega)_x) + |D|(\epsilon G(y_x, \omega)_x)\,,$$
$$\partial_{tt}(\omega) + \Omega_0^2 \partial_{xx}(\omega) = \epsilon(|D|(F(y_x, \omega)_x) + \partial_t(\epsilon G(y_x, \omega)_x)\,.$$

The above example not a genuine solution of the Kelvin Helmholtz problem because of $-\partial_{x_2} u_1(x_2)$. With convenient $u_1(x_2)$ $\Gamma(t)$ is a non analytic *"limited regularity"* surface. It satisfies the Lebeau-Wu hypothesis. "Catch" $\Gamma(t) = \{x_3 = \gamma(x_1, x_2, t)\}$, and *small perturbation of the stationnary state* $x_3 = 0, \quad \tilde\omega^0(x_1, x_2) = (\tilde\omega_1^0, \tilde\omega_2^0, 0)$.

Leading part of the perturbed equation $\partial_t \begin{pmatrix} \hat{x_3} \\ \hat\omega_1 \\ \hat\omega_2 \\ \hat\omega_3 \end{pmatrix} = \mathcal{A} \begin{pmatrix} \hat{x_3} \\ \hat\omega_1 \\ \hat\omega_2 \\ \hat\omega_3 \end{pmatrix}$

with $k = |k|(\cos\theta, \sin\theta)$ and $\mathcal{A} =$

$$= \begin{pmatrix} 0 & \frac{i}{2}\sin\theta & -\frac{i}{2}\cos\theta & 0 \\ -\frac{i}{2}|k|^2|\omega^0|^2\sin\theta & 0 & 0 & \frac{1}{2}(k\cdot\omega^0)\sin\theta \\ \frac{i}{2}|k|^2|\omega^0|^2\cos\theta & 0 & 0 & -\frac{1}{2}(k\cdot\omega^0)\cos\theta \\ 0 & -\frac{1}{2}(k\cdot\omega^0)\sin\theta & \frac{1}{2}(k\cdot\omega^0)\cos\theta & 0 \end{pmatrix}$$

eigenvalues of the matrix $\mathcal{A} = (0, 0, -\frac{1}{2}|k\wedge\omega^0|, \frac{1}{2}|k\wedge\omega^0|)$.

$\partial_t - \mathcal{A}$ is no more elliptic a basic reason why a smooth (with limited regularity) singular support may persist without being in fact analytic. $3d$ more stable than $2d$!!!

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$$= \begin{pmatrix} 0 & \frac{i}{2}\sin\theta & -\frac{i}{2}\cos\theta & 0\\ -\frac{i}{2}|k|^2|\omega^0|^2\sin\theta & 0 & 0 & \frac12(k\cdot\omega^0)\sin\theta\\ \frac{i}{2}|k|^2|\omega^0|^2\cos\theta & 0 & 0 & -\frac12(k\cdot\omega^0)\cos\theta\\ 0 & -\frac12(k\cdot\omega^0)\sin\theta & \frac12(k\cdot\omega^0)\cos\theta & 0 \end{pmatrix}$$

$$\text{eigenvalues of the matrix } \mathcal{A}=(0,0,-\frac12|k\wedge\omega^0|,\frac12|k\wedge\omega^0|)\,.$$

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The Shear Flow and the Energy conservation.

Onsager conjecture : for weak solutions of the 3d Euler equation the Energy decay for weak solutions of 3d Euler $\Leftrightarrow$ loss of regularity.

Critical value 1/3.

Constantin E and Titi : $u \in \mathcal{B}^{\frac{1}{3}}_{3+\epsilon,\infty} \Rightarrow$Energy conservation .

DeLellis and Szekelyhidi: No conservation for some solutions in $C_{weak}(\mathbb{R}_t L^2(\mathbb{R}^3))$ and conservation for some "wild" solutions.

Eyink: a function $u_0(x) \in C^{0,\frac{1}{3}}$ which cannot be the initial data of any weak solution which conserves the energy.

This is not however a *full* counter example because the existence of solutions with such initial data is an open problem.

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The Shear Flow and the Energy conservation.

Theorem

In a periodic box for $u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2)))$ *The energy conservation remains true under the only assumption* $u(x,0) \in L^2((\mathbb{R}/\mathbb{Z})^3)$.

Lemma

For $u_1(x_2), u_3(x_1) \in L^2((\mathbb{R}/Z)) \times L^2((\mathbb{R}/Z))$ *and any test functions* $\phi_i\,, i = 1,2$ *the following standard formula*

$$\iiint_\Omega |u_3(x_1 - tu_1(x_2))|^2 \phi_1(x_1)\phi_2(x_2) dx_1 dx_2$$
$$= \iiint_\Omega |u_3(x_1)|^2 \phi_1(x_1 + tu_1(x_2))\phi_2(x_2) dx_1 dx_2$$

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The Shear Flow and the Energy conservation.

Shvydkoy considers the energy conservation for weak solutions the Euler equation with singularities on a curve (in $2d$) and on a surface (in $3d$). This class of solutions includes the Kelvin Helmholtz and our example above.
In agreement with this observation we propose the following example. Consider for $u_1(s)$ a periodic function which coincide near 0 with the function $\sin \frac{1}{s}$ and for $u_3(s)$ a periodic function which near 0 coincides with $\mathrm{sgn}(s)$ the shear flow

$$u(x,t) = (u_1(x_1), 0, u_3(x_1 - tu_1(x_2))$$

is a weak solution which conserves the energy and which does not satisfies the hypothesis given by Shvydkoy.

Viscosity limit, estimates and energy dissipation.

The viscous limit of Leray solution of 3d Navier-Stokes is an open problem. It is also an open problem in 2d in the presence of no slip boundary. It is also a common, belief that the appearance of rough solution of Euler equation is related to energy dissipation
The only $3d$ general result is the fact that the limit is a dissipative solution in the sense of P.L. Lions.
However DeLellis and Szekelyhidi have shown that the notion of dissipative solution is not a criteria for stability and uniqueness!

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Viscosity limit, estimates and energy dissipation

$$\partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu + \nabla p_\nu = 0\,, \quad \nabla \cdot u_\nu = 0\,,$$
$$\frac{1}{2}|u_\nu(t)|^2 + \epsilon(t) \le \frac{1}{2}|u_\nu(0)|^2\,, \epsilon(t) = \nu \int_0^t \int |\nabla u_\nu(x,t)|^2 dxdt\,.$$

For the shear flow the solution is given by

$$u_\nu(x_1,x_2,x_3) = ((u_\nu)_1(x_2,t), 0, (u_\nu)_3(x_1,x_2,t))$$
$$\partial_t (u_\nu)_1(x_2,t)) - \nu \partial^2_{x_2}(u_\nu)_1(x_2,t)) = 0\,,$$
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Proposition

With $L^2((\mathbb{R}/\mathbb{Z})^3)$ initial data $(u_\nu)_1(x_2,t)$ converges strongly (in $C(\mathbb{R}^+_t; L^2((\mathbb{R}/\mathbb{Z})))$ to $u_2(x_2,0)$ and u_ν converges in to the shear flow solution.

Viscosity limit.

In the Torus $(\mathbb{R}/\mathbb{Z})^3$:

Proposition

For $\nu > 0$ *and* $u_0 = u(x,0) \in X$ *with*

$$X \in \{H^{\frac{1}{2}} \subset L^3 \subset \mathcal{B}_{1,\infty}^{-1+\frac{1}{p}} (1 \leq p < \infty) \subset BMO^{-1}\} \tag{10}$$

there exists a time $< T_\nu(u_0)^* \leq \infty$ *and a constant* $C(\nu, u_0)$ *such that: there exist a unique solution* $u_\nu(x,t) \in C(0, T^*; X)$ *of the* $\nu-$*Navier-Stokes equation which satisfies the estimate:*

$$\textit{for } u_0 \in X \textit{ and } 0 \leq t \leq T^*_\nu \ \|u(x,t)\|_X \leq C(\nu, u_0) \tag{11}$$

The previous examples and computations on the shear flow show that for $X = H^{\frac{1}{2}}, \ H^s, s > 0$ the statement of the above proposition has no chance to be true with $u_0 \in X$ and $T^*_\nu, C(\nu, u_0)$ ν-independent.

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converges to the shear flow solution in $C(\mathbb{R}^+_t; L^2_{weak}((\mathbb{R}/\mathbb{Z})^3)$. Since the limit conserves the energy the convergence is strong and the energy dissipation goes to zero. A weak solution with only L^2 regularity, viscous limit of Leray solutions with vanishing energy dissipation and energy conservation

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Conclusions.

All examples: solutions with some singularities in the initial data. What happen for large time with smooth initial data remains a fully open problem.

Wild initial data may generate non uniqueness (even of dissipative solutions) DeLellis and Szekelyhidi and with oscillation limits which are not solutions of the Euler equation...

Shear flow solutions show that C^1 is a "sharp critical space " for the well posedness of the Cauchy problem for the $3d$ Euler Equation.

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