

Phase transitions in Kinetic Theory

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Outline

- The particle model.
- Kinetic description.
- Equilibrium, phase transition.
- Results:
 - Stability for V-B and V-F-P in \mathbb{R} ;
 - Instability for V-B in \mathbb{R} .

Joint works with Yan Guo and Rossana Marra:

Arch.Rat.Mech.Anal. **195**, 75–116 (2010),

Commun.Math.Phys. **296**, 1–33 (2010).

Work in progress in a finite interval.

The model

Model introduced by Bastea and Lebowitz (1997).

System of N_1 **red** and N_2 **blue** hard spheres of radius $a = 1$ with unit mass in $\Lambda \subset \mathbb{R}^3$.

Interactions:

- elastic collisions (color blind);
- long range Kac-type interaction between red and blue particles with a two body potential:

$$\phi_\ell(x) = A_\ell U\left(\frac{|x|}{\ell}\right), \quad U(r) = 0 \text{ for } r > 1.$$

U is assumed **smooth, non negative, decreasing and normalized** ($\int dx U(|x|) = 1$) and $A_\ell \rightarrow 0$ as $\ell \rightarrow \infty$ suitably.

Boltzmann-Grad+Vlasov limit

$$|\Lambda| = \mathcal{O}(\ell^3)$$

$$\text{Mean free path } \lambda = \frac{|\Lambda|}{(N_1 + N_2)a^2}.$$

$$\gamma = \frac{\lambda}{\ell} \text{ fixed.}$$

$$A_\ell = \gamma^3 a^2 \lambda^{-2} = \mathcal{O}(N_1 + N_2).$$

Note that $N_1 + N_2 = \mathcal{O}(\ell^2)$, low density.

By scaling space and time variables (with λ^{-1}) and taking the **formal** limit in the BBGKY hierarchy one gets the kinetic equations below.

Warning: The analog of Lanford theorem is not been proved.

Boltzmann-Vlasov equation – B-V

$$\partial_t f^1 + v \cdot \nabla_x f^1 + F^1[f^2] \cdot \nabla_v f^1 = B(f^1, f^1) + B(f^1, f^2),$$

$$\partial_t f^2 + v \cdot \nabla_x f^2 + F^2[f^1] \cdot \nabla_v f^2 = B(f^2, f^2) + B(f^2, f^1),$$

$f^1 = f^{\text{red}}$ and $f^2 = f^{\text{blue}}$ probability distribution functions on the phase space $\Omega \times \mathbb{R}^3$, $\Omega = \lambda^{-1} \Lambda$;

Self-consistent forces $F^1[f^2]$, $F^2[f^1]$:

$$F^i[f^j](x, t) = -\nabla_x \int_{\Omega} dx' \gamma^3 U(\gamma|x - x'|) \int_{\mathbb{R}^3} dv f^j(x', v, t), \quad i \neq j.$$

Collisions:

$$B(f, g) = \int_{\mathbb{R}^3} dv_* \int_{|\omega|=1} d\omega |(v - v_*) \cdot \omega| [f(v')g(v'_*) - f(v)g(v_*)],$$

Remark

Why not just one species?

The interesting case for phase transitions, with one species, is the attractive one: $U(r) \leq 0$.

In order to have thermodynamic stability one needs to keep the hard core size finite on the kinetic scale.

Not compatible with the Grad-Boltzmann limit.

Vlasov-Fokker-Planck eq.n – V-F-P

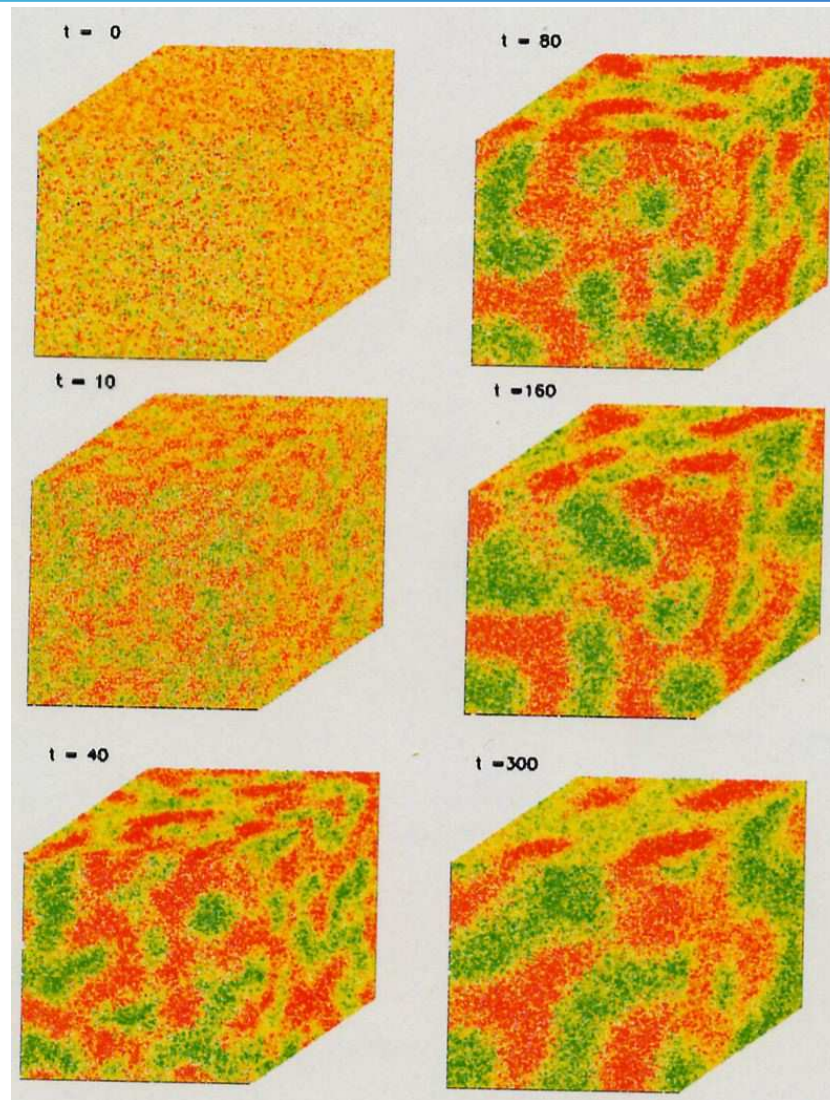
No elastic collisions; system in contact with a thermal bath at inverse temperature $\beta > 0$, modeled by white noise force added to the deterministic evolution with long range interaction.
Standard mean field arguments: limiting system

$$\partial_t f^1 + v \cdot \nabla_x f^1 + F^1[f^2] \cdot \nabla_v f^1 = L f^1,$$

$$\partial_t f^2 + v \cdot \nabla_x f^2 + F^2[f^1] \cdot \nabla_v f^2 = L f^2,$$

$$L f(v) = \nabla_v \cdot \left(\mu_\beta \nabla_v \left(\frac{f}{\mu_\beta} \right) \right), \quad \mu_\beta = \frac{\exp[-\beta v^2]}{(2\pi\beta^{-1})^{3/2}}.$$

Segregation



Free energy functional

$$\mathcal{F}_{\Omega}^{\text{kin}}(f^1, f^2) = \mathcal{E}_{\Omega}(f^1, f^2) + \beta^{-1} \mathcal{H}_{\Omega}(f^1, f^2),$$

$$\mathcal{E}_{\Omega}(f^1, f^2) = \int_{\Omega} dx \int_{\mathbb{R}^3} dv \frac{1}{2} v^2 \left[f^1(x, v) + f^2(x, v) \right]$$

$$+ \int_{\Omega} dx \int_{\Omega} dx' U(|x - x'|) \rho_1(x) \rho_2(x'),$$

$$\rho_i(x) = \int_{\mathbb{R}^3} dv f^i(x, v),$$

$$\mathcal{H}_{\Omega}(f^1, f^2) = \sum_{i=1,2} \int_{\Omega} dx \int_{\mathbb{R}^3} dv f^i(x, v) \ln f^i(x, v).$$

The functional $\mathcal{F}_{\Omega}^{\text{kin}}$ is non increasing for any $\beta > 0$ under the V-B dynamics and for β given by the thermal bath inverse temperature for the V-F-P evolution.

Entropy-energy competition.

- β small: entropy dominates. Disordered state.
- β large: energy dominates. Segregated states.

Equilibrium

Equilibrium solutions (both V-B and V-F-P):

$$f^1(x, v) = \rho_1(x)\mu_\beta(v), \quad f^2(x, v) = \rho_2(x)\mu_\beta(v),$$

$$\beta^{-1} \ln \rho_1(x) + \int_{\Omega} dx' U(|x - x'|) \rho_2(x') = C_1,$$

$$\beta^{-1} \ln \rho_2(x) + \int_{\Omega} dx' U(|x - x'|) \rho_1(x') = C_2.$$

Euler-Lagrange equations for the *spatial free energy* functional

$$\begin{aligned} \mathcal{F}_{\Omega}[\rho_1, \rho_2] = & \beta^{-1} \int_{\Omega} dx [\rho_1(x) \ln \rho_1(x) + \rho_2(x) \ln \rho_2(x)] \\ & + \int_{\Omega} dx \int_{\Omega} dx' U(|x - x'|) \rho_1(x) \rho_2(x'). \end{aligned}$$

Local free energy

Homogeneous minimizers: minimizers of the local free energy

$$\varphi(\rho_1, \rho_2) = \beta^{-1} [\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2] + \rho_1 \rho_2$$

Indeed,

$$\begin{aligned} \mathcal{F}_\Omega[\rho_1, \rho_2] &= \int_\Omega dx \varphi(\rho_1(x), \rho_2(x)) \\ &+ \frac{1}{2} \int_\Omega dx \int_\Omega dx' U(|x - x'|) (\rho_1(x) - \rho_1(x')) (\rho_2(x') - \rho_2(x)). \end{aligned}$$

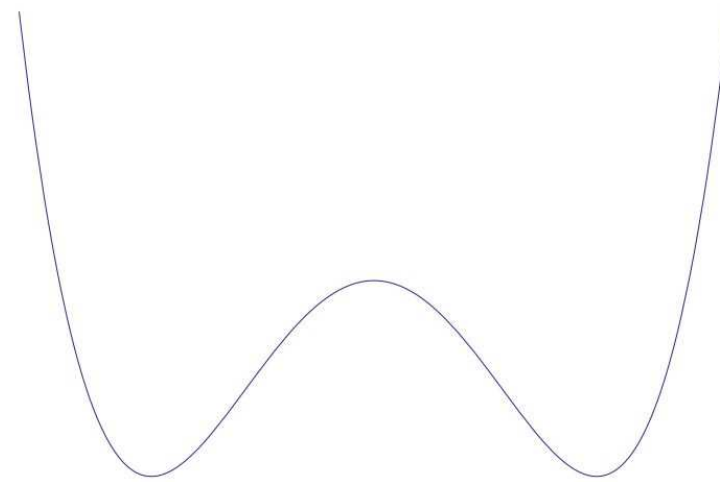
By rearrangement arguments (U decreasing) the non local term is non negative on minimizers, so minimizers are spatially homogeneous as much as they can.

Homogeneous equilibrium states

$$\text{Set } \rho = \rho_1 + \rho_2; m = \frac{\rho_1 - \rho_2}{\rho}.$$

$$\varphi(\rho_1, \rho_2) = \beta^{-1} \rho \ln \rho + \frac{1}{4} \rho^2 + \rho^2 f_\rho(m)$$

$$f_\rho(m) = -\frac{m^2}{4} + (\rho\beta)^{-1} \left[\frac{1-m}{2} \ln \frac{1-m}{2} + \frac{1+m}{2} \ln \frac{1+m}{2} \right]$$



Graph of $m \rightarrow f_\rho(m)$ for $\rho\beta > 2$.

Homogeneous equilibrium states

The critical points for f have to solve

$$m = \tanh(\sigma m),$$

with $\sigma = \frac{1}{2}\rho\beta$.

- If $\sigma \leq 1$: $m = 0$ is the only solution.
- If $\sigma > 1$: again $m = 0$ solution;
Moreover
 $\exists m_\sigma > 0$ such that $m = \pm m_\sigma$ is solution.

Set $\rho^\pm := \frac{1}{2}\rho(1 \pm m_{\frac{1}{2}\rho\beta})$.

Homogeneous equilibrium states

Therefore:

$\rho\beta \leq 2$: $m = 0$: Minimizer of φ : $\rho_1 = \rho_2$ (mixed phase).

$\rho\beta > 2$:

- Minimizer: $\rho_1 = \rho^+$, $\rho_2 = \rho^-$, (red rich phase);
- Minimizer: $\rho_1 = \rho^-$, $\rho_2 = \rho^+$, (blue rich phase);
- Maximizer (local): $\rho_1 = \rho_2$.

Note: the minimizing total density is always unique.

Non homogeneous equilibrium states

Conservation of masses \implies Minimizers with the constraints

$$\frac{1}{|\Omega|} \int_{\Omega} dx \rho_i(x) = n_i, \quad i = 1, 2$$

n_i given by the initial conditions.

If $\rho^- n_i < \rho^+$, $i = 1, 2$ **non homogeneous minimizers**, regions **blue rich** and **red rich** separated by interfaces.

Assume Ω a torus of size L .

From rearrangement arguments (using U monotone), for L sufficiently large, the minimizers are **symmetric monotone** with values close to ρ^+ and ρ^- in regions separated by a

small interface [Carlen, Carvalho, R.E., Lebowitz, Marra, *Nonlinearity* **16**, 1075–1105 (2003)].

Equilibrium states in \mathbb{R}

$\Omega = \mathbb{R}$. Fix $\rho = 2 \implies \sigma = \beta$; Critical point $\beta_c = 1$.

For $\beta \leq 1$: the only possible equilibrium is the homogeneous mixed phase, with $\rho_1 = \rho_2 = 1$.

For $\beta > 1$: three homogeneous states: the mixed one, $\rho_1 = \rho_2$, the **red** rich phase $\rho_1 = 1 + m_\beta$, $\rho_2 = 1 - m_\beta$ and the **blue** rich phase $\rho_1 = 1 - m_\beta$, $\rho_2 = 1 + m_\beta$.

For $\beta > 1$, non spatially homogeneous solutions (phase coexistence) by forcing the asymptotic values to ρ_\pm at $\mp\infty$: Two half-lines of **red rich** and **blue rich** phases separated by an **interface** of finite size.

Equilibrium states in \mathbb{R}

Define

$$\hat{\rho}_1(x) = \begin{cases} \rho^-, & x < 0 \\ \rho^+, & x > 0 \end{cases}, \quad \hat{\rho}_2(x) = \begin{cases} \rho^+, & x < 0 \\ \rho^-, & x > 0 \end{cases}.$$

Excess free energy:

$$\hat{\mathcal{F}}[\rho_1, \rho_2] = \lim_{\ell \rightarrow \infty} \left[\mathcal{F}_{(-\ell, \ell)}[\rho_1, \rho_2] - \mathcal{F}_{(-\ell, \ell)}[\hat{\rho}_1, \hat{\rho}_2] \right].$$

Remark:

$\hat{\mathcal{F}}[\rho_1, \rho_2]$ is not finite if $\lim_{x \rightarrow \pm\infty} \rho_1 \neq \rho^\pm$ or $\lim_{x \rightarrow \pm\infty} \rho_2 \neq \rho^\mp$.

Front solution

Theorem[Carlen, Carvalho, R.E., Lebowitz, Marra;
ARMA **194**, 823–847 (2009)]

Let $\beta > 1$. There is a unique (up to translations) minimizer to the excess free energy $\hat{\mathcal{F}}$. Let $\bar{\rho} = (\bar{\rho}_1, \bar{\rho}_2)$ be the one such that $\rho_1(0) = \rho_2(0)$.

- $\bar{\rho}$ is smooth; $\rho^- < \bar{\rho}_i(x) < \rho^+$, $i=1,2$;
- $\bar{\rho}_1$ is increasing and $\bar{\rho}_2$ is decreasing;
- $\beta^{-1} \ln \bar{\rho}_1 + U * \bar{\rho}_2 = C = \beta^{-1} \ln \bar{\rho}_2 + U * \bar{\rho}_1$;

Front solution

Moreover

- $\beta^{-1} \bar{\rho}'_1 + \bar{\rho}_1 U * \bar{\rho}'_2 = 0 = \beta^{-1} \bar{\rho}'_2 + \bar{\rho}_1 U * \bar{\rho}'_1;$

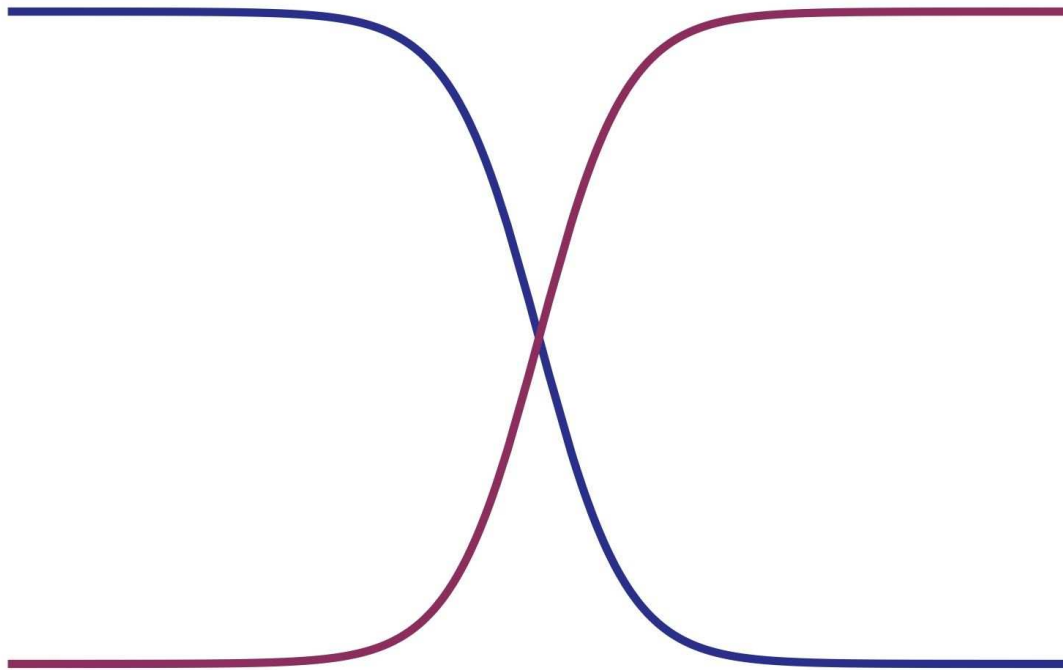
- $\bar{\rho}_1(x) = \bar{\rho}_2(-x), \quad \bar{\rho}'_1(x) = -\bar{\rho}'_2(-x);$

- $\exists \alpha > 0:$

$$|\bar{\rho}_1(x) - \rho^\pm| e^{\alpha|x|} \rightarrow 0, \quad x \rightarrow \pm\infty;$$

$$|\bar{\rho}_2(x) - \rho^\mp| e^{\alpha|x|} \rightarrow 0, \quad x \rightarrow \pm\infty.$$

Front solution



Stability

Question: Are the equilibrium states stable w.r.t. perturbations of the initial conditions in the time evolutions V-B and V-F-P?

Expected answer: Minimizers are stable, maximizers are unstable.

Results: Case $\Omega = \mathbb{R}$;

- V-F-P evolution: asymptotic stability of the minimizers.
- V-B evolution: stability of the minimizers, instability of the maximizer.

Open problems:

- Instability of the maximizer for V-F-P and asymptotic stability of the minimizers for V-B.
- $\Omega = [-L, L]$, periodic b.c., L large enough.

Stability for the V-F-P

Main problem: the force cannot be small: small force \implies no phase transition.

For small force, convergence to equilibrium and rate by Villani.

This would cover the high temperature regime.

For low temperatures, multiple equilibrium states.

Difficult case: front. Homogeneous minimizers are simpler.

Assume $f_0^i(x, v) = \bar{\rho}_i(x)\mu_\beta(v) + h_i(x, v, 0)$, $x \in \mathbb{R}$, $v \in \mathbb{R}^3$.

Define $h_i(x, v, t) := f^i(x, v, t) - \bar{\rho}_i(x)\mu_\beta(v)$.

Stability for the V-F-P

Equations for h_i :

$$\begin{aligned}\partial_t h_i + G_i h_i &= L h_i - F_i[h] \partial_{v_x} h_i, \\ G_i h_i &= v_x \partial_x h_i - (U * \bar{\rho}'_j) \partial_{v_x} h_i \\ &+ \left(U * \partial_x \int_{\mathbb{R}^3} dv h_j(\cdot, v, t) \right) \beta v_x M \bar{\rho}_i\end{aligned}$$

Norms: $\| \cdot \|$ denotes the $L^2(\mathbb{R} \times \mathbb{R}^3)$ with weight $(\bar{\rho}_i \mu_\beta)^{-1}$:

$$\|f\|^2 = \sum_{i=1,2} \int_{\mathbb{R} \times \mathbb{R}^3} dx dv \frac{f(x, v)^2}{\bar{\rho}(x) \mu_\beta(v)}$$

Stability for the V-F-P

Null space of L : $\text{kern}(L) = \{(u_1, u_2) = (a_1\mu_\beta, a_2\mu_\beta)\}$.

P projector on $\text{kern}(L)$, $P^\perp = 1 - P$; $\partial = (\partial_x, \partial_t)$.

Dissipation norm: $\|f\|_D^2 = \|P^\perp f\|^2 + \|\nabla_v P^\perp f\|^2$.

Weighted norms:

$$z_\gamma = (1 + |x|^2)^\gamma, \quad \|f\|_\gamma = \|z_\gamma f\|; \quad \|f\|_{D,\gamma} = \|z_\gamma f\|_D$$

Symmetry: Assume $h_1(x, v, 0) = h_2(-x, Rv, 0)$ with $Rv = (-v_x, v_y, v_z)$.

Note that this is the same symmetry of the front.

The symmetry is preserved in time.

Stability for the V-F-P

Theorem[ARMA 195,75-116 (2010)] *There is $\delta > 0$ s.t. if $\|h(\cdot, 0)\| + \|\partial h(\cdot, 0)\| < \delta$, then there is a unique global solution to the equation for h . Moreover there is $K > 0$ such that*

$$\begin{aligned} & \frac{d}{dt} \left(\|h(t)\|^2 + \|\partial_t h(t)\|^2 + \|\partial_x h(t)\|^2 \right) \\ & + K \left(\|h(t)\|_D^2 + \|\partial_t h(t)\|_D^2 + \|\partial_x h(t)\|_D^2 \right) \leq 0. \end{aligned}$$

If, for $\gamma > 0$ sufficiently small, $\|h(\cdot, 0)\|_\gamma + \|\partial h(\cdot, 0)\|_{\gamma+\frac{1}{2}} < \delta$ then the same norms are bounded at any time by the initial data and we have the decay estimate

$$\|h(t)\|^2 + \|\partial h(t)\|^2 \leq C \left[1 + \frac{t}{2\gamma} \right]^{-2\gamma} \left[\|h(0)\|_\gamma^2 + \|\partial h(0)\|_{\frac{1}{2}+\gamma}^2 \right].$$

Main tools

Spectral gap for L : there is $\nu_0 > 0$ s.t.

$$(g, Lg) \leq -\nu_0 \|P^\perp g\|_D^2.$$

This is used to control $P^\perp h(t)$. Control $P h(t) = (g_1 \mu_\beta, g_2 \mu_\beta)$:

$$(Ag)_1 = \beta^{-1} \frac{g_1}{\bar{\rho}_1} + U * g_2, \quad (Ag)_2 = \beta^{-1} \frac{g_2}{\bar{\rho}_2} + U * g_1 .$$

Note that

$$(g, Ag) = \frac{d^2}{ds^2} \hat{\mathcal{F}}(\bar{\rho} + sg) \Big|_{s=0} .$$

Main tools

The operator A has a null space $\text{kern}(A) = \{\alpha(\bar{\rho}'_1, \bar{\rho}'_2)\}$. Let \mathcal{P} is the projector on $\text{kern}(A)$ and $\mathcal{P}^\perp = 1 - \mathcal{P}$. (Transl. invar.)

Spectral gap for A : there is $\lambda > 0$ s.t.

$$(g, Ag) \geq \lambda(\mathcal{P}^\perp g, \mathcal{P}^\perp g).$$

The component of $h(t)$ in $\text{kern}(A)$ is not under control. The symmetry condition ensures that $h(t)$ is orthogonal to $\text{kern}(A)$.

Lower bound for $\|(Au)'\|$: There is $C > 0$ such that

$$\|(Au)'\|^2 \geq C\|Qu'\|^2.$$

where Q is the orthogonal projection on the orthogonal complement of $\bar{\rho}''$.

Remarks

- The same results hold for the mixed phase above the critical temperature.
- We do not have the instability of the mixed phase below the critical temperature. The construction of the growing mode presented below for V-B, fails for V-F-P because of the unboundedness of the F-P operator L .

Results for V-B

Theorem[CMP 296,1-33 (2010)]: Assume $\rho = 2$.

- $\beta < 1$: The unique equilibrium $(f_1, f_2) = (\mu_\beta, \mu_\beta)$ is stable.
- $\beta > 1$:
 - the homogeneous equilibrium states $(f_1, f_2) = (\rho^+ \mu_\beta, \rho^- \mu_\beta)$ and $(f_1, f_2) = (\rho^- \mu_\beta, \rho^+ \mu_\beta)$ are stable;
 - the equilibrium $(f_1, f_2) = (\bar{\rho}^1(x) \mu_\beta, \bar{\rho}^2(x) \mu_\beta)$ is stable w.r.t. symmetric perturbations;
 - the homogeneous equilibrium $(f_1, f_2) = (\mu_\beta, \mu_\beta)$ is unstable.

Here stability and instability are in $L^\infty(\mathbb{R} \times \mathbb{R}^3)$ and in $H^1(\mathbb{R} \times \mathbb{R}^3)$. Symmetric perturbation means again $h_1(x, v) = h_2(-x, Rv)$, where $Rv = (-v_x, v_y, v_z)$.

Remarks

- No convergence to the equilibrium is stated. This has to be compared with the Vlasov-Fokker-Plank case where there is an algebraic rate of convergence. **No instability result for VFP.**

- In order to have phase transitions: **force not small.**

Treating the force terms as perturbations does not work.

Strategy based on entropy-energy arguments: L^2 estimates promoted to L^∞ by analysis of the characteristics. Crucial step: **spectral gap** for the second derivative of the free energy.

- The instability is based on the construction of a **growing mode for the linear collisionless case,** perturbation arguments and persistence of the growing mode at non linear level.

Free energy functional

Given the equilibrium state $(M_1, M_2) = (\rho_1 \mu_\beta, \rho_2 \mu_\beta)$, let $g = (g_1, g_2)$ with $g_i = \frac{f_i - M_i}{\sqrt{M_i}}$ be the deviation from the equilibrium. Define: $\mathcal{M}_i(g) = \int_{\mathbb{R}} dx \int_{\mathbb{R}^3} dv \sqrt{M_i} g_i(x, v)$,

$$\mathcal{H}(g) = \sum_{i=1}^2 \int_{\mathbb{R}} dx \int dv \left[f_i \log f_i - M_i \log M_i \right],$$

$$\mathcal{E}(g) = \sum_{i=1}^2 \int_{\mathbb{R}} dx \int dv \frac{v^2}{2} g_i \sqrt{M_i}$$

$$+ \int_{\mathbb{R} \times \mathbb{R}} dx dy U(|x - y|) \left(\rho_{f_1}(x) \rho_{f_2}(y) - \rho_1(x) \rho_2(y) \right),$$

$$\rho_{f_i} = \int dv f_i(x, v).$$

Free energy functional

The free energy functional is

$$\mathcal{F}(g) = \mathcal{H}(g) + \beta \mathcal{E}(g) - \left(C + 1 + \log \left(\frac{\beta}{2\pi} \right)^{3/2} \right) \sum_{i=1}^2 \mathcal{M}_i(g),$$

The free energy functional does not increase:

$$\mathcal{F}(g(t)) \leq \mathcal{F}(g(0)) \text{ for any } t > 0.$$

Quadratic approximation. The coefficients have been chosen to cancel the linear part.

For some $\tilde{f}_i = \alpha f_i + (1 - \alpha)M_i$, $\alpha \in (0, 1)$:

$$\begin{aligned} \mathcal{F}(g) &= \sum_{i=1}^2 \int_{\mathbb{R}} dx \int_{\mathbb{R}^3} dv \frac{(f_i(t) - M_i)^2}{2\tilde{f}_i} \\ &+ \beta \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy U(|x - y|) (\rho_{f_1}(t, x) - \rho_1(x)) (\rho_{f_2}(t, y) - \rho_2(y)). \end{aligned}$$

Free energy functional

Lemma. *If $u_i = \rho f_i - \rho_i$ s.t. $\mathcal{P}(u_1, u_2) = 0$, then there are $\alpha > 0$ and κ sufficiently small so that*

$$\alpha \sum_{i=1,2} \int_{\mathbb{R}} dx \int_{\mathbb{R}^3} dv \left\{ \frac{(f_i(t) - M_i)^2}{M_i} \mathbf{1}_{\{|f_i(t) - M_i| \leq \kappa M_i\}} + |f_i(t) - M_i| \mathbf{1}_{\{|f_i(t) - M_i| \geq \kappa M_i\}} \right\} \leq \mathcal{F}(g(0)).$$

Remark: It is crucial that the initial perturbation is orthogonal to the null space of A . This is trivial for the spatially homogeneous equilibrium, while it is ensured by the symmetry condition for the phase coexisting equilibrium.

Main proposition

Theorem: Let $w(v) = (\Sigma + |v|^2)^\gamma$, with Σ sufficiently large and $\gamma > \frac{3}{2}$. If $\|wg(0)\|_\infty + \sqrt{\mathcal{F}(g(0))} < \delta$ for δ sufficiently small, then there is $T_0 > 0$ such that

$$\|wg(T_0)\|_\infty \leq \frac{1}{2}\|wg(0)\|_\infty + C_{T_0}\sqrt{\mathcal{F}(g(0))}.$$

The stability follows by iteration on the time interval.

Growing mode

Idea: Without collisions there is a growing mode.

Collisions do not destroy the growing mode.

The linearization of the equation for the perturbation is:

$$\partial_t g + \mathcal{L}g = 0,$$

Notation: $\mu = \mu_\beta$; ξ first component of the velocity $v = (\xi, \zeta)$,

$$(\mathcal{L}g)_i = \xi \partial_x g_i - \beta F(\sqrt{\mu} g_{i+1}) \xi \sqrt{\mu} - \alpha L_i g,$$

$$\alpha = 1 \text{ and } L_i g = \frac{1}{\sqrt{\mu}} \left(B(\sqrt{\mu} g_i, 2\mu) + B(\mu, \sqrt{\mu}(g_1 + g_2)) \right).$$

Seek for a growing mode of the form

$$g_1(t, x, \xi, \zeta) = e^{\lambda t} e^{\mathbf{i}kx} q(\xi, \zeta), \quad g_2(t, x, \xi, \zeta) = e^{\lambda t} e^{-\mathbf{i}kx} q(-\xi, \zeta).$$

Eigenvalue problem

$$\{\lambda + \mathbf{i}\xi k\}q - \beta k \mathbf{i} \hat{U}(k) \left\{ \int_{\mathbb{R}^3} q \sqrt{\mu} dv \right\} \xi \sqrt{\mu} - \alpha Lq = 0.$$

Proposition 1: *Let $\beta > 1$. There exists sufficiently small $\alpha > 0$ such that there is an eigenfunction $q(v)$ and the eigenvalue λ with $\Re \lambda > 0$.*

Proof. First assume $\alpha = 0$. λ is found by O. Penrose criterion,

$$\beta \int_{\mathbb{R}^3} \frac{\xi^2 \hat{U}(k) k^2 \mu(v)}{\lambda^2 + k^2 \xi^2} dv = 1.$$

Indeed, since $\beta \hat{U}(0) > 1$, by continuity there is $k_0 > 0$ such that $\beta \hat{U}(k_0) > 1$ and hence a $\lambda > 0$ so that this is satisfied.

Use Kato perturbation theorem to extend to $\alpha > 0$ small.

Eigenvalue problem

Proposition 2: *Let α_0 be the supremum of the α 's such that Proposition 1 is true. Then $\alpha_0 = +\infty$.*

Proof. Indeed, if $\alpha_0 < \infty$ then, $\lambda_0 = \lim_{\alpha \rightarrow \alpha_0} \lambda_\alpha$ exists (up to subsequences) and is a purely imaginary eigenvalue. It can be shown that the corresponding eigenfunction must be in the null space of L and this implies $\lambda = 0$. Moreover, collisions disappear for such an eigenfunction and we can use again the Penrose criterion which implies $\beta \hat{U}(k_0) = 1$. This is in contradiction with the definition of k_0 .

This provides a linear growing mode for any $\alpha > 0$.

Remark: It is crucial that L is a bounded perturbation. It does not work with the Fokker-Plank operator which is unbounded.

Non linear analysis. Bootstrap argument. Very technical.

Instability theorem

Theorem. Assume $\beta > 1$. There exist constants $k_0 > 0$, $\theta > 0$, $C > 0$, $c > 0$ and a family of initial $\frac{2\pi}{k_0}$ -periodic data $f_i^\delta(0) = \mu + \sqrt{\mu}g_i^\delta(0) \geq 0$, with $g^\delta(0)$ satisfying

$$\|\nabla_{x,\xi}g^\delta(0)\|_{L^2} + \|wg^\delta(0)\|_{L^\infty} \leq C\delta,$$

for δ sufficiently small, but the solution $g^\delta(t)$ satisfies

$$\sup_{0 \leq t \leq T^\delta} \|wg^\delta(t)\|_{L^\infty} \geq c \sup_{0 \leq t \leq T^\delta} \|g^\delta(t)\|_{L^2} \geq c\theta > 0.$$

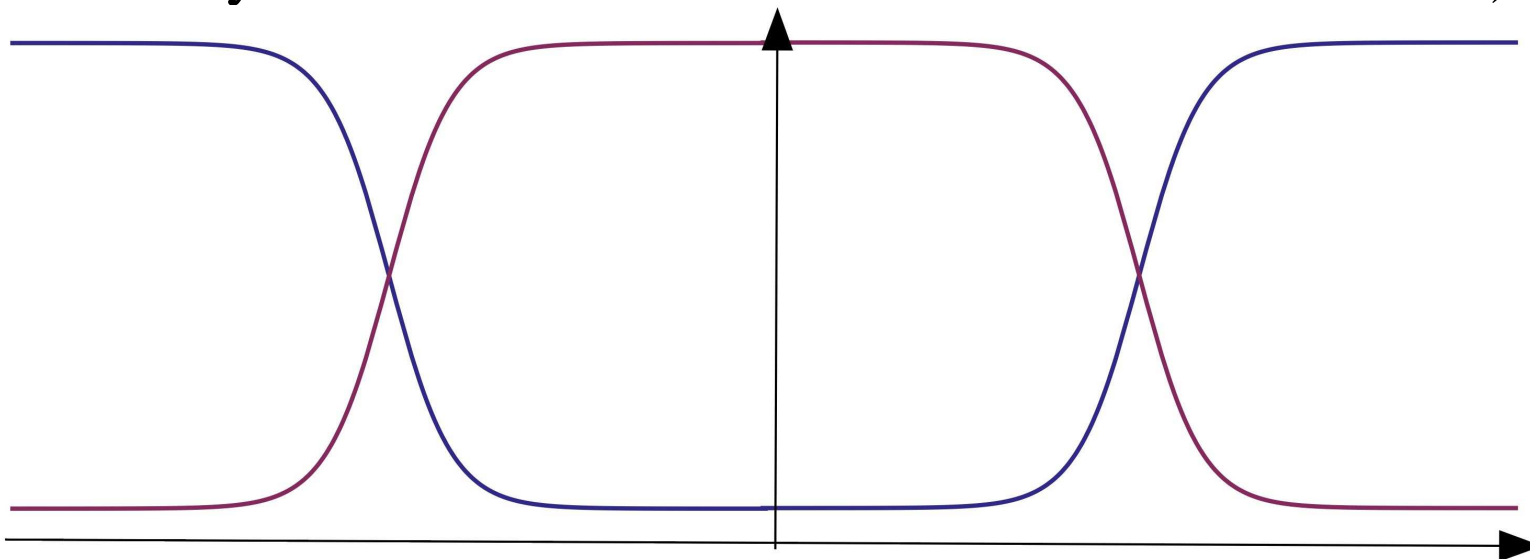
Here the escape time is $T^\delta = \frac{1}{\Re\lambda} \ln \frac{\theta}{\delta}$,

Note that the growing mode is symmetric.

The instability does not depend on the absence of symmetry.

Work in progress

- Finite volume, **1d torus** (with Y. Guo and R. Marra)
Stability of the non constant minimizer “double front”



- Operator A on $L^2(T_L)$, T_L the 1-d torus of size L .
Derivative of the front is in the null. **Null space? Spectral gap?**

Spectral gap

A has a **negative eigenvalue**. Indeed, let $w = (w_1, w_2)$ be a front on the torus such that $Aw' = 0$. Define $\tilde{w} = (|w'_1|, -|w'_2|)$

$$\begin{aligned}(\tilde{w}, A\tilde{w})_w &= \int_{T_L} |w'_1| \left(\frac{|w'_1|}{w_1} - U * |w'_2| \right) + |w'_2| \left(\frac{|w'_2|}{w_2} - U * |w'_1| \right) \\ &= -2 \int_0^L w'_1 U * (|w'_2| + w'_2) - 2 \int_{-L}^0 w'_2 U * (|w'_1| + w'_1) < 0\end{aligned}$$

We have used the E-L equations

$$\frac{w'_1}{w_1} = -U * w'_2, \quad x \geq 0; \quad \frac{w'_2}{w_2} = -U * w'_1, \quad x \leq 0$$

Show that the **mass constraint** kills the negative eigenvalue.

Neumann b.c.

Spectral gap true for **anti-symmetric** (by reflection) functions.

Problem on the torus for **symmetric** functions reduced to the case of **Neumann boundary conditions on $[0, L]$** :

$$(\hat{A}g)_1 = \beta^{-1} \frac{g_1}{w_1} + \hat{U} * g_2, \quad (\hat{A}g)_2 = \beta^{-1} \frac{g_2}{w_2} + \beta \hat{U} * g_1$$

$$\hat{U}(z, z') = U(z, z') + U(z, R_0 z') + U(z, R_L z')$$

R_0 reflection around zero and R_L reflection around L .

$\lambda_L < 0$ minimum eigenvalue and \hat{e} its eigenfunction.

Spectral gap

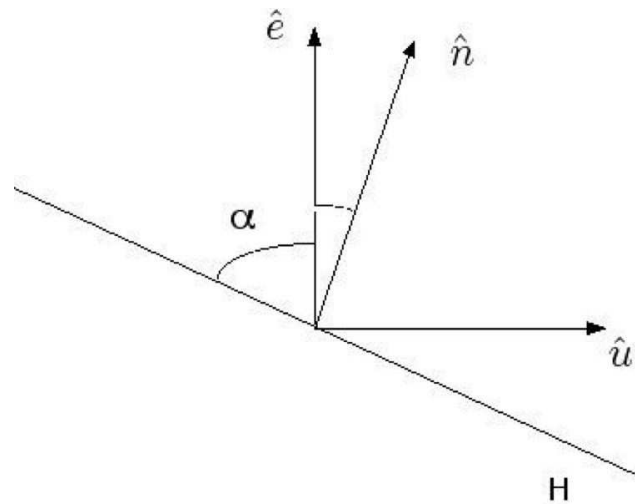
We need spectral gap for functions in the hyperplane

$$H = \{h : \int_0^L h = 0\}.$$

We have spectral gap for functions in the orthogonal to \hat{e} .

$$(u, \hat{A}u) \geq \delta(u, u), \quad \text{if } (u, \hat{e}) = 0.$$

If the angle α between \hat{e} and H is too small we are in trouble:



Spectral gap

Decompose $h \in H$ as $a\hat{e} + b\hat{u}$ with \hat{u} orthogonal to \hat{e}

$$(h, \hat{A}h) = a^2 \lambda_L + b^2 (\hat{u}, \hat{A}\hat{u})$$

$a^2 = \cos^2 \alpha$, $b^2 = \sin^2 \alpha$. with $\sin \alpha = \frac{1}{\sqrt{L}} \int_0^L \hat{e}$.

If \hat{e} decays fast enough $b^2 \approx \frac{1}{L}$. λ_L is negative

Competition $(h, \hat{A}h) \geq -|\lambda_L| + \frac{c}{L}\delta$

If λ_L decays faster than $\frac{1}{L}$ we can prove spectral gap for **L large**

$$(h, \hat{A}h) > d(h, h)$$

G. Manzi (2007)

Spectrum

Analysis of the spectrum using **Markov chains**. Generalize method by De Masi, Olivieri, Presutti (1998), Ising case.

- bound on the minimum eigenvalue λ_L

$$-c_1 e^{-\gamma L} \leq \lambda_L \leq c_2 e^{-\gamma L}$$

- exponential bound on the minimum eigenfunction \hat{e}

$$-c e^{-\gamma|L-x|} \leq \hat{e} < 0$$

- **spectral gap** in a suitable weighted L_∞ for functions u in the orthogonal to \hat{e} . Implies spectral gap in L_2 .

Spectral gap

Operator S $(Su)_i = w_i \hat{U} * u_j, \quad i \neq j, i = 1, 2$

$$(u, \hat{A}u) = \sum_i \int dx w_i u_i (\hat{A}u)_i = (u, u) + (u, Su)$$

$$(u, S^2u) = \sum_i \int u_i w_i \hat{U} * (w_j \hat{U} * u_i) = (u, Tu)$$

Negative eigenvalue for \hat{A} means eigenvalue for S greater than 1. We study the operators

$$T_1 h = w_1 \hat{U} * (w_2 \hat{U} * h), \quad T_2 h = w_2 \hat{U} * (w_1 \hat{U} * h).$$

Markov chain

$$\hat{J}(x, x') = \int dz \hat{U}(x - z) \frac{w_2(z)}{w_2(x)} \int dy \hat{U}(z - x')$$

$$M(x, x') = p(x) \hat{J}(x, x'); \quad p(x) = w_1(x) w_2(x)$$

For $\lambda_L > 0$ and $\hat{e}(x)$ positive, define the Markov kernel

$$K(x, y) = \frac{M(x, y) \hat{e}(y)}{\lambda_0 \hat{e}(x)}.$$

The End
Thanks!