A number of nonconservative hyperbolic models have been introduced in fluid dynamics to serve as (simplified) models of two-phase or two-layer flows. Our objective in the present paper is to address the fundamental question whether finite difference schemes for nonconservative systems converge toward correct weak solutions containing shock waves. The nonconservative hyperbolic systems under consideration have the general form
\[ u_t + A(u) u_x = 0, \quad u = u(t, x) \in \mathbb{R}^N, \]
where \( u \) is the vector-unknown and \( A = A(u) \) is a smooth, \( N \times N \) matrix-valued map \( A \) which admits real eigenvalues \( \lambda_1 < \ldots < \lambda_N \) and a basis of eigenvectors \( r_1, \ldots, r_N \). We are interested in solving initial value problems.

The solutions of nonlinear hyperbolic systems are generally discontinuous; due to the non-divergence form of the equations the notion of solutions in the sense of distributions can not be used, and weak solutions to nonconservative systems are defined in the sense introduced by Dal Maso, LeFloch, and Murat [4].

Generally speaking, weak solutions to nonconservative systems depend upon regularization mechanisms [6]; for instance, different approximation schemes may converge toward different solutions, and for this reason in developing the well-posedness theory, higher-order regularization effects such as viscosity, capillarity, relaxation terms, must be taken into account in the modeling. The Rankine-Hugoniot relations for shock waves are determined from the given regularization.

In the present work, we show that while, for certain simplified models, solutions are actually stable upon regularization, for general systems the general DLM theory is necessary. Still, as pointed out by Hou and LeFloch [5] —who focused attention on the same issues for nonconservative formulations of scalar hyperbolic equations— the effects of the regularization may be difficult to pinpoint in practice. In view of the fact that the models under study are derived from modeling approximation assumptions, this fully justifies the use of a numerical strategy based on a direct discretization of the nonconservative hyperbolic models, as developed by Berthon and
Coquel [1], Parés [8], Muñoz-Ruiz and Parés [7], Castro, Gallardo, and Parés [3], and Berthon, Coquel, and LeFloch [2].

Here, we consider the family of formally path-consistent schemes associated with a given family of paths, which was originally introduced by Pares [8] and called therein “path-conservative” schemes. The extensions to non-conservative systems of Godunov, Roe, and Lax-Friedrichs schemes are particular cases of this family. The convergence of the schemes of this family is studied. We show that, if the approximations converge uniformly in the sense of graphs to some function as the mesh is refined, then this function is a weak solution of the system. Nevertheless, we must take into account that the convergence in the sense of graphs is rather strong and usually fails in practice for general systems (although it holds for Glimm and front tranking schemes). When the approximations converge only almost everywhere, the limit is a solution of a system containing an error source-term which is a measure supported on the shocks. The equivalent equations associated with the schemes will be used to explain the presence of this error source-term.

The core part of this work is an extensive body of numerical experiments which we have performed and allow us to compare exact and numerical Rankine-Hugoniot curves for some nonconservative systems arising in applications to fluid dynamics. Our conclusions justify to search for robust and efficient high-order schemes for the approximation of nonconservative systems.

References


