Traffic Flow on Networks: Conservation Laws Models

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The aim of the talk is to present recent developments for macroscopic traffic flow models on networks; more precisely we focus on the approach based on conservation laws. We will address modelling, analysis, numerics and application issues.

The underlaying equations for the models we consider are hyperbolic systems of conservation laws in one dimension:

$$u_t + f(u)_x = 0,$$

where $x \in \mathbb{R}$, $u \in \mathbb{R}^n$ and $Df(u)$ is assumed to have real distinct eigenvalues. The main mathematical novelty is to describe the dynamics on a network, represented by a directed topological graph, instead of a real line. The more advanced results are available for the scalar case, i.e. $n = 1$.

Modelling. Let us focus on vehicular traffic first. Various fluidodynamic models were developed in the literature: they treat traffic from a macroscopic point of view considering the evolution of macroscopic variables, such as density and average velocity of cars. The Lighthill-Whitham-Richards (LWR) model introduced in the 50s is based on the solely conservation of the number of cars, giving rise to a scalar conservation law. Then several models of more equations were introduced, such as the Aw-Rascle model [1], the Colombo phase transition model [9] and the Helbing model [18].
More recently, a growing attention was devoted to extend the same models to networks, see for instance [3, 8, 14, 19, 21, 22].
The interest was also motivated by other applications: data networks [13], supply chain [12, 16], air traffic management [4], gas pipelines [2, 10], irrigation channels [17] etc.

The main interest is in the Cauchy problem for a complex network. The dynamics happens to be undertermined at nodes. Indeed the only conservation of u through a node is not sufficient to describe a unique solution. Think for example to a junction with one incoming road filled with cars and two empty outgoing roads: there are many feasible solutions depending on the direction taken by drivers at the junction.
To construct solutions to Cauchy problems, we consider wave-front tracking approximations [5], thus it is natural to define Riemann problem at a node a Cauchy problem with constant initial data on each arc entering or existing the node. Then one relies on the notion of Riemann solver, i.e. a map providing solutions to Riemann problems as function of the initial data.
The Riemann solver proposed in [8] is based on two rules:

(A) there are some fixed coefficients, that represent drivers preferences. These coefficients indicate the distribution of the traffic from the incoming to the outgoing roads. We can collected them in a traffic distribution matrix:

$$A = \{\alpha_{ji}\}_{j=n+1,\dots,n+m,\ i=1,\dots,n} \in \mathbb{R}^{m\times n},$$

such that

$$0 < \alpha_{ji} < 1, \quad \sum_{j=n+1}^{n+m} \alpha_{ji} = 1,$$

where α_{ji} is the percentage of drivers who, arriving from the i-th incoming road, take the j-th outgoing road;

(B) respecting (A), drivers behave so as to maximize the flux through the junction.

The first model of [22] was based only on rule (B). There are some observations to do on this definition: rule (A) implies the conservation of u through the node but is not sufficient to determine a unique solution; given rule (A), rule (B) is an "entropy like" condition; (A) and (B) together correspond to a Linear Programming problem written for the incoming fluxes. To determine a unique solution one has to further use priority parameters for nodes having more entering than exiting arcs.

A second solver was proposed in [13] for data networks: it is essentially equivalent to invert the order of rules (A) and (B), i.e. the through flux is maximized and then priority and traffic distribution rules are used.

Let us mention some interesting mathematical problems showing up in other applications. The model for supply chains proposed in [16] consists of a scalar conservation law on a network with a queue in front of each arc: this gives rise to a mixed ode-pde model. The approach of [12], also for supply chains, gives rise to a system on a mixed discrete-continuous space.
Finally, another mixed model is that describing a car trajectory on a loaded network: it consists of a conservation law on a network for the load evolution and an ode for the car trajectory, see [11]

Analysis. As usual, the construction of wave-front tracking approximations relies on three estimates: the number of waves, the number of wave interactions and total variation of the solution. While these estimates are straightforward on a real line (see [5]), they becomes difficult to prove on complex networks (see [15]). In particular one has to rely on estimates on the total variation of the flux of the solution.
We provide a general strategy to overcome the technical problems: three key properties of Riemann solvers are defined, which guarantee the needed bounds and thus existence of solutions to Cauchy problems. One interesting technical point is the estimate of the total variation (in space) of the solution flux via bounds on the positive variation (in time) of incoming fluxes at junctions. The three key properties are shared by various Riemann solvers proposed in the literature, in particular by those proposed in [8, 13].

The continuous dependence of solutions with respect to initial data is an open problem in the general case. For instance the Lipschitz continuous dependence with respect to initial data may fail; see [8]. However, it was proved for the solver of [13], by viewing L^1 as a Finsler manifold and considering "generalized tangent vectors". This was method used in [6] for systems of conservation laws.

Regarding the ode-pde model of [16], estimates are obtained for the flux variation of the solution and the variation (in time) of queue buffer occupancy. Also in this case a generalized tangent vector method works to prove uniqueness and Lipschitz continuous dependence of solutions on initial data, see [20].

Numerics. Thanks to the formulation of the dynamics via Riemann solver at nodes, it is easy to develop numerical methods on networks, based on schemes using a Riemann solver. For instance the Godunov scheme is easily generalized. The model using rules (A) and (B) gives directly the values needed to compute the numerical fluxes, see [7].

Some ad-hoc methods were also developed for having fast algorithms on networks, based on theoretical results on the structure of solutions starting from empty networks. The latter is a reasonable assumption for vehicular traffic and permits to deal with complex networks of thousand arcs.

Then numerical schemes can be also developed for the mixed models cited above.

Applications. We will illustrate various applications, mainly for vehicular traffic. In collaboration with the Rome traffic society (A.T.A.C. spa), we developed various tests on the urban network of Rome using data from radars and coils. Such models are of help both for info-mobility (providing reconstructions of the networkd load) as well as for traffic managing (giving forecasts by simulations and control strategies for traffic regulation).

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