

On Krein-like theorems for noncanonical Hamiltonian systems with continuous spectra: application to Vlasov-Poisson

P. J. Morrison

*Department of Physics and Institute for Fusion Studies
The University of Texas at Austin*

`morrison@physics.utexas.edu`

`http://www.ph.utexas.edu/morrison/`

FRG Brown University May 13, 2010

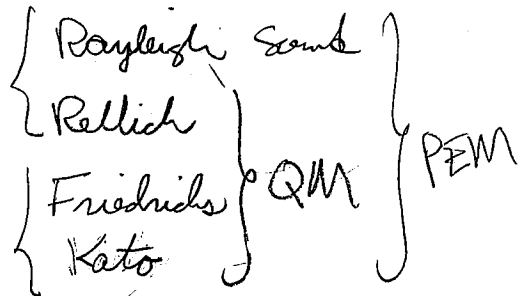
Collaborator: George Hagstrom

Goal: Do for infinite degree-of-freedom Hamiltonian systems that which can be done for finite. Krein's theorem? Discrete spectrum easy. What about the **continuous spectrum**? Requires some analysis.

BLACK BOARD

Perturbation of Spectra

Brillouin, ...



Grillakis NEM canonical CS
Special

Krein's Theory

Ham. Sys.

$$z = (q, p) \quad \dot{z} = J_c \nabla H$$

$$= [z, H]$$

$$J_c = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

$$[f, g] = \tilde{\nabla} f \cdot J_c \cdot \nabla g$$

$$H(z; \lambda)$$

parametric dep.

mass, charge, phenom. interaction etc

Equilibria

$$\nabla H(z_e, \lambda) = 0 \Rightarrow z_e(\lambda)$$

linearization

$$z = z_e + \delta z$$

$$H_L = \frac{1}{2} \delta z^T \cdot \frac{\partial^2 H}{\partial z \partial z} \cdot \delta z = \frac{1}{2} \delta z^T \cdot \underbrace{J_c A(\lambda)}_{\text{Ham. matrix}} \cdot \delta z$$

$$\delta \dot{z} = J_c \nabla H_L = J_c A \cdot \delta z$$

$$\delta z = \tilde{z} e^{i\omega t}$$

$\lambda = i\omega$

$$\det(i\omega I - A) = 0 \Rightarrow$$

$$\omega(\lambda)$$

~~...~~

Ham. Spectra

i) $\omega = \pm \omega_R$

$\omega_R \in \mathbb{R}$

ii) $\omega = \pm i\gamma$

$\gamma \in \mathbb{R}$

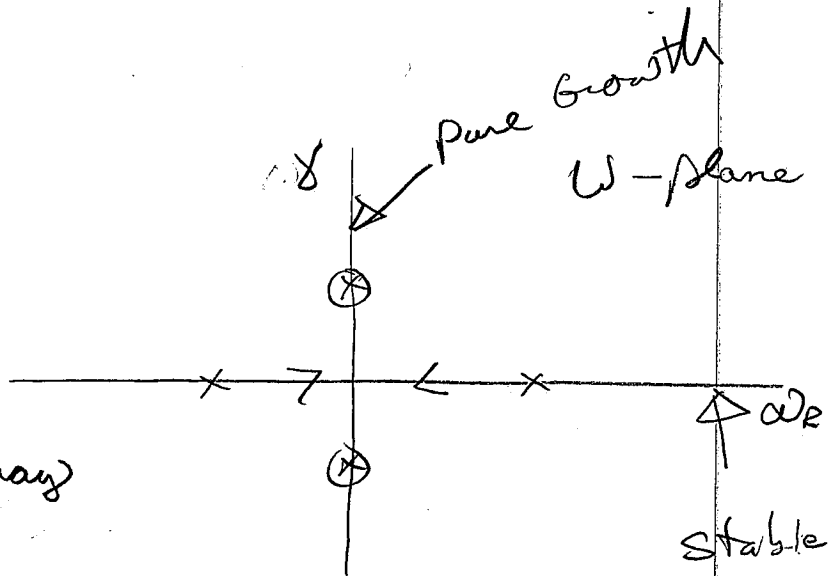
iii) $\omega = \pm \omega_R \pm i\gamma$

Two Bifurcations

Var $\lambda \Rightarrow$

Evans' wave and may collide

steady state



$H = \frac{p^2}{2} + \frac{\lambda q^2}{2}$

$\lambda > 0$ stable

$\lambda < 0$ unstable

Krein-Hopf (Krein)

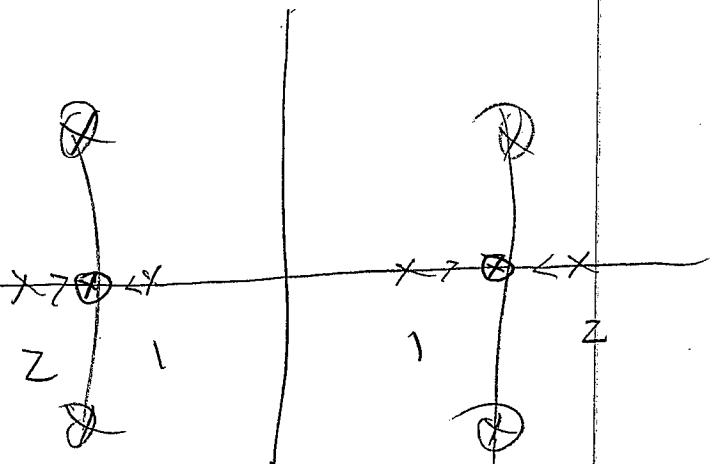
Krein Crash

Signature - Stable Normal Form

$H = \sum_n \omega_n \left(\frac{q_n^2 + p_n^2}{2} \right) G_n$

$G_n \in \{-1, 1\}$

Signature



(Krein - Moser - Sturrock)
1940's 1950's

Krein (Moser & Sturrock)

Opposite sig. is necessary for bifurcation to instability.

$$\text{Pf} \quad H = \sum \omega_i \left(\frac{q_i^2 + p_i^2}{2} \right) + \omega_2 \left(\frac{q_2^2 + p_2^2}{2} \right)$$

$$\omega_1(\lambda)$$

$$\omega_1(\lambda_k) = \omega_2(\lambda_k) > 0$$

Have Liapunov stability (w/ $H = \text{const}$ defining sub nhbds.) in nhbd of λ_k .

PDE w/ CS (expect)

$$H = \int \omega(u) \left[\frac{Q^2(u) + P^2(u)}{2} \right] \delta(u) \, du$$

What is $\delta(u)$ for Vlasov?

Vlasov-Poisson System

Phase space density (1 + 1 + 1 field theory):

$$f(x, v, t) \geq 0$$

Conservation of phase space density:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{e}{m} \frac{\partial \phi[x, t; f]}{\partial x} \frac{\partial f}{\partial v} = 0$$

Poisson's equation:

$$\phi_{xx} = 4\pi \left[e \int_{\mathbb{R}} f(x, v, t) dv - \rho_B \right]$$

Energy:

$$H = \frac{m}{2} \int_{\Pi} \int_{\mathbb{R}} v^2 f dx dv + \frac{1}{8\pi} \int_{\Pi} (\phi_x)^2 dx$$

How Hamiltonian?

Where are the canonical fields (ψ, π) ?

$$\{F, G\} = \int d^4x \left(\frac{\delta F}{\delta \psi} \frac{\delta G}{\delta \pi} - \frac{\delta F}{\delta \pi} \frac{\delta G}{\delta \psi} \right)$$

$$\psi_t = \{\psi, H\} \quad \text{and} \quad \pi_t = \{\pi, H\}$$

For Vlasov the phase space density $f(x, v, t)$ is not a canonical variable and not in 1-1 correspondence with canonical variables (ψ, π) . Yet, it is still Hamiltonian in a deeper sense: dynamics generated by PB that is a Lie algebra realization on phase space functionals and therefore canonizable.

Vlasov-Poisson shares this Hamiltonian form with a class of continuous media models that includes:

2D Euler, Charney-Hasegawa-Mima, other plasma reduced fluid models,

Grand Unified 2+1 Media Theory (1-field)

Phase space density of something :

$$\zeta : \mathcal{Z} \times \mathbb{R} \rightarrow \mathbb{R}, \quad z = (q, p) \in \mathcal{Z} = \text{symplectic manifold}$$

Conservation of phase space density:

$$\frac{\partial \zeta}{\partial t} + [\mathcal{E}, \zeta] = 0 \quad [f, g] := \partial_q f \partial_p g - \partial_p f \partial_q g$$

Constraint/elliptic equation:

$$\mathcal{E} = \frac{\delta H}{\delta \zeta} = h_1(z) + \int_{\mathcal{Z}} h_2(z, z') \zeta(z') d^2 z' + \dots$$

Energy:

$$H[\zeta] = H_1 + H_2 = \int_{\mathcal{Z}} h_1(z) \zeta(z) d^2 z + \frac{1}{2} \int_{\mathcal{Z}} \int_{\mathcal{Z}} \zeta(z) h_2(z, z') \zeta(z') d^2 z d^2 z'$$

Noncanonical Hamiltonian Structure

Hamiltonian structure of media in Eulerian variables

Kinematic Commonality:

energy, momentum, Casimir conservation; dynamics is measure preserving rearrangement; continuous spectra; ... \rightarrow Krein's theorem

Noncanonical Poisson Bracket:

$$\{F, G\} = \int_{\mathcal{Z}} \zeta \left[\frac{\delta F}{\delta \zeta}, \frac{\delta G}{\delta \zeta} \right] dqdp = \int_{\mathcal{Z}} \frac{\delta F}{\delta \zeta} \mathcal{J} \frac{\delta G}{\delta \zeta} dqdp$$

Cosymplectic Operator:

$$\mathcal{J} \cdot = - \left(\frac{\partial \zeta}{\partial q} \frac{\partial \cdot}{\partial p} - \frac{\partial \cdot}{\partial q} \frac{\partial \zeta}{\partial p} \right)$$

Equation of Motion:

$$\frac{\partial \zeta}{\partial t} = \{\zeta, H\} = \mathcal{J} \frac{\delta H}{\delta \zeta} = -[\zeta, \mathcal{E}].$$

Organizing principle. Do one do all!

Linear Vlasov-Poisson System

Expand about Stable Homogeneous Equilibrium:

$$f = f_0(v) + \delta f(x, v, t)$$

Linearized EOM:

$$\frac{\partial \delta f}{\partial t} + v \frac{\partial \delta f}{\partial x} + \frac{e}{m} \frac{\partial \delta \phi[x, t; \delta f]}{\partial x} \frac{\partial f_0}{\partial v} = 0$$

$$\delta \phi_{xx} = 4\pi e \int_{\mathbb{R}} \delta f(x, v, t) dv$$

Linearized Energy (Kruskal-Oberman):

$$H_L = -\frac{m}{2} \int_{\Pi} \int_{\mathbb{R}} \frac{v (\delta f)^2}{f'_0} dv dx + \frac{1}{8\pi} \int_{\Pi} (\delta \phi_x)^2 dx$$

Linear Hamiltonian Theory

- Because noncanonical must expand f -dependent Poisson bracket as well as Hamiltonian. \Rightarrow

Linear Poisson Bracket:

$$\{F, G\}_L = \int f_0 \left[\frac{\delta F}{\delta \delta f}, \frac{\delta G}{\delta \delta f} \right] dx dv ,$$

where δf is the new dynamical variable and the Hamiltonian is the Kruskal-Oberman energy, H_L . The LVP system has the following Hamiltonian form:

$$\frac{\partial \delta f}{\partial t} = \{\delta f, H_L\}_L ,$$

with variables noncanonical and H_L not diagonal.

Hamiltonian Integral Transform Solution

Assume

$$\delta f = \sum_k f_k(v, t) e^{ikx}, \quad \delta \phi = \sum_k \phi_k(t) e^{ikx}$$

Linearized EOM:

$$\frac{\partial f_k}{\partial t} + ikv f_k + ik\phi_k \frac{e}{m} \frac{\partial f_0}{\partial v} = 0, \quad k^2 \phi_k = -4\pi e \int_{\mathbb{R}} f_k(v, t) dv$$

Three methods:

1. Laplace Transforms (Landau and others 1946)
2. Normal Modes (Van Kampen, Case,... 1955)
3. Coordinate Change \iff Integral Transform (PJM, Pfirsch, Shadwick, Balmforth 1992)

Canonization & Diagonalization

Fourier Linear Poisson Bracket:

$$\{F, G\}_L = \sum_{k=1}^{\infty} \frac{ik}{m} \int_{\mathbb{R}} f'_0 \left(\frac{\delta F}{\delta f_k} \frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k} \frac{\delta F}{\delta f_{-k}} \right) dv$$

Linear Hamiltonian:

$$\begin{aligned} H_L &= -\frac{m}{2} \sum_k \int_{\mathbb{R}} \frac{v}{f'_0} |f_k|^2 dv + \frac{1}{8\pi} \sum_k k^2 |\phi_k|^2 \\ &= \sum_{k,k'} \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(v) \mathcal{O}_{k,k'}(v|v') f_{k'}(v') dv dv' \end{aligned}$$

Canonization:

$$q_k(v, t) = f_k(v, t), \quad p_k(v, t) = \frac{m}{ik f'_0} f_{-k}(v, t) \quad \implies$$

$$\{F, G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\frac{\delta F}{\delta q_k} \frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k} \frac{\delta F}{\delta p_k} \right) dv$$

Dynamical Accessibility

Definition A phase space function k is dynamically accessible from a phase space function h , if g is an area-preserving rearrangement of h ; i.e., in coordinates $k(x, v) = h(X(x, v), V(x, v))$, where $[X, V] = 1$. A perturbation δh is linearly dynamically accessible from h if $\delta h = [G, h]$, where G is the infinitesimal generator of the canonical transformation $(x, v) \leftrightarrow (X, V)$.

Remark Dynamically accessible perturbations come about by perturbing the particle orbits under the action of some Hamiltonian; hence, dynamically accessible. For VP $\delta f = G_x f'_0$.

Lemma Continuous rearrangements preserve the 'topology' of level sets.

Integral Transform

Definition:

$$f(v) = \mathcal{G}[g](v) := \varepsilon_R(v) g(v) + \varepsilon_I(v) H[g](v),$$

where

$$\varepsilon_I(v) = -\pi \frac{\omega_p^2}{k^2} \frac{\partial f_0(v)}{\partial v}, \quad \varepsilon_R(v) = 1 + H[\varepsilon_I](v),$$

and the Hilbert transform

$$H[g](v) := \frac{1}{\pi} \mathcal{P} \int \frac{g(u)}{u - v} du,$$

with \mathcal{P} denoting Cauchy principal value of $\int_{\mathbb{R}}$.

Transform Properties

Theorem (G1) $\mathcal{G}: L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$, $1 < p < \infty$, is a bounded linear operator; i.e.

$$\|\mathcal{G}[g]\|_p \leq B_p \|g\|_p,$$

where B_p depends only on p .

Theorem (G2) If $f'_0 \in L^q(\mathbb{R})$, stable, Hölder decay, then $\mathcal{G}[g]$ has a bounded inverse,

$$\mathcal{G}^{-1}: L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R}),$$

for $1/p + 1/q < 1$, given by

$$\begin{aligned} g(u) &= \mathcal{G}^{-1}[f](u) \\ &:= \frac{\varepsilon_R(u)}{|\varepsilon(u)|^2} f(u) - \frac{\varepsilon_I(u)}{|\varepsilon(u)|^2} H[f](u). \end{aligned}$$

where $|\varepsilon|^2 := \varepsilon_R^2 + \varepsilon_I^2$.

Diagonalization

Mixed Variable Generating Functional:

$$\mathcal{F}[q, P'] = \sum_{k=1}^{\infty} \int_{\mathbb{R}} q_k(v) \mathcal{G}[P'_k](v) dv$$

Canonical Coordinate changes $(q, p) \longleftrightarrow (Q', P')$:

$$p_k(v) = \frac{\delta \mathcal{F}[q, P']}{\delta q_k(v)} = \mathcal{G}[P_k](v), \quad Q'_k(u) = \frac{\delta \mathcal{F}[q, P']}{\delta P_k(u)} = \mathcal{G}^\dagger[q_k](u)$$

New Hamiltonian:

$$H_L = \frac{1}{2} \sum_{k=1}^{\infty} \int_{\mathbb{R}} du \sigma_k(u) \omega_k(u) [Q_k^2(u) + P_k^2(u)]$$

where $\sigma_k = -\text{sgn}(uf'_0)$ and $\omega_k(u) = |ku|$

$(Q', P') \longleftrightarrow (Q, P)$ is trivial.

Hamiltonian Spectrum

Hamiltonian Operator:

$$f_{kt} = -ikv f_k + \frac{if'_0}{k} \int_{\mathbb{R}} d\bar{v} f_k(\bar{v}, t) =: -T_k f_k,$$

Complete System:

$$f_{kt} = -T_k f_k \quad \text{and} \quad f_{-kt} = -T_{-k} f_{-k}, \quad k \in \mathbb{R}^+$$

Lemma *If λ is an eigenvalue of the Vlasov equation linearized about the equilibrium $f'_0(v)$, then so are $-\lambda$ and λ^* . Thus if $\lambda = \gamma + i\omega$, then eigenvalues occur in the pairs, $\pm\gamma$ and $\pm i\omega$, for purely real and imaginary cases, respectively, or quartets, $\lambda = \pm\gamma \pm i\omega$, for complex eigenvalues.*

Spectral Stability

Definition The dynamics of a Hamiltonian system linearized around some equilibrium solution, with the phase space of solutions in some Banach space \mathcal{B} , is spectrally stable if the spectrum $\sigma(T)$ of the time evolution operator T is purely imaginary.

Theorem *If for some $k \in \mathbb{R}^+$ and $u = \omega/k$ in the upper half plane the plasma dispersion relation*

$$\varepsilon(k, u) := 1 - k^{-2} \int_{\mathbb{R}} dv \frac{f'_0}{u - v} = 0,$$

then the system with equilibrium f_0 is spectrally unstable. Otherwise it is spectrally stable.

Theorem (Penrose) *If there exists a point u such that*

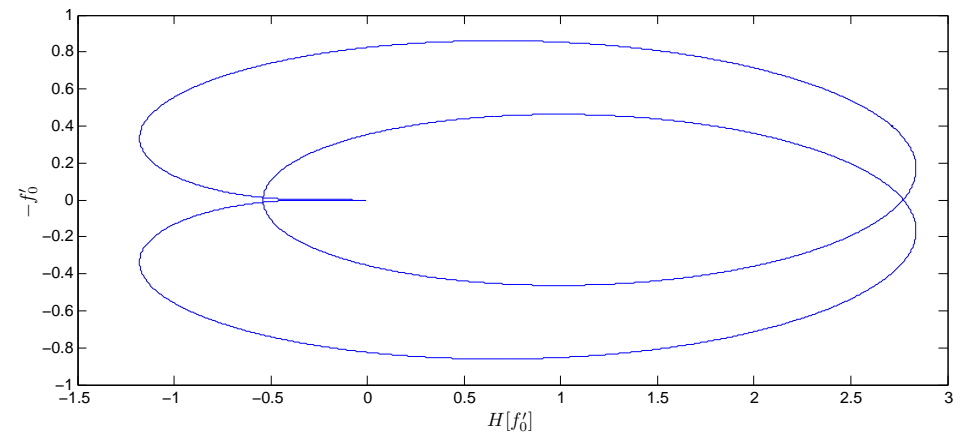
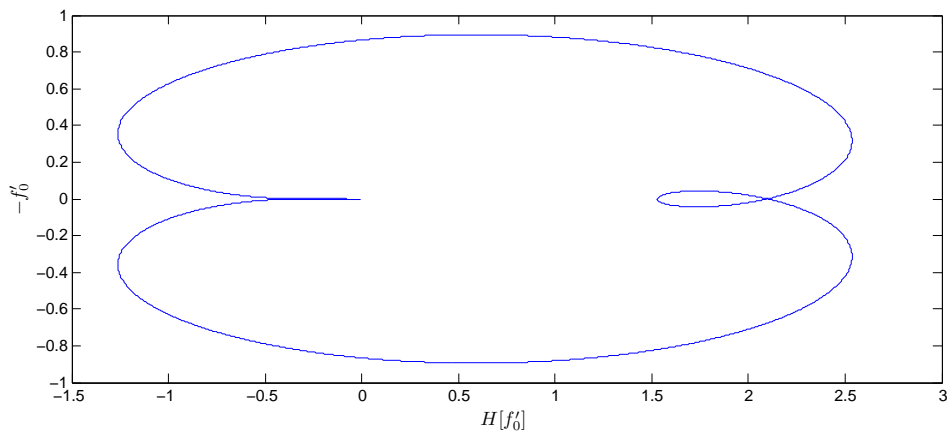
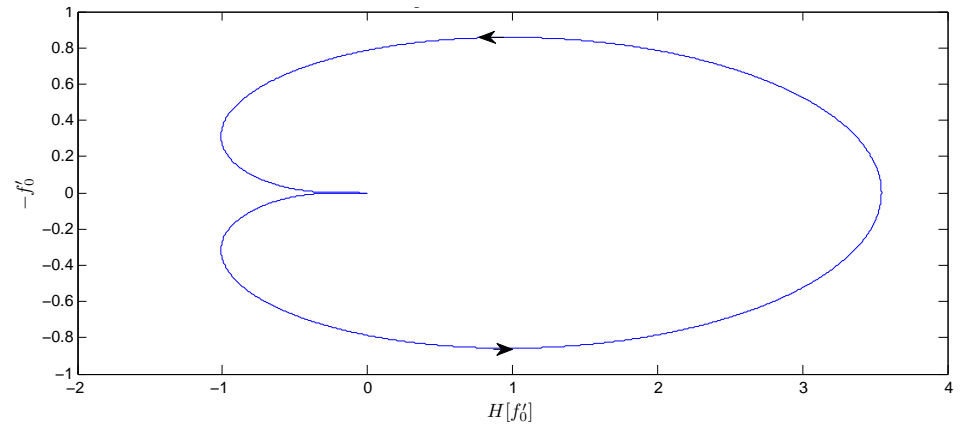
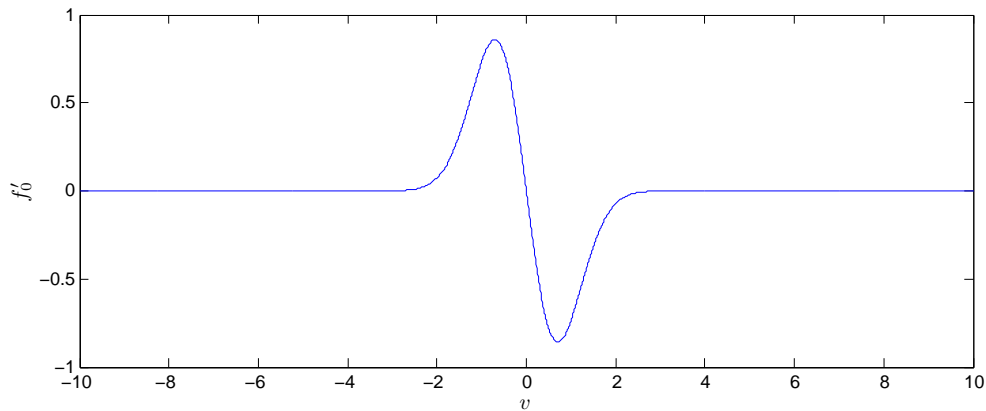
$$f'_0(u) = 0 \quad \text{and} \quad \int dv \frac{f'_0(v)}{u - v} < 0,$$

with f'_0 traversing zero at u , then the system is spectrally unstable. Otherwise it is spectrally stable.

Penrose Criterion

Winding number of $u \in \mathbb{R} \mapsto \varepsilon$, or

$$\lim_{u \rightarrow 0^+} \frac{1}{\pi} \int_{\mathbb{R}} dv \frac{f'_0}{v - u} = H[f'_0](u) - i f'_0(u),$$



Spectral Theorem

Set $k = 1$ and consider $T: f \mapsto ivf - if'_0 \int f$ in the space $W^{1,1}(\mathbb{R})$.

$W^{1,1}(\mathbb{R})$ is Sobolev space containing closure of functions $\|f\|_{1,1} = \|f\|_1 + \|f'\|_1 = \int_{\mathbb{R}} dv(|f| + |f'|)$. Contains all functions in $L^1(\mathbb{R})$ with weak derivatives in $L^1(\mathbb{R})$. T is densely defined, closed, etc.

Definition Resolvent of T is $R(T, \lambda) = (T - \lambda I)^{-1}$ and $\lambda \in \sigma(T)$.

(i) λ in point spectrum, $\sigma_p(T)$, if $R(T, \lambda)$ not injective. (ii) λ in residual spectrum, $\sigma_r(T)$, if $R(T, \lambda)$ exists but not densely defined. (iii) λ in continuous spectrum, $\sigma_c(T)$, if $R(T, \lambda)$ exists, densely defined but not bounded.

Theorem Let $\lambda = iu$. (i) $\sigma_p(T)$ consists of all points $iu \in \mathbb{C}$, where $\varepsilon = 1 - k^{-2} \int_{\mathbb{R}} dv f'_0 / (u - v) = 0$. (ii) $\sigma_c(T)$ consists of all $\lambda = iu$ with $u \in \mathbb{R} \setminus (-i\sigma_p(T) \cap \mathbb{R})$. (iii) $\sigma_r(T)$ contains all the points $\lambda = iu$ in the complement of $\sigma_p(T)$ that satisfy $f'_0(u) = 0$.

cf. e.g. P. Degond (1986). Similar but different.

Structural Stability

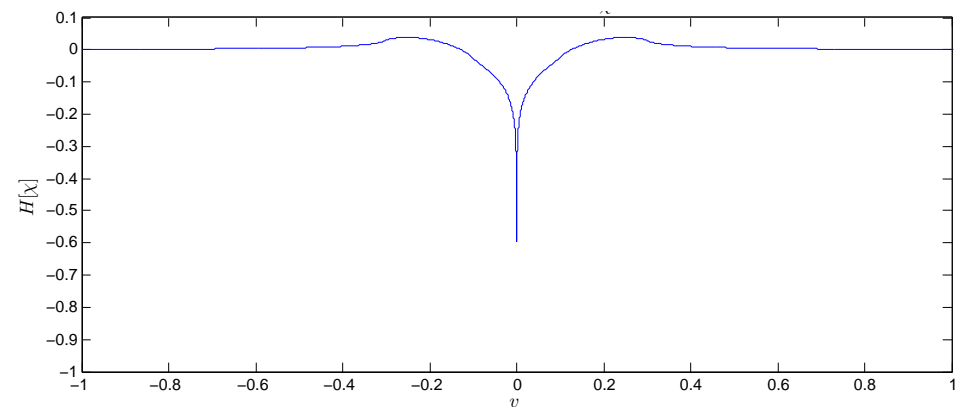
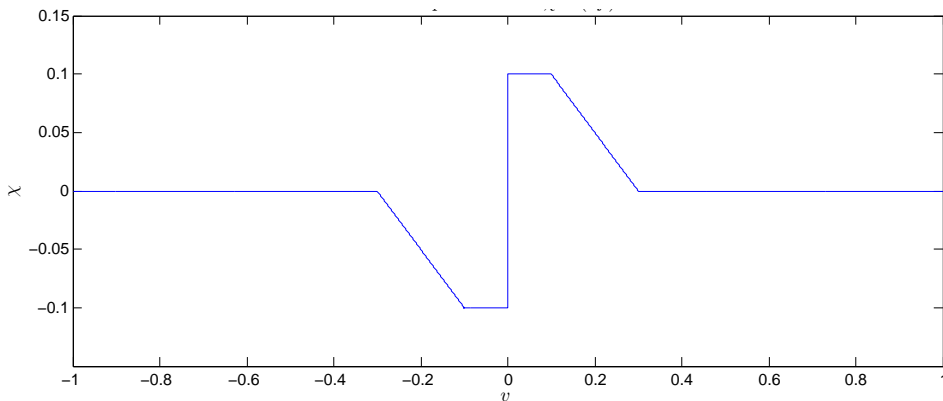
Definition Consider an equilibrium solution of a Hamiltonian system and the corresponding time evolution operator T for the linearized dynamics. Let the phase space for the linearized dynamics be some Banach space \mathcal{B} . Suppose that T is spectrally stable. Consider perturbations δT of T and define a norm on the space of such perturbations. Then we say that the equilibrium is structurally stable under this norm if there is some $\delta > 0$ such that for every $\|\delta T\| < \delta$ the operator $T + \delta T$ is spectrally stable. Otherwise the system is structurally unstable.

Definition Consider the formulation of the linearized Vlasov-Poisson equation in the Banach space $W^{1,1}(\mathbb{R})$ with a spectrally stable homogeneous equilibrium function f_0 . Let $T_{f_0 + \delta f_0}$ be the time evolution operator corresponding to the linearized dynamics around the distribution function $f_0 + \delta f_0$. If there exists some ϵ depending only on f_0 such that $T_{f_0 + \delta f_0}$ is spectrally stable whenever $\|T_{f_0} - T_{f_0 + \delta f_0}\| < \epsilon$, then the equilibrium f_0 is structurally stable under perturbations of f_0 .

All f_0 are Structurally Unstable in $W^{1,1}$

True in space where Hilbert transform unbounded, e.g. $W^{1,1}$.
Small perturbation \Rightarrow big jump in Penrose plot.

Theorem *A stable equilibrium distribution is structurally unstable under perturbations of f'_0 in the Banach spaces $W^{1,1}$ and $L^1 \cap C_0$.*

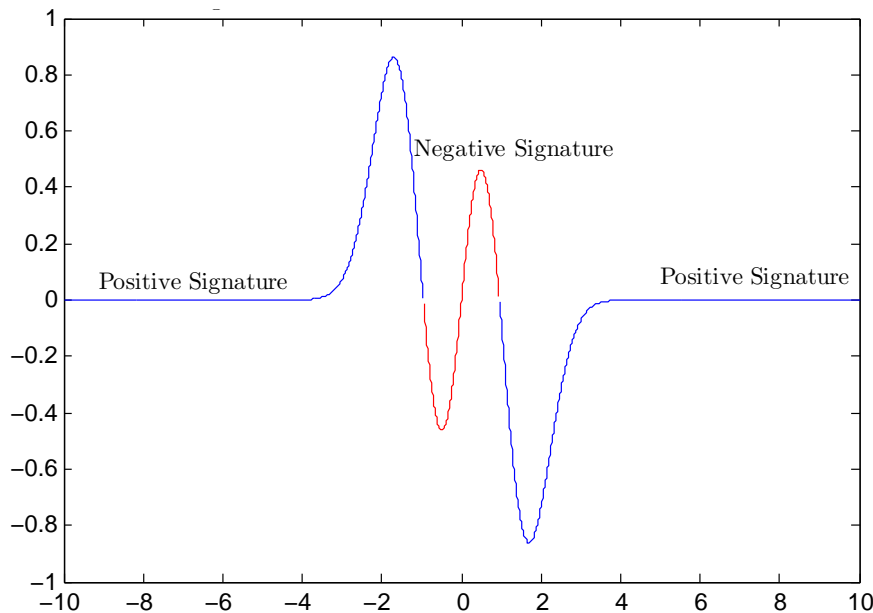


Easy to make 'bumps' in f_0 that are small in norm. What to do?

Krein-Moser (Sturrock)

Theorem (KMS) *Let H define a stable linear finite-dimensional Hamiltonian system. Then H is structurally stable if all the eigenfrequencies are nondegenerate. If there are any degeneracies, H is structurally stable if the associated eigenmodes have energy of the same sign. Otherwise H is structurally unstable.*

Definition The signature of the point $u \in \mathbb{R}$ is $-\text{sgn}(uf'_0(u))$.



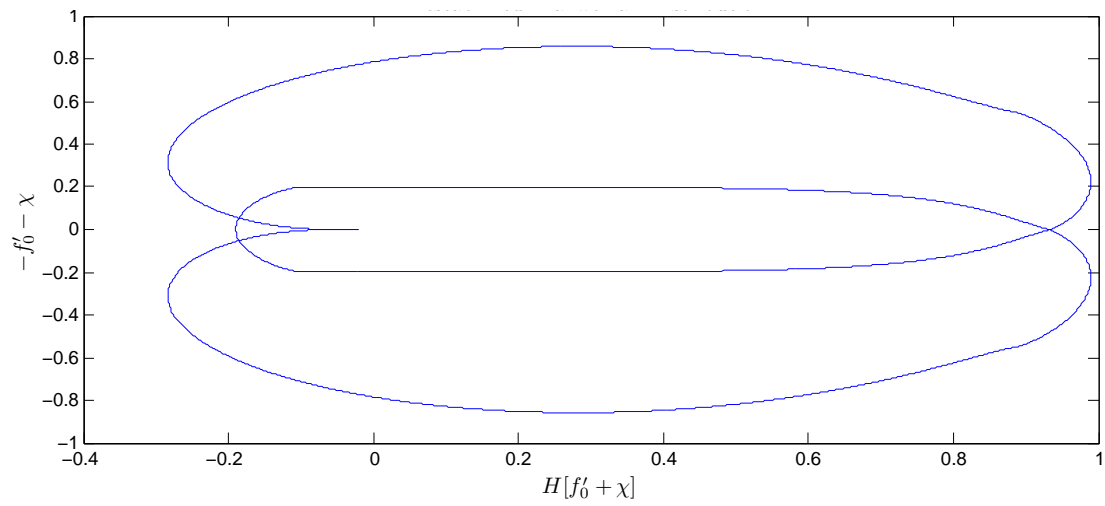
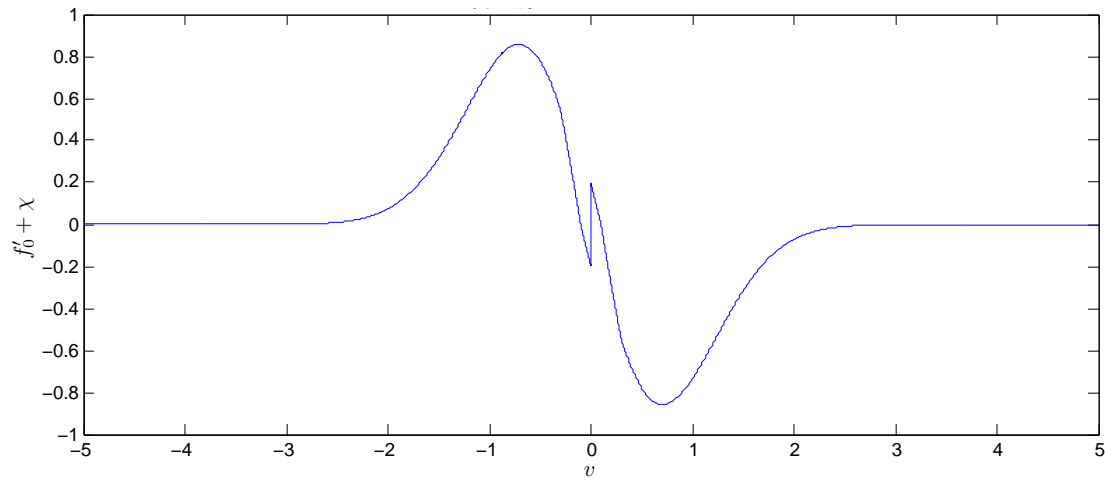
f'_0 for a bi-Maxwellian

Krein-Like Theorem for VP

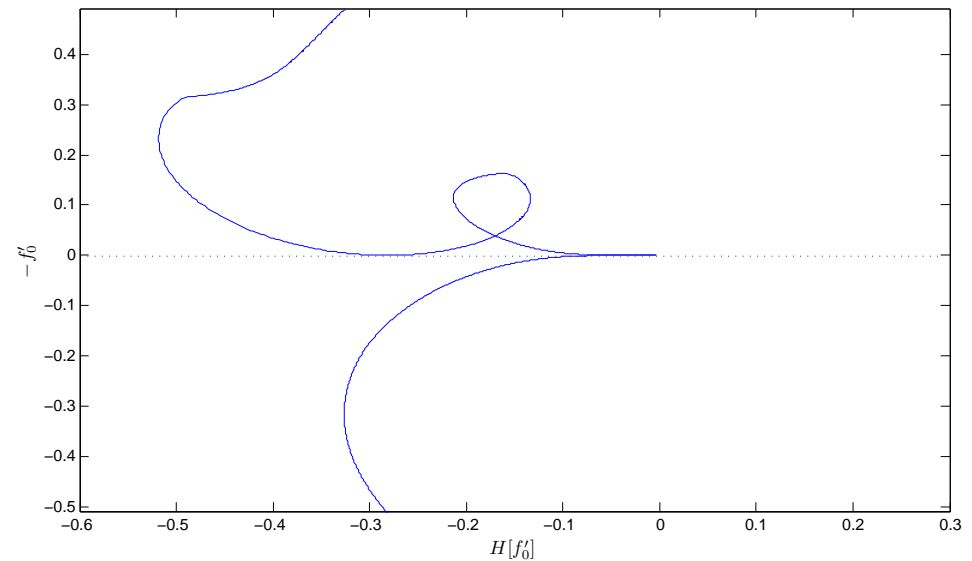
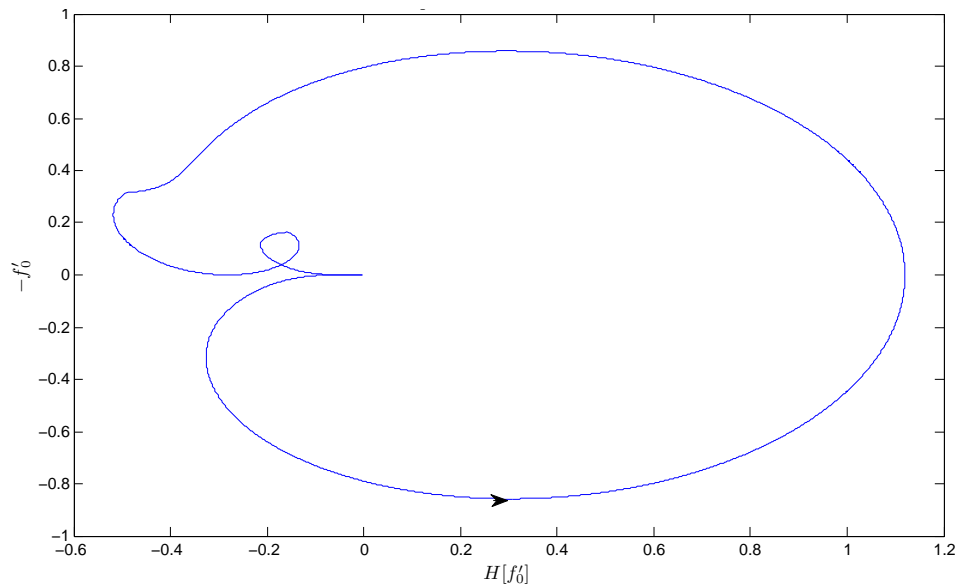
Theorem *Let f_0 be a stable equilibrium distribution function for the Vlasov equation. Then f_0 is structurally stable under dynamically accessible perturbations in $W^{1,1}$, if there is only one solution of $f'_0(v) = 0$. If there are multiple solutions, f_0 is structurally unstable and the unstable modes come from the roots of f'_0 that satisfy $f''_0(v) < 0$.*

Remark A change in the signature of the continuous spectrum is a necessary and sufficient condition for structural instability. The bifurcations do not occur at all points where the signature changes, however. Only those that represent valleys of the distribution can give birth to unstable modes.

DA Perturbation to Instability



Bifurcation at $k \neq 0$



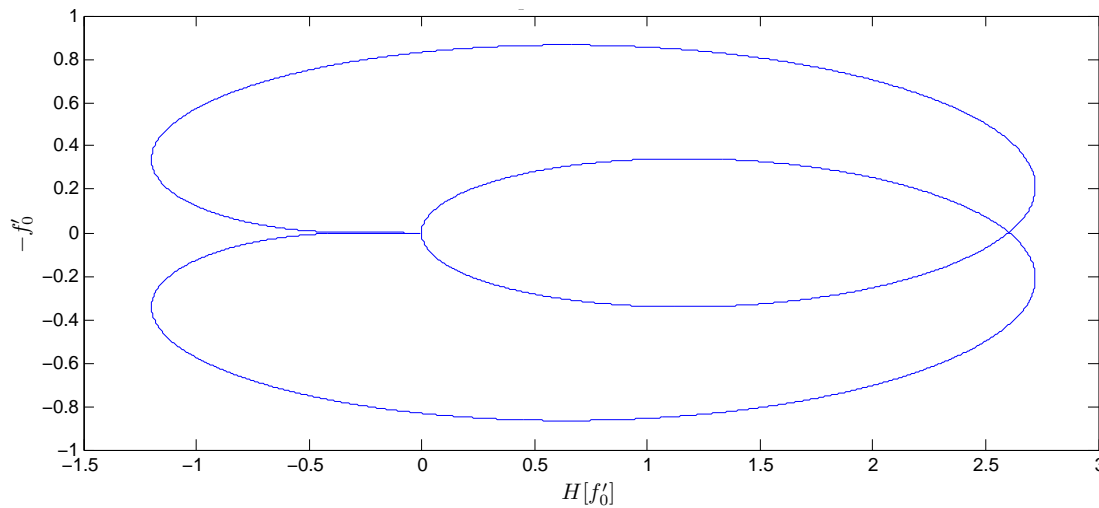
Theorem *Let f'_0 and f''_0 be Hölder continuous. If f_0 is stable there are no discrete modes (elements of the point spectrum) with signature the same as the signature of the continuum, where the signature of an embedded mode is $\text{sgn}(u \partial \varepsilon / \partial u) = \text{sgn}(\omega \partial \varepsilon / \partial \omega)$.*

Theorem *Let f'_0 be the derivative of an equilibrium distribution function with a discrete mode embedded in the continuous spectrum. Then there exists an infinitesimal function with compact support in the C^n norm for each n such that $f'_0 + \delta f'$ is unstable.*

Little Big Man Theorem

Theorem *Let f'_0 be the derivative of an equilibrium distribution function that has three discrete modes with real frequency. Consider a reference frame where all of the modes have positive frequency. Then represent the energies of the three modes as a triplet $(\pm \pm \pm)$ where the plus and minus signs correspond to the signature each mode, with the first mode being the one with the lowest frequency and the last the one with the highest frequency. Then if the triplet is of the form $(+ - +)$ or $(- + -)$ there is no reference frame in which all the modes have the same signature. If the triplet has any other form there is a reference frame in which all the modes have the same signature.*

Bifurcation at $k = 0$



Theorem *Suppose that f'_0 is a stable equilibrium distribution function that has a root at u of both f'_0 and $H[f'_0]$. Then f'_0 is structurally unstable under perturbations bounded by the C^n norm for all n .*

Summary-Conclusions

- Reviewed Hamiltonian structure of continuous media models
- Linearized and canonized (dynamical accessibility)
- Diagonalized by integral trans and obtained signature for continuous spectrum
- Hamiltonian spectrum
- Spectral theorem
- Structural stability— all unstable unless DA
- Variety of Krein-like theorems
- Whole new area of study. Tools \exists to do shearflow, etc.