Multiscale Dynamics of Hyperbolic PDEs in Bounded Domains with Ill-prepared Initial Data

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FRG Workshop
Brown University

May 10, 2010

In connection with Kinetic Theory

- Kinetic eqs are partially hyperbolic.
- Multiscale justification of fluid equations as singular limits of Boltzmann eqs, especially when far away from Maxwellian.
- Multiscale computation of kenetic equations.
- Boundary conditions: kinetic description of boundary layer.

Why initial-boundary value problem of Hyperbolic PDEs?

Simplest examples.

- Linear transport equation.

$$\partial_t u + \partial_x u = 0, \quad x \in [0, 1].$$

  Well-posedness of classical solutions?

- Linear wave equation on Ω =ellipsoid,

$$\partial_{tt}\phi - \nabla\phi = 0, \quad \nabla\phi{\cdot}\vec{n}\big|_{\partial\Omega} = 0.$$

  Waves emitting from one focus converge simultaneously to the other focus (“whisper chamber”).

Compressible Euler equations for barotropic fluids

Spatial domain $\Omega \subset \mathbb{R}^3$: bounded, smooth. Velocity u, density $\hat{\rho} \approx 1$

Mach number $\varepsilon \ll 1$ inducing fast scale.

$$\begin{cases} \hat{\rho}_t + \nabla\cdot(\hat{\rho} u) = 0 \\ \rho(u_t + u\cdot\nabla u) + \dfrac{\nabla p(\hat{\rho})}{\varepsilon^2} = 0 \end{cases}$$

eq. of state: pressure=$p(\hat{\rho})$ with $p(\cdot) \in C^\infty$, $p'(1) = 1$.

density perturbation $\rho := (\hat{\rho} - 1)/\varepsilon$

$$\begin{cases} \rho_t + \nabla\cdot(\rho u) + \dfrac{\nabla\cdot u}{\varepsilon} = 0 \\ u_t + u\cdot\nabla u + \dfrac{p'(1+\varepsilon\rho)}{1+\varepsilon\rho}\,\dfrac{\nabla\rho}{\varepsilon} = 0 \end{cases}$$

Solid-wall boundary conditions: $u\cdot\vec{n}\big|_{\partial\Omega} = 0$.

- $\mathbf{X}_m := \left\{ U = \binom{\rho}{u} \in (H^m(\Omega))^4 \;\middle|\; u\cdot\vec{n}\big|_{\partial\Omega} = 0, \quad \int_\Omega \rho\, dx = 0 \right\}$
- $\mathcal{L}[U] := \binom{\nabla\cdot u}{\nabla p}$ and L^2 projection $\mathcal{P} : (H^m)^4 \mapsto \overline{\mathbf{X}_m \cap \mathrm{Ker}\mathcal{L}}^{L^2}$.

$$\mathcal{P}[U] := \begin{pmatrix} 0 \\ u - \nabla\phi \end{pmatrix} \quad \text{where } \phi \text{ solves} \quad \begin{cases} \Delta\phi = \nabla\cdot u \\ \nabla\phi\cdot\vec{n} = u\cdot\vec{n} \text{ on } \partial\Omega \end{cases}$$

Theorem (BC 2010). Fix integer $m \geq 6$ and initial data $U_0 \in \mathbf{X}_m$, compatible with the boundary condition "$\partial_t^k U_0$"$\cdot\vec{n}\big|_{\partial\Omega} = 0$ for $k < m$. Then, there exist constants $\bar{\varepsilon}$, T, C continuously dependent on $\|U_0\|_{H^m}$ s.t. for all $\varepsilon \in (0, \bar{\varepsilon}]$ and $t \in [0, T]$,

$$\|\mathcal{P}[U](t,\cdot) - U^{inc}(t,\cdot)\|_{H^{m-3}(\Omega)} \leq C\varepsilon\,.$$

$U^{inc} = \binom{0}{v}$ solves the incomp. Euler eqs. $\partial_t v + v\cdot\nabla v + \nabla p = 0$, $\nabla\cdot v = 0$, i.e. $\partial_t U^{inc} + \mathcal{P}[B(U^{inc}, \nabla U^{inc})] = 0, \quad U_0^{inc} = \mathcal{P}[U_0]$.

Available results on existence and regularity,

- Linear.
  - L^2 theory: Friedrichs (58'), Lax & Phillips (60').
  - H^m theory: Rauch & Massey (74', nonsingular boundary matrix), Rauch (85', singular boundary matrix, tangential regularity).
- Quasilinear.
  - Schochet (86'a, compressible Euler; 86'b, general case with vorticity equation) — full regularity.

$$\partial_t U + B(U, \nabla U) + \frac{1}{\varepsilon}\mathcal{L}[U] = 0, \quad u\cdot\vec{n}\big|_{\partial\Omega} = 0$$

$B(\cdot,\cdot)$ vanishes at 0 of 2nd order and does *not* depend on ε^{-1}.

In this talk ...

- Estimates on $\mathcal{P}[U] - U^{inc}$.
  - Vorticity equation $\rightarrow \mathcal{P}$[fast-fast interaction] $\equiv 0$
  - Average in time $\int_0^T B(U, \nabla U) - B(\mathcal{P}[U], \nabla\mathcal{P}[U])dt$ and perform integration by parts.
- Uniform *a priori* estimates (i.e. T is independent of ε).

Utilize ellipitic estimates [Admon, Douglis, Nirenberg, 64'].

Elliptic operators

(a) $\exists$ "vorticity operator" $\mathcal{K}$ s.t. $\mathcal{K}\mathcal{L} \equiv 0$.

$$\text{Here,} \quad \mathcal{K}[U] = \nabla \times u.$$

(b) Ellipitic estimates *a la* Nirenberg, etc

$$\|U\|_{H^m} \lesssim \|\mathcal{L}[U]\|_{H^{m-1}} + \|\mathcal{K}[U]\|_{H^{m-1}} + \|U\|_{L^2}.$$

$$\text{Here, for } U \in \mathbf{X}_m, \begin{cases} \|u\|_{H^m} \lesssim \|\nabla \cdot u\|_{H^{m-1}} + \|\nabla \times u\|_{H^{m-1}} \\ \|\rho\|_{H^m} \lesssim \|\nabla \rho\|_{H^{m-1}} \end{cases}$$

Claim. In addition to the definition, $\mathrm{Img}\mathcal{P} \cap \mathbf{X}_m = \mathrm{Ker}\mathcal{L}\big|_{\mathbf{X}_m}$, there is a dual characterization of $\mathcal{P}$,

$$\mathrm{Ker}\mathcal{P}\big|_{\mathbf{X}_m} = \mathrm{Ker}\mathcal{K}|_{\mathbf{X}_m} = \mathrm{Img}\mathcal{L} \cap \mathbf{X}_m.$$

(not true for no-slip boundary condition).

Estimates on $\mathcal{P}[U] - U^{inc}$

Let $U^P = \mathcal{P}[U]$, $U^Q = U - U^P$.

$$\begin{cases} \partial_t U^P + \mathcal{P}[B(U,\nabla U)] = 0 & \text{slow} \\ \partial_t U^Q + (I-\mathcal{P})[B(U,\nabla U)] = \varepsilon^{-1} L[U^Q] & \text{fast} \end{cases}$$

We need to compare

$$-\partial_t U^{inc} = \mathcal{P}[B(U^{inc}, \nabla U^{inc})]$$

v.s.

$$\begin{aligned} -\partial_t U^P &= \mathcal{P}[B(U,\nabla U)] \\ &= \mathcal{P}[B(U^P,\nabla U^P)] + \underbrace{\mathcal{P}[B(U^Q,\nabla U^Q)]}_{I} + \\ &\quad \underbrace{\mathcal{P}[B(U^P,\nabla U^Q)] + \mathcal{P}[B(U^Q,\nabla U^P)]}_{II} \end{aligned}$$

Is it really I, $II \sim O(1)$ or otherwise?

Claim. Fast-fast interaction has no contribution to the slow dynamics,

$$\mathcal{P}[B(U^Q, \nabla U^Q)] \equiv 0 \quad \text{for} \quad U^Q \in \mathbf{X}_m.$$

- Recal $U^Q = \begin{pmatrix} \rho \\ \nabla\phi \end{pmatrix}$ for ϕ solves $\Delta\phi = \nabla\cdot u$ with $\nabla\phi\cdot\vec{n}\big|_{\partial\Omega} = 0$.

$$B(U^Q, \nabla U^Q) = \begin{pmatrix} \nabla\cdot(\rho\nabla\phi) \\ \nabla\phi\cdot\nabla(\nabla\phi) \end{pmatrix} = \begin{pmatrix} \cdots \\ \nabla|\nabla\phi|^2 \end{pmatrix}$$

$$\to \mathcal{P}[\text{potential flow}] = 0 \to \quad \text{Claim.}$$

- $\text{Ker}\mathcal{K}\big|_{H^m} = \text{Ker}\mathcal{P}\big|_{H^m}$ implies $\omega = 0$ iff $U = U^Q$. By vorticity eq.

$$\underbrace{\omega_t}_{\mathcal{K}[U_t]} + \underbrace{\nabla\cdot(\omega u) + \omega\cdot\nabla u}_{\mathcal{K}[B(U,\nabla U)]} = \underbrace{0}_{\frac{1}{\varepsilon}\mathcal{K}\mathcal{L}[U]}.$$

Set $U = U^Q$ and thus $\omega = 0$ in $\mathcal{K}[B(U^Q, \nabla U^Q)] = 0 \to$ Claim.

II: fast-slow interaction: average in time

Ignoring regularity issues, take for example a term $U^P U^Q$.

Claim. $\int_0^T U^P U^Q \, dt \sim O(\varepsilon)$

Proof. Upon performing integration by parts in in time, we estimate

- Slow $\partial_t U^P \sim O(1)$ by the slow eq.
- Fast part averaged in time $U^Q = \partial_t W$ with $W(T, \cdot) := \int_0^T U^Q(t, \cdot) dt \sim O(\varepsilon)$. Why? Elliptic estimates.

$$\text{Since } \mathcal{L}\mathcal{P} = 0, \quad \varepsilon^{-1}\mathcal{L}[U^Q] = U_t + B(...)$$

$$\text{Integrate in time,} \quad \varepsilon^{-1}\mathcal{L}[W] = U\Big|_0^T + \int_0^T B(...)dt$$

$$\mathcal{P}[U^Q] = 0 \rightarrow \mathcal{K}[U^Q] = 0 \rightarrow \mathcal{K}[W] = 0$$

Finally, estimate $\int_0^T U^P U^Q \, dt = U^P W\Big|_0^T - \int_0^T \partial_t U^P W \, dt \sim O(\varepsilon)$

Error equation of $U^P - U^{inc}$

Now let $R(T,\cdot) := \int_0^T (I + II)\,dt$ so that upon *a priori* estimates of U, we have $\|R\|_{\mathbf{X}_{m-2}} \sim O(\varepsilon)$ controlled in spatial norm.

$$\partial_t U^P + \mathcal{P}[B(U^P, \nabla U^P)] + \partial_t R = 0$$

To avoid unwanted time differentiation, let $V := U^P + R$,

$$\partial_t V + \mathcal{P}[B(V, \nabla V)] + R_1 = 0, \quad \|R_1\|_{\mathbf{X}_{m-3}} \sim O(\varepsilon),$$

compared with

$$\partial_t U^{inc} + \mathcal{P}[B(U^{inc}, \nabla U^{inc})] = 0$$

This comparison uses standard energy estimates and does NOT involve estimating time derivatives. Boundary conditions are O.K.

Corollary. (fast oscillations and weak* convergence)

$$\left| \int_0^T \int_\Omega (U - U^{inc}) \phi(t,x)\, dxdt \right| \le C\varepsilon (\|\phi\|_{L_t^\infty L_x^2} + \|\phi_t\|_{L_{t,x}^2})$$

for any smooth testing function $\phi \in C^1([-1, T+1] \times \Omega)$.
Proof. $U - U^{inc} = U^Q + (U^P - U^{inc})$ and $\|U^P - U^{inc}\| \sim O(\varepsilon)$. So it suffices to estimate U^Q. Define $W(T,\cdot) := \int_0^T U^Q(t,\cdot)dt \sim O(\varepsilon)$ as before and perform integration by parts,

$$\int_0^T U^Q \phi(t,x)\, dt = W\phi\big|_0^T - \int_0^T W \partial_t \phi\, dt.$$

The rest is just integrate the RHS over Ω and estimate.

Uniform *a priori* estimates

Main trouble is that $\int_\Omega U{\cdot}\mathcal{L}[U] = 0$ only when $u{\cdot}\vec{n} = 0$ on $\partial\Omega$.

- $\partial_t^k U$ satisfies the solid-wall boundary condition $\to$ growth in L^2 norm of $\partial_t^k U$ estimated by energy method.
- $\mathcal{K}[U]$ satisfies vorticity equation w/o $\mathcal{L}$ $\to$ energy method.
- $-\varepsilon^{-1}\mathcal{L}[U] = \partial_t U + B(..)$ $\to$ estimate $\mathcal{L}[U]$ statically.

Use elliptic estimates to close the argument.

Symmetrization and rescaling

Eq. of state: density$= r$(pressure).

Let pressure$= \varepsilon p$ so that $1 + \varepsilon\rho = r(\varepsilon p)$,

$$\begin{cases} \dfrac{r'(\varepsilon p)}{r(\varepsilon p)}\,(p_t + u\cdot\nabla p) + \dfrac{\nabla\cdot u}{\varepsilon} = 0 \\ r(\varepsilon p)\,(u_t + u\cdot\nabla u) + \dfrac{\nabla p}{\varepsilon} = 0 \end{cases}$$

As a symmetric hyperbolic PDE system for $V = \binom{p}{u}$,

$$A_0(\varepsilon V)\partial_t V + \sum_{j=1}^{3} A_j(V)\partial_{x_j} V + \frac{1}{\varepsilon}\mathcal{L}[V] = 0, \quad u\cdot\vec{n}\big|_{\partial\Omega} = 0$$

replace V with εV and t with $\varepsilon^{-1} t$

$$A_0(V)\partial_t V + \sum_{j=1}^{3} A_j(V)\partial_{x_j} V + \mathcal{L}[V] = 0$$

Space-time norm

$$\mathbf{X}_{m,T} := \{V = \begin{pmatrix} p \\ u \end{pmatrix} \quad \text{s.t.} \quad u\cdot\vec{n}\big|_{\partial\Omega} = 0, \ \frac{1}{|\Omega|}\int_\Omega r(p)dx = 1,$$
$$\partial_t^\alpha \partial_x^\beta V \in L_t^\infty([0,T]; L_x^2(\Omega)) \quad \text{for } \alpha + |\beta| \le m\}$$

Claim. For initial data $V_0 \in \mathbf{X}_m$ and $\tilde{V}$, $V \in \mathbf{X}_{m,T}$, the system

$$A_0(\tilde{V})\partial_t V + \sum_{j=1}^{3} A_j(\tilde{V})\partial_{x_j} V + \mathcal{L}[V] = 0$$

is equipped with Gronwall's inequallity,

$$\|V\|_{\mathbf{X}_{m,T}} \lesssim g(\|V_0\|_{\mathbf{X}_m}) + f(\|\tilde{V}\|_{\mathbf{X}_{m,T}}, \|V\|_{\mathbf{X}_{m,T}}) + \int_0^T f(\|\tilde{V}\|_{\mathbf{X}_{m,t}}, \|V\|_{\mathbf{X}_{m,t}})\, dt$$

for smooth function $g(a) \sim a$ and $f(a,b) \sim ab$ near origin.

Corollary. For small ε and $\|\tilde{V}\| \sim O(\varepsilon)$, solution size $\|V\|$ stays $O(\varepsilon)$ for time $\sim 1/\varepsilon$. Here, $V_0 = \varepsilon U_0$.

A recursive program to close estimates

Step a Estimate the growth of $A(t) := \sum_{\alpha+|\beta|<m} \|\partial_t^\alpha \partial_x^\beta U\|_{L^2(\Omega)}$.

Step b Estimate the growth of $B(t) := \sum_{\alpha+|\beta|=m-1} \|\partial_t^\alpha \partial_x^\beta \omega\|_{L^2(\Omega)}$ using $\partial_t\omega + u\cdot\nabla\omega + \omega\nabla\cdot u + \omega\cdot\nabla u = 0$.

Step c_1 Estimate the growth of $C_1(t) := \|\partial_t^m U\|_{L^2}$ (boundary ok). $k := 1$.

Step d_k Estimate $D_k := \|\partial_t^{m-k}\partial_x^{k-1}\mathcal{L}[U]\|_{L^2}$ directly from $A_0(V)\partial_t V + \sum_{j=1}^3 A_j(V)\partial_{x_j} V = -\mathcal{L}[V]$. Plug in estimate of C_k here for the highest time derivative.

Step c_{k+1} Apply elliptic estimate together with D_k and Step b to estimate $C_{k+1} := \|\partial_t^{m-k}\partial_x^k U\|_{L^2}$. Set $k = k+1$. Go to Step d_{k+1}... ...

What to be included in a general framework?

A "vorticity" operator $\mathcal{K}$ so that ...

- $\mathcal{K}$ annihilates $\varepsilon^{-1}\mathcal{L}$;
- elliptic estimate $\|U\|_{H^m} \lesssim \|\mathcal{L}[U]\|_{H^{m-1}} + \|\mathcal{K}[U]\|_{H^{m-1}} + ...$
- $\mathcal{K}[U]$ satisfies a vorticity equation that preserves $\omega = 0$;
- $\text{Ker}\mathcal{P} = \text{Ker}\mathcal{K}$ in proper solution space.

Last two conditions are crucial for studying fast-fast interactions.

Domains:

- Periodic domains, whole space, exterior domains;
- Manifolds. Hodge decomposition and Poincare Lemma.

Models:

- Rotating fluid equations, e.g. 2D rotating shallow water eqs;
- Viscous fluids w/o boundary;
- Baroclinic fluids w/ small entropy perturbation.

Fluid equations with more than one fast scales ([BC, HYP 2008]).
Any implication in designing numerical schemes for multiscale problems? Hierarchy of reduced equations?

Thank you!