# Global existence for the ion dynamics in the Euler-Poisson equations

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FRG Meeting

May 2010

# Abstract


We prove global existence for solutions of the Euler-Poisson/Ion dynamics equation

$$\begin{aligned} \partial_t n + \nabla \cdot (nv) &= 0 \\ n\left(\partial_t v + v \cdot \nabla v\right) &= -\nabla n - n\nabla\phi \\ \Delta\phi &= 4\pi\left(n_0 \exp\left(\phi\right) - n\right). \end{aligned}$$

in $\mathbb{R} \times \mathbb{R}^3$ which are small perturbation of a constant background.

This is a joint work with Y. Guo.

# Modeling a plasma

In order to model a plasma composed with 2 species of ions and electrons of charge $\pm e$, with collision parameter $\kappa$,

- For $\kappa$ large, the ions and electrons form only one fluid and one considers the Magneto-hydrodynamics equation.
- At medium $\kappa$, the dynamics of the ions and electrons decouple due to their very different masses, and one uses a 2 **fluids** (compressible Euler-Maxwell isothermal) **model**.
- At small $\kappa$, one switches to a kinetic model, and, neglecting the collisions, one studies the Vlasov-Maxwell equations.

We focus now on the intermediate case.

# The 2 fluids model

The 2 fluid equations are

$$\begin{aligned} \partial_t n_\pm + \nabla \cdot (n_\pm v_\pm) &= 0 \\ n_\pm m_\pm(\partial_t v_\pm + v_\pm \cdot \nabla v_\pm) + T_\pm \nabla n_\pm &= \mp e n_\pm \nabla \phi \\ -\Delta \phi &= 4\pi e(n_+ - n_-). \end{aligned} \tag{1}$$

Here $n_\pm, v_\pm, m_\pm, T_\pm$ are the ion $(+)$ and electron density, velocity, masses and effective temperatures. Besides, $E = \nabla\phi$ is the electric field $e$ is the charge of an electron.

Note that we have neglected the magnetic field (possible if the initial data are irrotational, or if the field is small).

# Observations

- The Zakharov equations describe (at first order) the behavior of some particular solutions of the above Euler-Maxwell system.
- In general, solutions of the compressible Euler equations will develop shocks (even with small smooth initial data, cf Sideris, 85). However it is possible that the presence of a feed-back due to coupling with a field might stabilize the system.

# The EP/electron equation

The electrons have much smaller mass, hence, as a first approximation, one can look at the fast-time dynamics and treat the ion as a fixed background. One then obtains the EP/electron equation

$$\begin{aligned} \partial_t n_- + \nabla \cdot (n_- v_-) &= 0 \\ n_- m_-(\partial_t v_- + v_- \cdot \nabla v_-) + T_- \nabla n_- &= e n_- \nabla \phi \\ \Delta \phi &= 4\pi e(n_- - n_0). \end{aligned}$$

**Global existence for small perturbation (Guo, 98)**

Irrotational initial data $(n_-, v_-) = (n_0 + \varepsilon\rho, \varepsilon v)$ lead to global solutions.

Hence in this case, when one turns on an electric field, the system is stabilized.

**Key observation**: in the case of pure Euler, the linearization around a fixed background satisfies the **pure wave equation**:

$$\partial_{tt}\rho - \Delta\rho = 0.$$

For this, linear solutions decay like $t^{-1}$ and have many quadratic resonances. whereas for the electron equation, the linearization yields a **Klein Gordon** equation:

$$\partial_{tt}\rho - \Delta\rho + \omega_e^2\rho = 0, \;\; \omega_e = \sqrt{\frac{4\pi e^2 n_0}{m_-}}$$

where $\omega_e$ is the "electron plasma frequency". For **KG**, solutions decay faster $t^{-\frac{3}{2}}$ and have no quadratic resonances (hence a normal form transformation eliminates the quadratic terms).

# The ion equation

In this work, we focus on the other extreme dynamics: the long time dynamics. In this case, the electron move so fast that they are constantly at the Boltzman equilibrium state:

$$n_- = n_0 \exp(\frac{e\phi}{T_-})$$

(also obtained by formally setting $m_-/m_+ = 0$ in the 2 fluid equation). In this case, the system is defined by the ion equation interacting with their self-consistent field **(EP/ion)**

$$\begin{aligned} \partial_t n_+ + \nabla \cdot (n_+ v_+) &= 0 \\ n_+ m_+ (\partial_t v_+ + v_+ \cdot \nabla v_+) &= -T_+ \nabla n_+ - n_+ e \nabla \phi \\ \Delta \phi &= 4\pi e \left( n_0 \exp\left(\frac{e\phi}{T_-}\right) - n_+ \right). \end{aligned}$$

# Previous results

This equation was studied in several context before.

- Cordier and Grenier, Peng and Wang, studied the quasi-neutral limit.
- Feldman, Ha, Slemrod studied the plasma sheath interface.
- Guo and Tahvildar-Zadeh prove formation of shocks for large amplitude solutions.
- Other works: Holm,Johnson and Lonngren, Liu and Tadmore, Peng and Wang, Perthame, Texier, Wang and Wang ...

# Main result

## Global existence for the ion equation

Suppose that $\mathrm{curl} v_0 = 0$ and that $(\rho, v)$ is small enough in $W^{8,\frac{10}{9}} \cap H^{10}$, then the solution with initial data

$$(n_-, v_-) = (1 + \rho, v)$$

exists globally and scatters in $L^2$.

- Remark: the regularity is not optimal.
- Here again, the electric field stabilize the system.
- Now, the two extreme dynamics for the 2-fluids system are controlled.

# Main difficulties-1

Here, the linearized operator yields

$$\partial_{tt}\rho - \left(\Delta + \frac{\Delta}{1-\Delta}\right)\rho = 0$$

which is really close to the wave equation: the **dispersion relation** is

$$\omega(k) = \pm|k|\sqrt{\frac{2+|k|^2}{1+|k|^2}} = \pm p(k).$$

This has 2 stationary points where $\omega'' = 0$: $1 < r_0 < 3$ and $\infty$. Fortunately, this still retains some curvature in the radial direction and the linear solutions decay like $t^{-\frac{4}{3}}$, i.e. **faster than** $t^{-1}$ and are integrable in time.

# Main difficulties-2

This is strongly resonant (especially around 0 where it is almost the wave equation). We overcome this by careful estimate of the resonance region

$$\{(\xi,\eta) : \Phi = \omega(\xi) \pm \omega(\xi-\eta) \pm \omega(\eta) = 0\} = \{|\xi||\xi-\eta||\eta| = 0\}$$

and $\Phi$ vanishes at 0 at second order. Normal form transformation produces bilinear terms with singular multipliers (not in $L^\infty_\xi H^{\frac{3}{2}}_\eta$, in particular, not of Coifman-Meyer type), and we need to deal with singular operators with singular kernels.

# Outline of the proof

1. we first study the linear flow (to get good decay rate)

   The we turn to the nonlinear term and we bound the norm

$$\|\rho\|_X = \sup_t \left\{ \||\nabla|^{-1}\rho(t)\|_{H^{2k+1}} + (1+t)^{\frac{16}{15}}\|\rho(t)\|_{W^{k,10}} \right\}.$$

2. we control $L^2$-scale bounds using standard energy estimates

   In order to control $L^{10}$-like norm,

3. we perform a normal form transformation (as in **Shatah**, **Gustafson**, **Nakanishi** and **Tsai**, **Germain**, **Masmoudi** and **Shatah**)
4. we study the singular multipliers coming from the transformation.

# The linear equation-1

To study the linear equation, we need to estimate the half-wave propagator

$$e^{it\omega(|\nabla|)}, \quad \omega(\xi) = |\xi|\sqrt{\frac{2+|\xi|^2}{1+|\xi|^2}}$$

This operator has a wave-like degeneracy at 0, and 2 stationary points ($\omega'' = 0$): $|\xi|_{stat} = \sqrt{1+\sqrt{7}}$ and $\infty$. We can counter the degeneracies at 0 and $\infty$ by paying some derivatives:

$$\| |\nabla|^{-\frac{1}{2}} e^{it\omega(|\nabla|)} P_0 \delta \|_{L^\infty} \lesssim (1+|t|)^{-\frac{3}{2}}$$

$$\| e^{it\omega(|\nabla|)} P_N \delta \|_{L^\infty} \lesssim N^{\frac{5}{2}} |t|^{-\frac{3}{2}}, \quad N > 10$$

# The linear equation-2

We need to have a decay faster than $t^{-1}$. Since our phase $\omega$ is radial, the stationary phase one the sphere automatically gives us a decay like $t^{-1}$. To get better, we exploit the curvature in the radial direction.

At the finite stationary point, we use the fact that $\omega^{(3)}(|\xi|_{stat}) \neq 0$ and only get a slower decay:

$$\|e^{it\omega(|\nabla|)} P_{stat}\delta\|_{L^\infty} \lesssim (1+|t|)^{-1-\frac{1}{3}}.$$

Finally, we get

**decay estimates**

$$\|e^{it\omega(|\nabla|)} f\|_{L^{10}} \lesssim (1+|t|)^{-\frac{16}{15}} \|f\|_{W^{\frac{12}{5},\frac{10}{9}}}.$$

# Reduction of the system-1

We want to consider the system

$$\begin{aligned}
\partial_t\rho + \mathrm{div}(v) &\quad + \mathrm{div}(\rho v) &&= 0\\
\partial_t v + \nabla\rho + \nabla\phi &\quad + (v\cdot\nabla)v - \nabla\frac{\rho^2}{2} &&= -\nabla\left[\ln(1+\rho) - \rho + \frac{\rho^2}{2}\right]\\
\rho = (1-\Delta)\phi &\quad + \frac{\phi^2}{2} &&+ \left[e^\phi - 1 - \phi - \frac{\phi^2}{2}\right].
\end{aligned}$$

The last line defines an operator $\rho \mapsto \phi(\rho)$ with

$$\phi(\rho) = (1-\Delta)^{-1}\rho - \frac{1}{2}(1-\Delta)^{-1}\left[(1-\Delta)^{-1}\rho\right]^2 + O(\rho^3).$$

and since $\nabla\times v = 0$, the second equation reduces to a scalar equation.

# Reduction of the system-2

Diagonalizing the linear part and introducing the eigenvector

$$\alpha = \rho - \frac{i}{q(|\nabla|)}\mathcal{R}^{-1}v, \qquad q(\xi) = \sqrt{\frac{2+|\xi|^2}{1+|\xi|^2}}$$

we are left with the equation

$$(\partial_t - i\omega(|\nabla|))\alpha = Q(\alpha) + \mathcal{N}$$

where

$$Q = -\mathrm{div}(\rho v) - \frac{i|\nabla|}{2q(|\nabla|)}\left\{(1-\Delta)^{-1}[(1-\Delta)^{-1}\rho]^2 - \rho^2 - |v|^2\right\}$$

$$\mathcal{N} = -\frac{i|\nabla|}{q(|\nabla|)}\left[\ln(1+\rho) - \rho + \frac{\rho^2}{2} - R(\rho)\right].$$

# The normal form transformation-1

It suffices to consider the quadratic terms. To bound them, we use a normal form transformation. First, we conjugate by the linear flow: consider

$$\beta(t) = e^{it\omega(|\nabla|)}\alpha(t)$$

Then, if one only takes into account only the **linear** terms, we get

$$\partial_t \beta = 0$$

# The normal form transformation-1

It suffices to consider the quadratic terms. To bound them, we use a normal form transformation. First, we conjugate by the linear flow: consider

$$\beta(t) = e^{it\omega(|\nabla|)}\alpha(t)$$

including the nonlinear terms, we get

$$\partial_t \beta = \mathfrak{F}(\beta), \quad \mathfrak{F}(\beta) = e^{-it\omega(|\nabla|)} F(e^{it\omega(|\nabla|)}\beta)$$

**key point:** no linear point, deriving in time gains nonlinear terms:

$$\partial_t \beta = O(\beta^2).$$

# The normal form transformation-2

Using that

$$\partial_t \beta = e^{-i\omega(|\nabla|)} Q(\alpha),$$

We write the Duhamel formula

$$\begin{aligned}\beta(t) &= \beta(0) + \int_0^t \mathfrak{F}(\beta(t-s))ds \\ &= \beta(0) + \mathcal{F}_\xi^{-1} \int_0^t \int_{\eta\in\mathbb{R}^3} e^{is\Phi} m(\xi,\eta)\hat{\beta}(\xi-\eta)\hat{\beta}(\eta)d\eta\end{aligned}$$

for $m(\xi,\eta) \simeq |\xi| n_1(\xi) n_2(\xi-\eta) n_3(\eta)$ a multiplier and the phase

$$\Phi = \omega(\xi) \pm \omega(\xi-\eta) \pm \omega(\eta).$$

Integrating by parts in $s$, we obtain

$$\begin{aligned}\beta(t) - \mathcal{B}[\beta,\beta] &= \beta_0 - \mathcal{B}[\beta,\beta](0) \\ &+ 2\mathcal{F}_\xi^{-1} \int_0^t \int_{\eta\in\mathbb{R}^3} \frac{e^{is\Phi}}{i\Phi} m(\xi,\eta)\hat{\beta}(\xi-\eta)\partial_t\hat{\beta}(\eta)d\eta.\end{aligned}$$

And now the integrand is **cubic**.

# The normal form transformation-3

Finally, going back to $\alpha$, we get

$$\hat{\alpha}(t) + \mathcal{F}_\xi \mathfrak{B}[\alpha(t), \alpha(t)] = e^{it\omega(\xi)} \left(\hat{\alpha}_0 + \mathcal{F}_\xi \mathfrak{B}[\alpha_0, \alpha_0]\right)$$
$$+ \int_0^t e^{i(t-s)\omega(\xi)} \int_{\mathbb{R}^6} \frac{m_1(\xi,\eta) m_2(\eta,\zeta)}{\Phi(\xi,\eta)} \hat{\alpha}(s, \xi-\eta)\hat{\alpha}(s,\eta-\zeta)\hat{\alpha}(s,\zeta)$$
$$d\zeta d\eta ds$$

where

$$\mathcal{F}\mathfrak{B}[\alpha,\alpha] = \int_{\mathbb{R}^3} \frac{m(\xi,\eta)}{\Phi(\xi,\eta)} \hat{\alpha}(\xi-\eta)\hat{\alpha}(\eta) d\eta.$$

is a bilinear form with singular multiplier $\frac{m}{\Phi}$ and $m(\xi,\eta) \sim |\xi|$.

# The $H^{-1}$-norm

Around 0, we have that

$$\Phi = \omega(\xi) - \omega(\xi - \eta) - \omega(\eta) \sim |\xi||\xi - \eta||\eta|,$$

so in the cubic term, the multiplier ($m(\xi,\eta) \sim |\xi|$)

$$\frac{m_1(\xi,\eta)m_2(\eta,\zeta)}{\Phi(\xi,\eta)}$$

is not even bounded. To account for that, we rewrite

$$\begin{aligned} &\frac{m_1(\xi,\eta)m_2(\eta,\zeta)}{\Phi(\xi,\eta)}\hat{\alpha}(s,\xi-\eta)\hat{\alpha}(s,\eta-\zeta)\hat{\alpha}(s,\zeta) \\ \simeq\ &\frac{|\xi||\eta||\xi-\eta|}{\Phi(\xi,\eta)}\frac{\hat{\alpha}(s,\xi-\eta)}{|\xi-\eta|}\hat{\alpha}(s,\eta-\zeta)\hat{\alpha}(s,\zeta) \end{aligned}$$

and look for a control of $\alpha$ in $\dot{H}^{-1}$ (possible since in the nonlinearity, all the terms have derivative- “null” structure).

# Boundedness for bilinear operator with singular multiplier

In order to control the various bilinear "pseudo product" operators, we use the following result from Gustafson, Nakanishi and Tsai. Consider the operator

$$B[f,g] = \mathcal{F}_\xi^{-1} \int_{\mathbb{R}^3} \mathfrak{m}(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta$$

The Coifman-Meyer theorem is too restrictive for our multiplier because

- they are not smooth
- it requires an homogeneous behavior (control on $|\partial_\xi^\alpha \partial_\eta^\beta \mathfrak{m}| \lesssim |\xi|^{-|\alpha|}|\eta|^{-|\beta|}$ is not enough (Grafakos and Kalton))

# Boundedness for bilinear operator with singular multiplier

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$$B[f,g] = \mathcal{F}_\xi^{-1} \int_{\mathbb{R}^3} \mathfrak{m}(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta$$

and the norm

$$\|\mathfrak{m}\|_{M^s_{\xi,\eta}} = \sum_{N\in 2^{\mathbb{Z}}} \|P^\eta_N \mathfrak{m}(\xi,\eta)\|_{L^\infty_\xi \dot{H}^s_\eta}$$

## Boundedness for multiplier operators

Assume $0 \le s \le 3/2$, $2 \le p, r \le 2n/(n-2s)$

$$\|B[f,g]\|_{L^{r'}} \lesssim \|\mathfrak{m}\|_{M^s_{\eta,\xi}} \|f\|_{L^2} \|g\|_{L^p}.$$

It remains to estimate the phases

$$\begin{aligned}\Phi_1(\xi,\xi-\eta,\eta) &= p(\xi)-p(\xi-\eta)-p(\eta)\\ \Phi_2(\xi,\xi-\eta,\eta) &= p(\xi)+p(\xi-\eta)+p(\eta)\\ \Phi_3(\xi,\xi-\eta,\eta) &= p(\xi)-p(\xi-\eta)+p(\eta)\\ \Phi_4(\xi,\xi-\eta,\eta) &= p(\xi)+p(\xi-\eta)-p(\eta).\end{aligned}$$

We remark that if $H=\max(|\xi|,|\xi-\eta|,|\eta|)$ $L$ is the min, and $\gamma$ is the angle between $H$ and $L$, then

$$|\Phi|\gtrsim L\left[(1-\cos\gamma)+\frac{H^2}{(1+H^2)(1+L^2)}\right]$$

and we get that

$$\mathfrak{M}=\frac{|\xi||\xi-\eta||\eta|}{\Phi\langle\xi-\eta\rangle^{\frac{9}{4}}\langle\eta\rangle^{\frac{9}{4}}}\in M^{\frac{5}{4}-\varepsilon}_{\eta,\xi}\cap M^{\frac{5}{4}-\varepsilon}_{\xi,\eta}$$

# Control on the cubic term

Using this, we can control the quadratic and singular cubic terms in the equation.

$$
\begin{aligned}
&\|\mathcal{F}^{-1}\int_{s,\eta,\zeta} e^{i(t-s)\omega(\xi)}\frac{m_1(\xi,\eta)m_2(\eta,\zeta)}{\Phi}\hat{\alpha}(\xi-\eta)\hat{\alpha}(\eta-\zeta)\hat{\alpha}(\zeta)\|_{W^{k,10}} \\
&\lesssim \int_0^t \frac{1}{(1+t-s)^{\frac{16}{15}}} \\
&\|\mathcal{F}_\xi^{-1}\int_{\mathbb{R}^6}\frac{m_1(\xi,\eta)m_2(\eta,\zeta)}{\Phi}\hat{\alpha}(s,\xi-\eta)\hat{\alpha}(\eta-\zeta)\hat{\alpha}(\zeta)d\eta d\zeta\|_{W^{k+\frac{12}{5},\frac{10}{9}}} ds \\
&\lesssim \|\alpha\|_X^3(1+t)^{-\frac{16}{15}}
\end{aligned}
$$

# Control on the quadratic term

The quadratic/integrated term leads to the big loss of derivatives

$$
\begin{aligned}
&\|\mathfrak{B}[\alpha,\alpha]\|_{W^{k,10}} \\
&\lesssim \|\mathcal{F}\int_{\mathbb{R}^3} \frac{|\xi||\xi-\eta||\eta|}{\langle\xi-\eta\rangle^r\langle\eta\rangle^r\Phi} \frac{\langle\xi-\eta\rangle^r\hat{\alpha}(\xi-\eta)}{|\xi-\eta|} \frac{\langle\eta\rangle^r\hat{\alpha}(\eta)}{|\eta|} d\eta\|_{W^{k+2,2}} \\
&\lesssim \|\mathfrak{M}\|_{M^{\frac{3}{2}-\varepsilon}_{\xi,\eta}} \|\alpha\|_{W^{k+\frac{11}{5}+,p}} \|\frac{(1-\Delta)^2}{|\nabla|}\alpha\|_{L^q} \\
&\lesssim \|\alpha\|_X \|\alpha\|_{W^{k,10}}
\end{aligned}
$$

with $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}+\varepsilon$. This is more problematic if the first term is at high frequencies and the second at low frequencies.

# The 2 fluid system-Linear part-1

The linearization of the system yields with $\varepsilon = m_e/m_i$ and $T = T_i/T_e$:

$$\partial_t \begin{pmatrix} n \\ h \\ \rho \\ g \end{pmatrix} + \begin{pmatrix} 0 & -|\nabla| & 0 & 0 \\ \frac{|\nabla|}{\varepsilon}(T + (-\Delta)^{-1}) & 0 & -\frac{|\nabla|^{-1}}{\varepsilon} & 0 \\ 0 & 0 & 0 & -|\nabla| \\ -|\nabla|^{-1} & 0 & |\nabla|(1 + (-\Delta)^{-1}) & 0 \end{pmatrix} \begin{pmatrix} n \\ h \\ \rho \\ g \end{pmatrix}$$

We set $y_1 = \left(\frac{\varepsilon(T+(-\Delta)^{-1})}{1+(-\Delta)^{-1}}\right)^{\frac{1}{4}} \left(n_e - i\sqrt{\frac{\varepsilon}{T+(-\Delta)^{-1}}}\mathcal{R}^{-1}v_e\right)$,
$z_1 = \rho_i - \frac{i}{\sqrt{1+(-\Delta)^{-1}}}\mathcal{R}^{-1}v_i$, $y_2 = \overline{w}_1$, $z_2 = \overline{z}_1$,

# The 2 fluid system-Linear part-2

writing $A = (y_1, y_2, z_1, z_2)$, we need to investigate the following equation

$$\partial_t A + iMA = 0$$

with

$$M = \begin{pmatrix} -H_\varepsilon & 0 & L & L \\ 0 & H_\varepsilon & -L & -L \\ L & L & -H_1 & 0 \\ -L & -L & 0 & H_1 \end{pmatrix}$$

with

$$H_\varepsilon = |\nabla| \sqrt{\frac{T + (-\Delta)^{-1}}{\varepsilon}} = \sqrt{\frac{1 - T\Delta}{\varepsilon}}$$

$$H_1 = |\nabla| \sqrt{1 + (-\Delta)^{-1}} = \sqrt{1 - \Delta}$$

$$2L = \frac{|\nabla|^{-1}}{[\varepsilon(T + (-\Delta)^{-1})(1 + (-\Delta)^{-1})]^{\frac{1}{4}}} = \frac{1}{[\varepsilon(1 - T\Delta)(1 - \Delta)]^{\frac{1}{4}}}.$$

# The 2 fluid system-Linear part-2

The eigenvalues are now given by

$$\Lambda_1 = \sqrt{\frac{(1+\frac{1}{\varepsilon}) - (1+\frac{T}{\varepsilon})\Delta + \frac{1}{\varepsilon}\sqrt{((1-\varepsilon)-(T-\varepsilon)\Delta)^2 + 4\varepsilon}}{2}}$$

$$= \frac{1}{\sqrt{\varepsilon}}\sqrt{1-T\Delta}\left(1+\mathcal{O}\left(\frac{\varepsilon}{1-\Delta}\right)\right)$$

$$\Lambda_3 = \sqrt{\frac{(1+\frac{1}{\varepsilon}) - (1+\frac{T}{\varepsilon})\Delta - \frac{1}{\varepsilon}\sqrt{((1-\varepsilon)-(T-\varepsilon)\Delta)^2 + 4\varepsilon}}{2}}$$

$$= \sqrt{-\Delta}\sqrt{\frac{T+1-T\Delta}{1-T\Delta}}\left(1+\mathcal{O}\left(\frac{\varepsilon}{1-\Delta}\right)\right)$$

$\Lambda_2 = -\Lambda_1$ and $\Lambda_4 = -\Lambda_3$