Diffusion and Homogenization limit of Boltzmann-Poisson system

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Outline

Fluid description: Drift-Diffusion equation

- $\rho := \rho(t,x)$: charge density

$$x \in \omega \subset \mathbb{R}^d, \quad d \geq 1, \quad t \geq 0$$

$$\partial_t \rho + \nabla_x . j = 0$$

$$j = -D \nabla_x \rho + \mu \rho \nabla_x \phi, \qquad D = \frac{k_B T}{e} \mu$$

$$-\Delta_x \phi = \rho$$

Kinetic description: Boltzmann Equation

- $f(t,x,v)dxdv$:= number of particles in the unit volume $dxdv$ arround (x, v) and at time t.

$$\frac{df}{dt} = \partial_t f + v.\nabla_x f + \frac{e}{m_p}\nabla_x \phi.\nabla_v f := S + Q(f)$$

- **Electron-phonon collisions**

$$Q(f)(v) = \int_{\mathbb{R}^d} \sigma(v,v')\,[M(v)\,f(v') - M(v')\,f(v)]dv'$$

$$M(v) = \left(\frac{m}{2\pi k_B T}\right)^{d/2} \exp\left(\frac{-m|v|^2}{2k_B T}\right)$$

Diffusion approximation

$$\varepsilon > 0 \qquad \text{"} \quad t \to \varepsilon^2\, t \qquad x \to \varepsilon\, x \quad \text{"}$$

$$f^\varepsilon : f^\varepsilon(t,x,v) \geq 0, \qquad (x,v) \in \Omega = \omega \times \mathbb{R}^d$$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}\left(v.\nabla_x f^\varepsilon - \nabla_x \Phi . \nabla_v f^\varepsilon\right) - \frac{Q(f^\varepsilon)}{\varepsilon^2} = 0$$

$$Q(f) = \int_{\mathbb{R}^d} \sigma(v,v')\left[\, M(v)\, f(v') - M(v')\, f(v)\,\right] dv'$$

$$M(v) = \frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}, \qquad 0 < \sigma_1 \leq \sigma(v,v') = \sigma(v',v) \leq \sigma_2$$

Proposition (Dynamics of collisions)

- $-Q$: *non-negative and selfadjoint on* $L^2(dv/M(v))$
- $\mathcal{N}(Q) = \mathbb{R}\, M(v)$
- *Solvability condition*

$$\mathcal{R}(Q) = \mathcal{N}(Q)^{\perp} := \left\{ g = g(v) \ / \ \langle g, M \rangle = \int_{\mathbb{R}^d} g(v) dv = 0 \right\}$$

- *Q is inversible on* $\mathcal{N}(Q)^{\perp}$.
- *H-Theorem*

$$-\int_{\mathbb{R}^d} Q(f) \log\left(\frac{f}{M}\right) dv \geq \frac{1}{2} \int_{\mathbb{R}^d} \left(\sqrt{f} - \sqrt{\rho M}\right)^2 dv$$

Review on the question

- Poupaud (90') :

$$\|f^{\varepsilon} - \rho\, M(v)\|_{L^{\infty}(0,T;\, L^1)} \leq C_T\, \varepsilon$$

For : $\Phi \in W^{2,\infty}_{t,x}$ & well prepared initial & inflow data

$$f^{\varepsilon}(t=0) = \rho_I(x) M(v)$$

$$f^{\varepsilon} = \rho_b(t,x) M(v), \qquad (x,v) \in \Gamma^{-},$$

$$\Gamma^{\pm} = \{(x,v) \in \partial\Omega, \quad \pm(v.n(x)) > 0\}$$

Sketch of the proof

1. Regular Hilbert expansion $\Longrightarrow$ control en mass thanks to contraction property.

2. Analysis of the Half-Space problem associated to Q

   to correct the incoming boundary layer of order ε^{-1}

   Main tool : • Entropy dissipation

   • Good decay of $Q^{-1}(vM)$ as $|v| >> 1$

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References : Bardos, Degond, Perthame, Schmeiser,...

Diffusion & Homogenization

Outline

- Free space : $(x, v) \in \mathbb{R}^{2d}$ & given potential

$$\Phi^\varepsilon := \Phi\left(t, x, \frac{x}{\varepsilon}\right)$$

Φ is periodic with respect to $y \in Y = [0,1]^d$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v.\nabla_x f^\varepsilon - \nabla_x \Phi^\varepsilon.\nabla_v f^\varepsilon) - \frac{Q(f^\varepsilon)}{\varepsilon^2} = 0$$

$$f^\varepsilon(t=0) = f_0^\varepsilon$$

References: Goudon, Mellet, Poupaud, Zhang, . . .

- The homogenized fluid system is associated with an effective Φ_e :

$$\Phi_e(t,x) = -\log\left(\int_Y e^{-\Phi(t,x,y)}dy\right)$$

Define

$$M_\Phi(t,x,y,v) = M(v)\exp(\Phi_e(t,x) - \Phi(t,x,y))$$

$$M_{\Phi^\varepsilon}(t,x,v) = M_\Phi\left(t,x,\frac{x}{\varepsilon},v\right)$$

Double-scale expansion

$$f^\varepsilon = f_0(t,x,x/\varepsilon,v) + \varepsilon f_1(t,x,x/\varepsilon,v) + \varepsilon^2 f_2(t,x,x/\varepsilon,v) + \cdots$$

f_i : $Y = [0,1]^d-$periodic with respect to the fast variable.

$$y := x/\varepsilon \qquad \nabla_x f_i \longrightarrow \nabla_x f_i + \frac{1}{\varepsilon}\nabla_y f_i$$

Identification

$$\begin{aligned}
\varepsilon^{-2} &: \quad Lf_0 := v.\nabla_y f_0 - \nabla_y\Phi.\nabla_v f_0 - Q(f_0) = 0\\
\varepsilon^{-1} &: \quad Lf_1 = -v.\nabla_x f_0 + \nabla_x\Phi.\nabla_v f_0\\
\varepsilon^{0} &: \quad Lf_2 = -\partial_t f_0 - v.\nabla_x f_1 + \nabla_x\Phi.\nabla_v f_1
\end{aligned}$$

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References: Allaire, E, Ngutseng,...

Proposition (Cell problem)

- *L is maximal monotone on* $L^2_{M_\Phi} := L^2(dydv/M_\Phi)$
- $\mathcal{N}(L) = \mathbb{R}\ M_\Phi$
- $\mathcal{R}(L) = \left\{ g = g(y,v) \in L^2_{M_\Phi}\ /\ \iint_{Y\times\mathbb{R}^d} g\ dydv = 0 \right\}$

  $\forall g \in \mathcal{R}(L) \quad \exists\ f \in D(L)\ /\ L f = g$

  f *is unique iff* $\iint_{Y\times\mathbb{R}^d} f\ dydv = 0$

$$f_0 = \rho(t,x)M(v)\exp(\Phi_e(t,x) - \Phi(t,x,y)) := \rho(t,x)\ M_\Phi$$

$$f_1 = -(\nabla_x\rho + \rho\nabla_x\Phi_e)(t,x)\,.\,L^{-1}(vM_\Phi)(t,x,y)$$

Homogenized fluid system:

$$\partial_t\rho + \nabla_x.J(\rho,\Phi_e) = 0$$
$$J(\rho,\Phi_e) = -\mathbb{D}(t,x)[\nabla_x\rho + \rho\nabla_x\Phi_e]$$
$$\mathbb{D}(t,x) = \iint_{Y\times\mathbb{R}^d} v\otimes L^{-1}(vM_\Phi)\ dydv$$

Remark

- Entropy inequality $\Longrightarrow$ $\mathbb{D}$ is nonnegative
- L is not selfadjoint $\Longrightarrow$ $\mathbb{D}$ is not necessary symmetric

Uniform estimates

- $x \in \mathbb{R}^d \quad \Longrightarrow \quad$ No boundary effect
- Total conservation of mass :

$$\frac{d}{dt}\|f^\varepsilon\|_{L^1} = 0$$

- $L^\infty-$ maximum principle: If

$$f^\varepsilon(t=0) \lesssim M(v)$$

$$\Longrightarrow$$

$$f^\varepsilon \lesssim exp\left(-\frac{|v|^2}{2} + \int_0^T \|\partial_t \Phi\|_{L^\infty}(s)ds\right)$$

Relative Entropy

$$(v.\nabla_x - \nabla_x\Phi^\varepsilon.\nabla_v)\, e^{-|v|^2/2-\Phi^\varepsilon} = Q(e^{-|v|^2/2-\Phi^\varepsilon}) = 0$$

Multiplying by $f^\varepsilon/e^{-|v|^2/2-\Phi^\varepsilon}$, and integrate by parts :

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}}\left[\frac{f^{\varepsilon 2}}{e^{-|v|^2/2-\Phi^\varepsilon}}\right] + \frac{\sigma_1}{\varepsilon^2}\int_{\mathbb{R}^{2d}}\frac{(f^\varepsilon - \rho^\varepsilon M)^2}{e^{-|v|^2/2-\Phi^\varepsilon}} \leq 0$$

$$\Downarrow$$

$$\rho^\varepsilon \quad \& \quad j^\varepsilon := \frac{1}{\varepsilon}\int_{\mathbb{R}^d} f^\varepsilon v\, dv \qquad \in_b \qquad L^2_{loc}(\mathbb{R}^+,\ L^2_x)$$

$\Longrightarrow$ We can deal with double scale convergence

Denote :

- $\bar{f},\ \bar{\rho},\ \bar{j}\ldots$: double-scale limits of $f^\varepsilon,\ \rho^\varepsilon,\ j^\varepsilon,\ \ldots$
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$$\varepsilon^2\partial_t f^\varepsilon + \varepsilon\, v\,.\nabla_x f^\varepsilon - \varepsilon\nabla_x\Phi^\varepsilon\,.\,\nabla_v f^\varepsilon - Q(f^\varepsilon) = 0$$

$$\Longrightarrow \qquad f^\varepsilon \quad \overset{2-s}{\rightharpoonup} \quad \bar{f} := \rho(t,x)\, M_{\Phi_H} \qquad \in \mathcal{N}(L)$$

- Entropy-Dissipation $\Longrightarrow$

$$f^\varepsilon = \rho^\varepsilon M(v) + O(\varepsilon)_{L^2}$$

- Expression of $j(t,x)$?

$$g^\varepsilon = \frac{1}{\varepsilon}(f^\varepsilon - \rho^\varepsilon M) \quad \in_b L^2 \quad \overset{2-s}{\rightharpoonup} \quad \bar{g} \quad \in L^2_{t,x,y,v}$$

Green formula with $\dfrac{\varepsilon\,\psi(t,x,x/\varepsilon,v)}{M_{\Phi^\varepsilon}}$ / $\nabla_y\,.\,\int_{\mathbb{R}^d} v\psi dv = 0$

$$j^\varepsilon \rightharpoonup J(\rho, \Phi_e)$$

Outline

- Remark 1 : Compacity method ?
  1. NO strong convergence in L^p & velocity averaging lemma
  2. div-curl Lemma FAILS
- Remark 2 : L^1-contraction ?

$$f^\varepsilon(t=0) = \rho(t,x) M_{\Phi^\varepsilon}(t=0) : \qquad \text{well prepared}$$

$$r^\varepsilon = f^\varepsilon - (f_0 + \varepsilon\, f_1 + \varepsilon^2\, f_2)(t,x,x/\varepsilon,v)$$

$$\begin{cases} \partial_t r^\varepsilon + \frac{1}{\varepsilon}\left(v.\nabla_x r^\varepsilon - \nabla_x \Phi^\varepsilon.\nabla_v r^\varepsilon\right) - \frac{Q(r^\varepsilon)}{\varepsilon^2} = -\varepsilon \partial_t f_1 - \varepsilon^2 \partial_t\, f_2 + \dots \\ r^\varepsilon(t=0) = -\varepsilon\, f_1 - \varepsilon^2\, f_2 \end{cases}$$

NEED : Good decay on the derivatives of $\chi = L^{-1}(vM)$

$$v.\nabla_y \chi - \nabla_y \Phi\,.\,\nabla_v \chi - Q(\chi) = v\, M_\Phi$$

$\Longrightarrow$: Difficulty even for linear free case !!

Outline

1. Introduction
2. Diffusion approximation
3. Diffusion & Homogenization
   - Linear free space
   - Remarks about convergence
   - Poisson Coupling & inflow boundary data
4. Extensions & questions

Poisson Coupling & inflow boundary data ??

(ε: fixed)

- If we regularize the Boltzmann-Poisson system we can only establish uniform estimate :

$$f \in L\, logL, \qquad \nabla_x \Phi \in L^2$$

  This DOES NOT give a sense for $\nabla_x \Phi \,.\, \nabla_v f$ in $\mathcal{D}'$!

- L^∞ estimate on f need $\partial_t \Phi \in L^\infty$!!
- Entropy-dissipation gives only $j \in L^1$ which does not imply $\partial_t \Phi \in L^\infty$... "NO L^p estimate" !!
- $j \in L^1 \qquad \not\Longrightarrow \qquad \partial_t \nabla_x \Phi \in L^1_{loc}$!

$(\varepsilon \to 0)$

- Entropy production terms due to the boundary and no direct conservation of mass
- We can not use the Hilbert expansion since the Boltzmann equation has no 'weak PDE sense'.
- Assume that we can use the Hilbert expansion...we need to control $\varepsilon \partial_t \nabla_x \Phi^\varepsilon$ in L^1 !! but
  for fixed ε

$$-\Delta_x \partial_t \Phi^\varepsilon = \partial_t \rho^\varepsilon = \nabla . j^\varepsilon \in W_x^{1,1} \qquad \not\Longrightarrow \partial_t \nabla_x \Phi^\varepsilon \in L^1_{loc}$$

- We need strong convergence on f^ε and the self consistent potential ! we need to pass to the limit in some $\beta(f^\varepsilon)$...

How to deal ??

For only macroscopic variation

- with N. Ben Abdallah : 1D case

  We constructed a weak solution of the 1D Boltzmann-Poisson system. We used a Hybrid Hilbert expansion and the relative entropy to analyze the entropy production terms due to the boundary and obtain in a first step uniform estimates. Then we pass to the limit like Poupaud and exhibit a rate convergence in ε.

- with N. Masmoudi : multi-D case

  We prove existence of renormalized solution and prove strong convergence using a "limiting" L^1-velocity averaging but without rate of convergence

$$f^\varepsilon := f^\varepsilon(t,x,v), \qquad (x,v) \in \Omega = \omega \times \mathbb{R}^d$$

$$\begin{cases} \partial_t f^\varepsilon + \frac{1}{\varepsilon}(v.\nabla_x f^\varepsilon - \nabla_x \Phi_T^\varepsilon . \nabla_v f^\varepsilon) - \frac{Q(f^\varepsilon)}{\varepsilon^2} = 0 \\ \Phi_T^\varepsilon(t,x) = \Phi_P^\varepsilon(t,x) + \Phi_H(x, \frac{x}{\varepsilon}) \\ f^\varepsilon_{|_{\Gamma^-}} = \rho_b(t,x) M_{\Phi_H}(x, \frac{x}{\varepsilon}) \qquad (\Gamma^- : v.n(x) < 0), \\ f^\varepsilon_{|_{t=0}} = f_I^\varepsilon \end{cases}$$

$$-\Delta_x \Phi_P^\varepsilon = \rho^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon dv, \qquad (\Phi_P^\varepsilon)_{|_{\partial\omega}} = 0$$

$$\int_{\Omega} f_I^{\varepsilon}(1 + |v|^2 + |\log f_I^{\varepsilon}|) \leq C \qquad \& \qquad \Phi_H \in W^{2,\infty}(\overline{\omega} \times Y)$$

Theorem

$$f^{\varepsilon} e^{\Phi_H(x, x/\varepsilon)} \to \rho\, M(v)\, e^{\Phi_e(x)} \qquad in \quad L^1_t(L^1_{x,v})$$

$$\nabla_x \Phi_P^{\varepsilon} \to \nabla_x \Phi_P \qquad in \quad L^2_t(L^p_x), \qquad p < 2$$

(ρ, Φ_P) : solution of the homogenized Drift-Diffusion-Poisson system associated with $\Phi_P + \Phi_e$.

$$\widetilde{f^\varepsilon} = e^{\Phi_H(x,x/\varepsilon)}\ f^\varepsilon$$

Definition & Theorem ε : fixed $\quad \exists\,(f^\varepsilon, \Phi_P^\varepsilon) \quad /$

1. $\forall\,\beta \in \mathcal{C}^1(\mathbb{R}^+), \quad |\beta(t)| \le C(\sqrt{t}+1) \quad \beta' \in L^\infty$

$$\partial_t\beta(\widetilde{f^\varepsilon}) + \frac{1}{\varepsilon}\left[v.\nabla_x\beta(\widetilde{f^\varepsilon}) - \nabla_v.(\nabla_x\Phi_T^\varepsilon\ \beta(\widetilde{f^\varepsilon}))\right] = \beta'(\widetilde{f^\varepsilon})\frac{Q(\widetilde{f^\varepsilon})}{\varepsilon^2}$$

2. $\forall\,\lambda > 0, \quad \theta_{\varepsilon,\lambda} = \sqrt{f^\varepsilon + \lambda\, M e^{-\Phi_H(x,x/\varepsilon)}}$

$$\varepsilon\ \partial_t\theta_{\varepsilon,\lambda} + v.\nabla_x\theta_{\varepsilon,\lambda} - \nabla_v.\left(\nabla_x\Phi_T^\varepsilon\ \theta_{\varepsilon,\lambda}\right) - \frac{Q(\widetilde{f^\varepsilon})}{2\ \varepsilon\ \theta_{\varepsilon,\lambda}}$$

$$= \frac{\lambda}{2\,\theta_{\varepsilon,\lambda}} vM.\nabla_x\Phi_P^\varepsilon\ e^{-\Phi_H(x,x/\varepsilon)}$$

3. Continuity equation

$$\partial_t \rho^\varepsilon + \nabla_x . j^\varepsilon = 0$$

4. Entropy inequality

$$\left[\int_\Omega f^\varepsilon \left(\Phi_H^\varepsilon + \frac{|v|^2}{2} + \log\ f^\varepsilon \right) + \frac{1}{2} \|\nabla_x \Phi_P^\varepsilon\|_{L^2}^2 \right]_0^t - \frac{1}{\varepsilon^2} \int_0^t \int_\Omega Q(f^\varepsilon) \log\ \frac{f^\varepsilon}{M}$$

$$\leq -\frac{1}{\varepsilon} \int_0^t \int_{\Gamma^+ \cup \Gamma^-} f^\varepsilon \left(2\Phi_H^\varepsilon + \frac{|v|^2}{2} + \log f^\varepsilon \right) (v.n(x))$$

Proposition

- $$\mathcal{M}^\varepsilon(t) + \mathcal{K}^\varepsilon(t) + \|\nabla_x \Phi_P^\varepsilon(t)\|_{L^2} + \frac{\mathcal{R}^\varepsilon(t)}{\varepsilon^2} + \int_0^t \|j^\varepsilon(s)\|_{L^1}^2 \, ds$$
  $$\in_b \quad L^\infty_{loc}(\mathbb{R}^+)$$

$$\mathcal{M}^\varepsilon(t) = \|f^\varepsilon(t)\|_{L^1(\Omega)} \quad \mathcal{K}^\varepsilon(t) = \left\| \frac{|v|^2}{2} f^\varepsilon(t) \right\|_{L^1(\Omega)}$$

$$\mathcal{R}^\varepsilon(t) = \int_0^t \int_\Omega \left(\sqrt{f^\varepsilon} - \sqrt{\rho^\varepsilon M} \right)^2$$

$$2\alpha\mathcal{M}^\varepsilon(t) + \mathcal{K}^\varepsilon(t) + \|\nabla_x \Phi_P^\varepsilon(t)\|_{L^2}^2 + \frac{\mathcal{R}^\varepsilon(t)}{\varepsilon^2}$$

$$\leq C_T - \frac{1}{\varepsilon}\int_0^t \int_{\Gamma^+} |v.n(x)|[f^\varepsilon(v) - f^\varepsilon(-v)](1+\mathcal{E}_F)$$

$\mathcal{E}_F(t,x) = \log\left(\frac{\rho_b(t,x)}{(2\pi)^{d/2}}\right)$: macroscopic quasi-Fermi level

$$(1+\mathcal{E}_F(t,x))(\partial_t \rho^\varepsilon + \nabla_x . j^\varepsilon) = 0$$

$$\left|\frac{1}{\varepsilon}\int_0^t \int_{\Gamma^+} |v.n(x)|[f^\varepsilon(v) - f^\varepsilon(-v)](1+\mathcal{E}_F)\right| \leq C_T + \alpha\,\mathcal{M}^\varepsilon(t) + \frac{\mathcal{R}^\varepsilon(t)}{2\varepsilon^2} + \int_0^t \{\mathcal{M}^\varepsilon + \mathcal{K}^\varepsilon\}(s)ds$$

$\Rightarrow$ **GRONWALL.**

- $f^\varepsilon \in_b L\,LogL, \qquad \nabla_x \Phi_P^\varepsilon \in_b L_x^2$
- $f^\varepsilon, \quad f^\varepsilon_{|_{\Gamma+}}$ and $\rho^\varepsilon \qquad$ w. r. compact in L^1
- $\dfrac{Q(f^\varepsilon)}{\varepsilon} \qquad$ w. r. compact in $L^1 \qquad$ *(only L^1!)*
- $$\left.\begin{array}{l} \nabla \Phi_P^\varepsilon \in L^2, \qquad \Delta \Phi_P^\varepsilon \in L^1 \\ \partial_t \nabla \Phi_P^\varepsilon := \nabla \Delta^{-1} \nabla . j^\varepsilon \in L_t^2(W_{loc}^{-s,p}) \\ p > 1,\ s > d - d/p \end{array}\right\} \Longrightarrow \begin{array}{c} \nabla_x \Phi_P^\varepsilon \text{ cpt in } L_t^2(L_x^p) \\ \forall p < 2. \end{array}$$

Velocity averaging lemma

Lemma

$h^\varepsilon \in_b L^2, \quad h_0^\varepsilon \in_b L^1 \quad$ *and* $\quad h_1^\varepsilon \in [L^1_{t,x}(L^1_{loc,v})]^d$ *bounded*

$$\varepsilon\, \partial_t h^\varepsilon + v \,.\, \nabla_x h^\varepsilon = h_0^\varepsilon + \nabla_v \,.\, h_1^\varepsilon$$

$$\rho_\psi^\varepsilon := \int_{\mathbb{R}^d} h^\varepsilon(t,x,v)\,\psi(v)\,dv \qquad \text{for} \quad \psi \in \mathcal{D}(\mathbb{R}^d_v)$$

Then,

$$\lim_{z\to 0}\left(\sup_{\varepsilon<1} \|\rho_\psi^\varepsilon(t,x+z) - \rho_\psi^\varepsilon(t,x)\|_{L^1_{t,x}}\right) = 0$$

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References: Golse, Jabin, Lions, St. Raymond, Tadmor,...

Application:

$$\beta(s) = \frac{s}{1+s}, \qquad \beta_\delta(s) = \frac{s}{1+\delta\, s}$$

1. $0 \le \beta_\delta \le min(1, 1/\delta))$
2. $|\beta_\delta(s) - \beta_\delta(t)| \le min\left(|s-t|, |\sqrt{s} - \sqrt{t}|/\delta \right)$
3. $|\beta'_\delta(s) - 1| \le 2\delta s$
4. $|s\beta'_\delta(s) - t\beta'_\delta(t)| \le min\left[(1+\delta)|t-s|, C_\delta|\sqrt{s} - \sqrt{t}| \right]$

δ fixed $\qquad h^\varepsilon := \beta_\delta(\widetilde{f^\varepsilon}) \qquad \in_b L^2$

satisfies the lemma with

$$h_0^\varepsilon : \quad = \quad \left[\frac{Q(\widetilde{f^\varepsilon})}{\varepsilon} + v.(\nabla_x \Phi_H)_\varepsilon \widetilde{f^\varepsilon}\right] \beta'_\delta(f^\varepsilon)$$

$$+ \quad \frac{v.(\nabla_y \Phi_H)\varepsilon}{\varepsilon}\left[\widetilde{f^\varepsilon}\beta'_\delta(\widetilde{f^\varepsilon}) - \widetilde{\rho^\varepsilon} M \beta'(\widetilde{\rho^\varepsilon} M)\right] \quad \in_b L^1$$

$$h_1^\varepsilon \quad = \quad \left[\nabla_x \Phi_P^\varepsilon + \nabla_x \Phi_H^\varepsilon\right] \beta_\delta(\widetilde{f^\varepsilon}) + (\nabla_x \Phi_H^\varepsilon)_\varepsilon \left(\frac{\beta_\delta(\widetilde{f^\varepsilon}) - \beta_\delta(\widetilde{\rho^\varepsilon} M)}{\varepsilon}\right)$$

$$\in_b \; L^1_{t,x}(L^2_{loc,v})$$

Lemma $\Longrightarrow \quad \int_{\mathbb{R}^d} \beta_\delta(\widetilde{f^\varepsilon})\psi dv$ bounded in $L^1(0, T;\ cpt(L^1_x))$

$$sup_{\varepsilon<1}\|\beta_\delta(\widetilde{f^\varepsilon}) - \widetilde{f^\varepsilon}\|_{L^1(0,T;\,L^1)} \quad \to 0 \qquad \text{as } \delta \to 0$$

- $\delta \to 0 \quad \Rightarrow \quad \int_{\mathbb{R}^d} \widetilde{f^\varepsilon}\psi \, dv$ bounded in $L^1(0,T;\; cpt(L^1_x))$
- Kinetic Energy $\quad \Rightarrow \quad$ we can take $\psi = 1$

$$\widetilde{\rho^\varepsilon} : \; bounded \; in \; L^1(0,T;\; cpt(L^1_x))$$

We have $x-$compactness for the modified density $\widetilde{\rho^\varepsilon}$:

- $\partial_t \rho^\varepsilon = -\nabla_x . j^\varepsilon$: bounded in $L^2(0,T;\; W_x^{-1,1})$

$$\Longrightarrow \qquad \widetilde{\rho^\varepsilon} \text{ compact in } L^1((0,T)\times\omega)).$$

- Remark 1. We only use time-compactness of ρ^ε because

$$\partial_t \widetilde{\rho^\varepsilon} + \nabla_x . \widetilde{j^\varepsilon} = \left(\nabla_x \Phi_H + \frac{1}{\varepsilon} \nabla_y \Phi_H \right)_\varepsilon . \widetilde{j^\varepsilon}$$

  It is not clear how we can obtain directly the time-compactness for $\widetilde{\rho^\varepsilon}$.

- Remark 2. We can not use the Div-curl lemma since $\widetilde{\rho^\varepsilon}$ and $\widetilde{j^\varepsilon}$ are only L^1

Consequences:

$$\exists\ \rho \in L^1_{loc}(\mathbb{R}^+;\ L^1_x)\ \&\ \Phi_P \in L^\infty_{loc}(\mathbb{R}^+;\ H^1_0(\omega))$$

1. $\|\rho^\varepsilon - \rho e^{\Phi_e(x)-\Phi_H(x,x/\varepsilon)}\|_{L^1_{t,x}} \to 0$
2. ρ^ε converges weakly in $L^1_{t,x}$ to ρ
3. $\sqrt{f^\varepsilon}\quad \&\quad \sqrt{\rho^\varepsilon M}\ \xrightarrow{2-s\ strongly}\ \sqrt{\rho M_{\Phi_H}}$ in $L^2_{t,x,v}$
4. $-\Delta_x \Phi_P = \rho,$

$$\nabla_x \Phi_P^\varepsilon \to \nabla_x \Phi_P \qquad in\ L^2_{loc}(\mathbb{R}^+;\ L^p_x), \qquad \forall p < 2$$

USE equation on $\sqrt{f^\varepsilon + \lambda M e^{-\Phi_H(x,x/\varepsilon)}}$ with an oscillating test function $\psi(t,x,x/\varepsilon,v)$ to identify the weak limit of j^ε

$(\rho,\ \phi)$ satifies the homogenized system

$$\partial_t\rho + \nabla_x . J(\rho, \Phi_P + \Phi_e) = 0,$$

$$J(\rho,\phi) = \sqrt{\rho}\,[-\mathbb{D}(x)(2\nabla_x\sqrt{\rho} + \nabla_x(\Phi_P + \Phi_e)\,\sqrt{\rho})],$$

$$\mathbb{D} = -\int_{Y\times\mathbb{R}^d} (v \otimes\ L^{-1}(vM_{\Phi_H}))dy\,dv > 0$$

$$L = v\,.\,\nabla_y - \nabla\Phi_H\,.\,\nabla_v - Q$$

$$M_{\Phi_H} = M(v)e^{\Phi_e(x)-\Phi_H(x,y)}$$

Moreover,

$$\begin{aligned} -\Delta\phi &= \rho, \\ \rho_{|_{t=0}} &= \lim \int_{\mathbb{R}^d} f_0^\varepsilon dv \qquad (L^1 - w), \\ \rho_{|_{\partial\omega}} &= \rho_b \end{aligned}$$

Regularity of $J(\rho, \Phi_P + \Phi_e))$

$$\begin{cases} \nabla_x \sqrt{\rho} + \frac{1}{2} \nabla_x \Phi_P \sqrt{\rho} & \in L^2_{t,x}, \\ -\Delta_x \Phi_P = \rho & \in L^\infty_t \left(L^1_x\right) \\ \nabla_x \Phi_P & \in L^\infty_t \left(L^2_x\right) \\ \rho = \rho_b \quad \text{in } H^{1/2}(\partial\omega) \end{cases}$$

$$\Longrightarrow \qquad \rho, \quad \nabla_x \sqrt{\rho} \quad \text{and} \quad \nabla \Phi_P \sqrt{\rho} \quad \in L^2_{t,x}$$

and then

$$J(\rho, \Phi_P + \Phi_e) = -\mathbb{D}(x) \left[\nabla_x \rho + \rho \nabla_x (\Phi_P + \Phi_e)\right] \quad \in L^1$$

- Decay of the solution of the cell problem ???

$$v.\nabla_y \chi - \nabla_y \Phi_H . \nabla_v \chi - Q(\chi) = g$$

- Is it possible to prove a double-scale averaging Lemma ??
- Extention of this analysis to the Fermi-Dirac model ?
- General electric field ? : No explicit relative Maxwellian !
- If σ is not symmetric ?? "No explicit entropy-dissipation"

THANK YOU