Global recovery of high frequency wave fields

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Computational high frequency wave propagation

- Semiclassical approximation of Schrödinger equations
- High frequency wave propagation in: geometrical optics, seismology, medical imaging, ...
- Math Theory: semiclassical analysis, Lagrangian path integral, wave dynamics in nonlinear PDEs...

Challenges: Propagation of oscillations of small wave length in both space and time causes mathematical and numerical challenges to solve high frequency wave propagation problems.
Direct numerical simulation of the wave dynamics can be very costly and approximate models for wave propagation are often used ...

Towards a recovery theory

- In past years various numerical techniques have been developed to compute phase and amplitude of a wave packet, as well as other physical observables in phase space.

- Our interest is to **recover** high frequency wave fields from computations in phase space?

Outline

- Introduction
- Phase space based computations
- Gaussian beams (GB)
- A global recovery theory
- summary

Two problems

- Quantum wave field

$$
\begin{aligned}
i\epsilon\partial_t\psi &= -\frac{\epsilon^2}{2}\Delta\psi + V_{ext}\psi, \quad x \in \mathbb{R}^n, \\
\psi(x,0) &= A_{\rm in}(x)e^{iS_{\rm in}(x)/\epsilon}.
\end{aligned}
$$

- Acoustic wave field

$$
\partial_t^2 u - c(x)^2\Delta_x u = 0,
$$

$$
u(x,0) = A_{\rm in}(x,\epsilon)e^{iS_{\rm in}(x)/\epsilon}, \quad u_t(x,0) = B_{\rm in}(x,\epsilon)e^{iS_{\rm in}(x)/\epsilon}.
$$

I. Schrödinger equation

$$i\epsilon\partial_t\psi = -\frac{\epsilon^2}{2}\Delta\psi + V_{ext}\psi, \quad \psi(x,0) = A_{\rm in}(x)e^{iS_{\rm in}(x)/\epsilon}, \quad x \in \mathbb{R}^n.$$

- If $V_{ext} = 0$, $A_{\rm in} = 1$ and $S_{\rm in} = -|x|^2/2$, then

$$\psi(t,x) = (1-t)^{-n/2}\exp\left(-\frac{i|x|^2}{2\epsilon(1-t)}\right).$$

- If $V_{ext} = 0$, $A_{\rm in} = g \in C_0^\infty$ and $S_{\rm in} = -|x|^2/2$, then

$$\psi(1,x) = e^{-\pi ni/4}\epsilon^{-n/2}\hat{g}(x/\epsilon)e^{i|x|^2/(2\epsilon)},$$

where $\hat{g}$ is the Fourier transform of g.

Bridging quantum to classical dynamics

- Madelung transform, Madelung (1927): polar coordinator

$$\psi = \sqrt{\rho} e^{iS/\epsilon}$$

$$\rho_t + \partial_x \cdot (\rho \partial_x S) = 0, \quad S_t + \frac{1}{2}|\partial_x S|^2 + V_{ext} = \frac{\epsilon^2}{2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}.$$

- The (modified) WKB method, Wentzel-Kramer-Brillouin (1926): allowing complex amplitude

$$\psi = A(t,x) e^{iS(t,x)/\epsilon}, \quad A \in \mathcal{C}$$

$$A_t + \partial_x S \cdot \partial_x A + \frac{1}{2} A \partial_x^2 S = \frac{i\epsilon}{2} \Delta A, \quad S_t + \frac{1}{2}|\partial_x S|^2 + V_{ext} = 0.$$

WKB vs. FIO

- The WKB method applies to both linear and nonlinear wave equations:

$$\partial_t^2 u - c(x)^2 \Delta_x u = 0$$

elastic waves, electromagnetic waves, etc.

- We always end up at a coupled WKB system

$$\partial_t S + H(\mathbf{x}, \nabla S) = 0, \quad (t, \mathbf{x}) \in R^+ \times R^n,$$
$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \nabla_{\mathbf{k}} H(\mathbf{x}, \nabla_{\mathbf{x}} S)) = 0.$$

- The common problem is at caustics where S becomes singular and $\rho = \infty$!
- Beginning with Keller[1958], Maslov[1965], the consideration of these difficulties led to the development of the theory of Fourier integral operators(FIO), Hörmander[1971].

Unfold 'caustics'

- Two approaches to circumventing caustic difficulties have emerged in literature.
  First, one abandons traditional WKB passage in favor of Wigner functions [1932], which map solutions of the underlying wave equation to functions on phase space
  Lions-Paul (93), Markowich-Mauser (93) ...
  Sparber-Markowich-Mauser(2003).
- Wigner (1932) phase space description

$$w^\epsilon(t,x,k) = \frac{1}{(2\pi)^n}\int e^{ik\cdot y}\psi(x-\frac{\epsilon y}{2})\bar{\psi}(x+\frac{\epsilon y}{2})dy.$$

The limiting Wigner equation

$$w_t+\nabla_k H\cdot\nabla_x w-\nabla_x H\cdot\nabla_k w = 0, \quad w(0,x,k) = \rho_0(x)\delta(k-\nabla_x S_0(x)).$$

Multi-valued solutions to WKB system

- Second, one seeks multi-valued solutions to the WKB system corresponding to crossing waves [Sparber-Markowich-Mauser(2003)].
- Classical viscosity/entropy solutions are inadequate and mutli-valued solutions are physically relevant. For one-dimensional Schrödinger equation with $(\rho, u) = (|A|^2, \partial_x S)$:

$$\partial_t u + u u_x = 0, \quad (t, x) \in R^+ \times R^1,$$
$$\partial_t \rho + \partial_x(\rho u) = 0.$$

- However, at caustics, neither gives correct prediction for the amplitude.

The Gaussian beam method

- The Gaussian beam method: allowing complex phase away from the central ray γ.

$$\psi(t,x) \sim A^\epsilon(t,x)e^{i\Phi(t,x)/\epsilon}.$$

  with

$$Im[\Phi(t,x)] \geq c\mathrm{dist}^2(x,\gamma).$$

- This approach is closely related to Fourier integral operators.
- Gaussian beams are asymptotic solutions concentrated on classical trajectories for the Hamiltonian $H(x,p)$.
- A history going back to at least the late 1960's.
  Initially used to study resonances in lasers [Babich and Buldyrev, 1972], and later to obtain results on the propagation of singularities in solutions of PDE's in Hörmander[1971] and Ralston [1982].

Semiclassical computation

Numerical Advances. In the last decade a considerable amount of work has been done to numerically capture multi-valued phases associated to the WKB system or solve the limiting Wigner equation.

⊙ Three stages:

- Efficient numerical algorithm for capturing multi-valued quantities
- Evaluation of physical observables
- Recovery of the wave field

⊙ Two main issues

- Multiple arrival of 'geometric optics'
- Caustics

Level set method

Recent level set methods for computing high frequency waves have provided a united approach.

$$\phi_t + H_p \cdot \phi_x - H_x \cdot \phi_p = 0.$$

The level set equation = the Liouville equation (1837)

- A geometric closure to the kinetic (Wigner) equation
- Set of multi-valued solutions to the Hamilton-Jacobi equation
- Global invariants of the Hamilton system – the characteristic ODEs (ray equations)

Moreover, one can compute many other quantities near the zero level set manifold, such as phase and its derivatives.

Level set description

In our development of the level set method for HFW problems, we have focused on two aspects:

- Capturing field statistics [multi-valued phase, phase gradient, etc]
- Evaluation of physical observables [density, energy , etc]

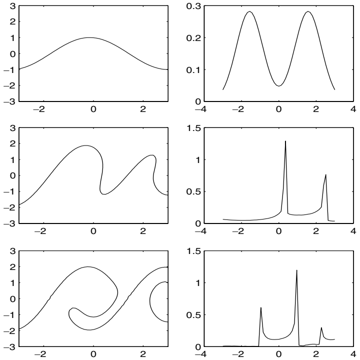
For several different applications:

- Phase space based level set method [Cheng-L-Osher(CMS 03), Jin-Osher(CMS 03)]

  Computing physical observables [Jin-L-Osher-Tsai(JCP 05)]
- Jet space based level set [Cheng-L-Osher(CMS 03), L-Cheng-Osher (JSC 06)]
- Field space based level set method [L-Wang(JCP 07)]
- Bloch-band based level set method[L-Wang(JCP 09)]

Evaluation of density — infinite at caustics!

$S_0 = sin(x + 0.15)$ and $V(x) = cos(2x + 0.4)$



$$\rho_0(x) = \frac{1}{\sqrt{2\pi}} \left[\exp\left(-(x - \frac{\pi}{2})^2 \right) + \exp\left(-(x + \frac{\pi}{2})^2 \right) \right].$$

Towards a global recovery theory

We consider the following equation

$$P\psi = 0, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R},$$

where $P = -i\epsilon\partial_t + H(x, -i\epsilon\partial_x)$ is a linear differential operator with a real principal symbol $\tau + H(x, p)$, subject to the highly oscillatory initial data

$$\psi(x, 0) = \psi_{in}(x) := A_{in}(x)e^{iS_{in}(x)/\epsilon}.$$

Our objective is to establish a global recovery theory:

i) to formulate globally valid asymptotic solutions ψ^ϵ;

ii) to estimate the error $\|\psi(t, \cdot) - \psi^\epsilon(t, \cdot)\| = O(\epsilon^r)$.

What else is needed for a global recovery?

- The level set method has been successful in resolving 'multi-arrival' issues.
- At present there is a considerable interest in using superpositions of beams to resolve high frequency waves near caustics.
- In geophysical applications, [Cervený-Popov (1982), Hill (1990)]

  Recent works in this direction include
  [Tanushev-Qian-Ralston (07)] on gravity waves
  [Leung-Qian(08), Jin-Wu-Yang(08)] on the semi-classical Schrödinger equation
  [Tanushev(07)] and [Motamed-Runborg (08)] on acoustic wave equations.

  Computation in phase space:
  Leung-Qian-Burridge(07), Leung-Qian(08), Jin-Wu-Yang(08)

The Gaussian beam ansatz

Let $X = (x, p)$ be phase space variables, the central ray (bi-characteristic curve) from X_0 is denoted by $X = X(t, X_0)$.
The GB construction adopts the following Ansatz:

$$\psi_{GB}(t, y; X_0) = A(t; X_0) \exp\left(\frac{i}{\epsilon}\Phi(t, y; X_0)\right),$$

where with $p(t, X_0) = \nabla_x \Phi(t, x(t, X_0))$:

$$\Phi(t, y; X_0) = S(t; X_0) + p(t, X_0)(y - x(t, X_0)) + \frac{1}{2}(y - x(t, X_0))^\top M(t; X_0)(y - x(t, X_0)).$$

2nd order Taylor expansion of Φ about y at the curve $x = x(t, X_0)$.

Construction of Gaussian beams [Ralston(1982)]

Along $X = X(t, X_0)$, the lowest order GB components $\{S, M, A\}$ solve

$$\frac{d}{dt}S(t; X_0) = p \cdot H_p - H(x, p), \quad S(0; X_0) = S_{\rm in}(x_0),$$
$$\frac{d}{dt}M(t; X_0) + H_{xx} + H_{xp}M + MH_{px} + MH_{pp}M = 0, \quad M(0; X_0) = M_{\rm in}(X_0),$$
$$\frac{d}{dt}A(t; X_0) = -\frac{A}{2}\left[Tr[H_{xp}] + Tr[MH_{pp}]\right], \quad A(0; X_0) = A_{\rm in}(x_0).$$

This ensures that the GB ansatz is an asymptotic solution.
The essential idea behind the GB method is to choose some complex Hessian $M_{\rm in}$ initially so that M remains bounded for all time, and its imaginary part is positive definite. This way the amplitude $A(t; X_0)$ is ensured to be also globally bounded!

Phase space GB ansatz

- (**Operator lifting**) Let the phase representative of $w(t;X_0)$ be $\tilde{w}(t,X)$ in the sense that they coincide on $X = X(t,X_0)$ for any $t > 0$, then
$$\frac{d}{dt}w(t;X_0) = \mathcal{L}\tilde{w}(t,X),$$
where $\mathcal{L}$ is the usual Liouville operator defined by
$$\mathcal{L} := \partial_t + V \cdot \nabla_X.$$

- First order Gaussian beam ansatz, defined through the volume preserving map $X = X(t,X_0)$,
$$\psi_{PGB}(t,y,X) = \tilde{A}(t,X)\exp\left(\frac{i}{\epsilon}\tilde{\Phi}(t,y,X)\right),$$
where
$$\tilde{\Phi}(t,y,X) = \tilde{S}(t,X) + p\cdot(y-x) + \frac{1}{2}(y-x)^\top \tilde{M}(t,X)(y-x).$$

Computation of Hessian and other phase derivatives

Let the Hamiltonian dynamics be expressed as

$$\frac{d}{dt}X(t, X_0) = v(X(t, X_0)), \quad X(0, X_0) = X_0,$$

where $X = (x, p)$ and the phase velocity $v = (H_p, -H_x)$. The bi-characteristics curve is expressed as

$$\Gamma = \{(t, X),\ \phi(t, X) = 0\}, \quad \phi = (\phi_1, \phi_2).$$

where

$$(\partial_t + v \cdot \nabla_X)\phi(t, X) = 0, \quad \phi(0, X) = X - X_0 \in R^{2n}. \qquad (1)$$

One may obtain any derivatives of the phase from this vector valued level set function:

$$M = -g_x(g_p)^{-1}, \quad g = (1, i\beta) \cdot \phi.$$

Superposition on subdomains

- The phase space based GB ansatz

$$\psi_{PGB}(t,y,X) := \psi_{GB}(t,y;X_0(t,X))$$

is no longer an asymptotic solution!
- The superposition of ψ_{PGB} over $\Omega(t) := X(t,\Omega(0))$ remains a correct asymptotic solution.

$$\psi^\epsilon(t,y) = Z(n,\epsilon)\int_{\Omega(t)} \psi_{PGB}(t,y,X)dX,$$

$\Omega(0) =$ an open domain in phase space from which we want to construct initial Gaussian beams
$Z(n,\epsilon) \sim \epsilon^{-n/2}$ is a normalization parameter chosen to match initial data $\psi_0(y)$ against the Gaussian profile.
- This can be seen by the following superposition of the Lagrangian construction

$$\psi^\epsilon(t,y) = Z(n,\epsilon)\int_{\Omega(0)} \psi_{GB}(t,y;X_0)dX_0.$$

Evolution error

- Asymptotic solution:

$$Z^{-1}P[\psi^\epsilon] = (-i\epsilon\partial_t + H(y, -i\epsilon\partial_y)) \int_{\Omega(t)} \psi_{PGB}(t, y, X) dX$$
$$= \int_{\Omega(t)} [P[\psi_{PGB}] - i\epsilon\nabla_X \cdot (V\psi_{PGB}))] \, dX. \qquad (2)$$

  The extra term in the integral gives an alternate way of seeing that the phase space super-position is an accurate solution of the PDE. This makes it possible to verify the accuracy without reference to the Lagrangian superposition.

- (Residual error)
  Let P be the linear Schrödinger wave operator of the form $P = -i\epsilon\partial_t + H(y, -i\epsilon\partial_y)$, where $H(y, p) = \frac{|p|^2}{2} + V_{ext}(y)$. If $Im(\tilde{M})$ is positive definite and $Z(n, \epsilon) \sim \epsilon^{-n/2}$, then ψ^ϵ is an asymptotic solution:

$$\|P[\psi^\epsilon](t, \cdot)\|_{L^2_y} \lesssim |\Omega(0)|\epsilon^{\frac{3}{2} - \frac{n}{4}}.$$

$$\|(\psi^\epsilon - \psi)(t,\cdot)\|_{L^2} \le \|\psi^\epsilon(0,\cdot) - \psi_{\rm in}(\cdot)\|_{L^2} + C|\Omega(0)|\epsilon^{\frac{1}{2}-\frac{n}{4}}.$$

Proof.

Let $e := \psi^\epsilon - \psi$, then from $P[\psi] = 0$ it follows

$$P[e] = P[\psi^\epsilon] - P[\psi] = P[\psi^\epsilon].$$

A calculation of $\int_{R^n}[e\overline{P[e]} - \bar{e}P[e]]dy$ leads to

$$\epsilon \frac{d}{dt}\int_y |e|^2 dy = \int_y Im(e\overline{P[\psi^\epsilon]})dy.$$

Integration over $[0,t]$ gives

$$\|e(t,\cdot)\|_{L^2_y} \le \|e(0,\cdot)\|_{L^2_y} + \frac{1}{\epsilon}\int_0^t \|P[\psi^\epsilon](\tau,\cdot)\|_{L^2} d\tau, \quad t \in [0,T]. \qquad (3)$$

□

Control of initial error

For highly oscillatory initial data we have

$$\psi_{\rm in}(y) = A_{\rm in}(y)e^{iS_{\rm in}(y)/\epsilon} = \int_x A_{\rm in}(y)e^{iS_{\rm in}(y)/\epsilon} K\left(x-y, \frac{\epsilon}{2}\right) dx.$$

Both the phase and amplitude in the integrand can be approximated by their Taylor expansion when $|x-y|$ is small, say $|x-y| < \epsilon^{1/3}$, and the integral will then be $O(\exp(-c/\epsilon^{1/3}))$ with some $c < \frac{1}{2}$ outside this neighborhood. Let $T_j^x[f](y)$ denote the j^{th} order Taylor polynomial of f about x at the point y. Then

$$\psi_{\rm in}(y) \sim g := \int_x A_{\rm in}(x)e^{\frac{i}{\epsilon}[T_2^x[S_{\rm in}](y)]} K\left(x-y, \frac{\epsilon}{2}\right) dx.$$

Tanushev[08] proved that

$$\|\psi_{\rm in}(\cdot) - g\|_{L^2} \lesssim \epsilon^{\frac{1}{2}}.$$

Main result

For the initial data of the form $\psi_{\rm in} = A_{\rm in}(x)e^{iS_{\rm in}(x)/\epsilon}$ we take $p = \nabla_x S_{\rm in}(x)$. The asymptotic solution is then represented as

$$\psi^\epsilon(t,y) = Z(n,\epsilon)\int_{\Omega(t)} \psi_{PGB}(t,y,X)\delta(w(t,X))dX, \quad (4)$$

where w is obtained from the Liouville equation

$$\partial_t w + H_p \cdot \nabla_x w - H_x \cdot \nabla_p w = 0, \quad w(0,X) = p - \nabla_x S_{\rm in}(x).$$

Theorem

Given $T > 0$, and let ψ be the solution of the Schrödinger equation and ψ^ϵ be the k^{th} order Gaussian beam superposition. Then

$$\|(\psi^\epsilon - \psi)(t,\cdot)\|_{L^2} \lesssim |{\rm supp}(A_{\rm in})|\epsilon^{\frac{k}{2}-\frac{n}{4}}.$$

A closer look at caustics

$$i\epsilon\partial_t\psi = -\frac{\epsilon^2}{2}\Delta\psi, \quad x \in \mathbb{R}^n,$$

with the initial data $\psi(0,x) = \exp(-i|x|^2/(2\epsilon))$, then

$$\psi(t,x) = (1-t)^{-n/2}\exp\left(-\frac{i|x|^2}{2\epsilon(1-t)}\right).$$

This solution becomes a multiple of the δ-function at $t = 1$. This suggested to use the initial data $\psi(x,0) = g(x)\exp\left(-i|x|^2/(2\epsilon)\right)$:

$$\psi(1,x) = \frac{1}{(2\pi i\epsilon)^{n/2}}\int_y g(y)e^{-ix\cdot y/\epsilon}dye^{i|x|^2/(2\epsilon)} = c\epsilon^{-n/2}\hat{g}(x/\epsilon)e^{i|x|^2/(2\epsilon)},$$

where $c = e^{-\pi ni/4}$ and $\hat{g}$ is the Fourier transform of g. As $\epsilon \to 0$, $\psi(1,x)$ diverges (pointwise) like $\epsilon^{-n/2}$ near $x = 0$, but goes rapidly to zero away from $x = 0$.

GB superposition

We now build a superposition of Gaussian beams approximation for the solution of the same problem. For the Gaussian beam superposition, the phase is obtained as

$$\Phi(t,x;y) = (t-1)\frac{|y|^2}{2} - x\cdot y + |y|^2(1-t) + \frac{\beta i - 1}{1+(\beta i - 1)t}\frac{|x-y(1-t)|^2}{2}.$$

For the amplitude we get

$$A(t,x(t;y)) = (1+(\beta i-1)t)^{-n/2}g(y), \quad x(t;y) = (1-t)y.$$

If we do the superposition with the normalization, we end up with

$$\psi^\epsilon(t,x) = \left(\frac{\beta}{2\pi\epsilon}\right)^{n/2}\int_y [1+(\beta i-1)t]^{-n/2}g(y)e^{i\Phi(t,x;y)/\epsilon}dy.$$

If we evaluate that at $t=1$, it becomes

$$\begin{aligned}\psi^\epsilon(1,x) &= \left(\frac{\beta}{2\pi\epsilon}\right)^{n/2}\int_y [\beta i]^{-n/2}g(y)e^{\frac{i}{\epsilon}[-x\cdot y+\frac{\beta i-1}{\beta i}\frac{|x|^2}{2}]}dy \\ &= c\epsilon^{-n/2}\hat{g}(x/\epsilon)e^{(\beta i-1)|x|^2/(2\beta\epsilon)} \\ &= \psi(1,x)e^{-|x|^2/(2\beta\epsilon)}.\end{aligned}$$

A remarkable accuracy

This shows that at the caustic $x = 0$, both become the same. We can see the error $\psi(1, x)\left(1 - e^{-|x|^2/(2\beta\epsilon)}\right)$ when measured in L^2-norm:

$$\|\psi^\epsilon(1, \cdot) - \psi(1, \cdot)\|_{L^2}^2 = \epsilon^{-n} \int |\hat{g}\left(\frac{x}{\epsilon}\right)|^2 \left(1 - e^{-|x|^2/(2\beta\epsilon)}\right)^2 dx$$
$$= \int |\hat{g}(z)|^2 \left(1 - e^{-\epsilon|z|^2/(2\beta)}\right)^2 dz.$$

That implies that

(a) For any $g \in L^2$ the Gaussian beam approximation converges to the true solution (at $t = 1$), but there is no uniform estimate on the difference in terms of the L^2-norm of g.

(b) If $\int |\hat{g}(z)|^2(1 + |z|^2)^2 dz < 1$, i.e. if $g \in H^2$, then the norm of the difference is $O(\epsilon)$.

No damage for exact GB solutions!

Theorem

Assume that the potential is a quadratic function. Then for $t \in [0, T]$ and $\epsilon \in (0, \epsilon_0)$ we have

- *If $S_{\rm in}$ is a quadratic function and $A_{\rm in} \in H^2$*

$$\|(\psi^\epsilon - \psi)(t, \cdot)\|_{L^2} \lesssim \epsilon.$$

- *If $S_{\rm in} \in C^\infty$ and $A_{\rm in} \in C_0^\infty$*

$$\|(\psi^\epsilon - \psi)(t, \cdot)\|_{L^2} \lesssim \epsilon^{1/2}.$$

No damage is done by caustics when GB solution solves the PDE exactly!

II. Acoustic wave equation

Our phase space based Gaussian beam superposition is

$$u^{\epsilon}(t,y) = Z(n,\epsilon)\left[\int_{\Omega^{+}(t)} u^{+}_{PGB}(t,y,X)dX + \int_{\Omega^{-}(t)} u^{-}_{PGB}(t,y,X)dX\right],$$

where $X=(x,p)$ denotes variables in phase space $\mathbb{R}^{2n}$, $\Omega^{\pm}(t)$ is the image of initial domain $\Omega(0)$ under the Hamiltonian flow for $H(x,p)=\pm c(x)|p|$.
The functions $u^{\pm}_{PGB}(t,y,X)$ are constructed using the phase space based Gaussian beams, and $Z(n,\epsilon)\sim\epsilon^{-n/2}$ is a normalization parameter.

Evolution error

For the $k-$th order phase space Gaussian beam superposition, the following estimate holds

$$\|(u^\epsilon - u)(t,\cdot)\|_E \lesssim \|u^\epsilon(0,\cdot) - u_{\rm in}(\cdot)\|_E + |\Omega(0)|\epsilon^{\frac{k}{2}},$$

where

$$\|e\|_E^2 := \frac{\epsilon^2}{2}\int_{\mathbb{R}^n}[c^{-2}|e_t|^2 + |\nabla_x e|^2]dx.$$

A refined superposition!

For the initial data of the form $(A_{\rm in}(x,\epsilon), B_{\rm in}(x,\epsilon))e^{iS_{\rm in}(x)/\epsilon}$ we need a superposition over an n-dimensional submanifold of phase space. The asymptotic solution is then represented as

$$u^\epsilon(t,y) = Z(n,\epsilon)\left[\int_{\Omega^+(t)} u^+_{PGB}\delta(w^+)dX + \int_{\Omega^-(t)} u^-_{PGB}\delta(w^-)dX\right],$$

where $w^\pm$ is obtained from the Liouville equation

$$\partial_t w + H_p\cdot\nabla_x w - H_x\cdot\nabla_p w = 0, \quad w(0,X) = p - \nabla_x S_{\rm in}(x),$$

with $H(x,p) = \pm c(x)|p|$, respectively.

Main result

Theorem

Given $T > 0$, and let u be the solution of the acoustic wave equation and u^ϵ be the k^{th} order Gaussian beam superposition. Then

$$\|(u^\epsilon - u)(t, \cdot)\|_E \lesssim \epsilon^{\frac{k}{2} + \frac{1-n}{4}}, \quad |t| \le T.$$

Here the exponent $k/2$ reflects the accuracy of the Gaussian beam in solving the PDE. It will increase when one uses more accurate beams. The exponent $\frac{1-n}{4}$ indicates the damage done by the caustics.

Summary

We have presented a systematic recovery theory of high frequency wave fields to both Schrödinger equations and acoustic wave equations.

- The support of initial data contributes to the error
- Initialization is essential for the error control
- The damage done by the caustics can be fixed by using more accurate GB solutions.
- If the GB superposition is exact, then no damage is done by caustics.
- All higher order derivatives of the phase can be evaluated by the level set functions!

A key estimate

Lemma

Assume that $Im(\tilde{\Phi}(t, y, X)) \geq c|y - x|^2$, $c > 0$, *and the Lebeque measure of the initial domain* $|\Omega(0)|$ *is bounded. Let* $B(t, y, X)$ *be a smooth function, satisfying*

$$|B| \leq C|y - x|^k, \quad k > 0.$$

Then we have

$$\left\| \int_{\Omega(t)} B(t, y, X) e^{i\tilde{\Phi}(t,y,X)/\epsilon} dX \right\|_{L^2_y} \lesssim |\Omega(0)| \epsilon^{\frac{k}{2} + \frac{n}{4}}.$$

Higher order GB Approximations

$$\psi_{kGB}(t, y; X_0) = \rho(y - x) \left[\sum_{l=0}^{\lfloor \frac{k-1}{2} \rfloor} \epsilon^l T^x_{k-1-2l}[A_l](y) \right] \exp\left(\frac{i}{\epsilon} T^x_{k+1}[\Phi](y) \right),$$

where $T^x_k[f](y)$ is the k^{th} order Taylor polynomial of f about x evaluated at y, and ρ is a cut-off function such that on its support the Taylor expansion of Φ still has a positive imaginary part.
Then the phase space based k^{th} order Gaussian beam Ansatz becomes

$$\psi_{kPGB}(t, y, X) := \psi_{kGB}(t, y; X_0(t, X)),$$

which is again no longer an asymptotic solution of the wave equation!

The kth order GB superposition is

$$\psi_k^\epsilon(t,y) = Z(n,\epsilon)\int_{\Omega(t)} \psi_{kPGB}(t,y,X)\delta(w(t,X))dX,$$

where $w(t,X)$ is the solution of the Liouville equation subject to $w(0,X) = p - \nabla_x S_{\rm in}(x)$. We show that

$$\|P[\psi_k^\epsilon](t,\cdot)\|_{L_y^2} \lesssim |\mathrm{supp}(A_{\rm in})|\epsilon^{\frac{k}{2}+1-\frac{n}{4}}.$$

Theorem

Given $T>0$, and let ψ be the solution of the Schrödinger equation subject to the initial data $\psi_{\rm in} = A_{\rm in}e^{iS_{\rm in}(x)/\epsilon}$, and ψ^ϵ be the k^{th} order approximation defined above:

$$\|(\psi^\epsilon - \psi)(t,\cdot)\|_{L^2} \lesssim |\mathrm{supp}(A_{\rm in})|\epsilon^{\frac{k}{2}-\frac{n}{4}}$$

for $t \in [0, T]$.