On the Dependence of Euler Equations on Physical Parameters

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OUTLINE:

1. Introduction/ Motivation
2. Our approach
3. Applications to Euler Equations

Hyperbolic Systems of Conservation Laws in one-space dimension:

$$\begin{cases} \partial_t U + \partial_x F(U) = 0 & x \in \mathbb{R} \\ U(0,x) = U_0, \end{cases} \tag{1}$$

where $U = U(t,x) \in \mathbb{R}^n$ is the conserved quantity and

$$F : \mathbb{R}^n \to \mathbb{R}^n \quad \text{smooth flux.}$$

Admissible/Entropy weak solution: $U(t,x)$ in BV.

$$\partial_t \eta(U) + \partial_x q(U) \leq 0 \qquad \text{in } \mathcal{D}'$$

Examples: **Isothermal Euler equations**

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho\, u) = 0 \\ &\partial_t(\rho\, u) + \partial_x(\rho\, u^2 + \kappa^2 \rho) = 0 \end{aligned} \tag{2}$$

where ρ is the density and u is the velocity of the fluid.

Isentropic Euler equations

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho\, u) = 0 \\ &\partial_t(\rho\, u) + \partial_x(\rho\, u^2 + \kappa^2 \rho^\gamma) = 0 \qquad \gamma > 1 \end{aligned} \tag{3}$$

where $\gamma > 1$ is the adiabatic exponent.

Relativistic Euler equations:

$$\begin{aligned} &\partial_t \left(\frac{(p + \rho\, c^2)}{c^2} \frac{u^2}{c^2 - u^2} + \rho \right) + \partial_x \left((p + \rho\, c^2) \frac{u}{c^2 - u^2} \right) = 0 \\ &\partial_t \left((p + \rho\, c^2) \frac{u}{c^2 - u^2} \right) + \partial_x \left((p + \rho\, c^2) \frac{u^2}{c^2 - u^2} + p \right) = 0, \end{aligned} \tag{4}$$

where $c < \infty$ is the speed of light.

- **Question:** As $\gamma \to 1$ and $c \to \infty$, can we pass

  from the **isentropic** to the **isothermal**

  and from the **relativistic** to the **classical**?

- **Question:** As $\gamma \to 1$ and $c \to \infty$, can we pass

  from the **isentropic** to the **isothermal**

  and from the **relativistic** to the **classical**?

- **In general, the Question is:**

  How do the admissible weak solutions depend on the physical parameters?

Systems of Conservation Laws in one-space dimension:

$$\begin{cases} \partial_t W^\mu(U) + \partial_x F^\mu(U) = 0 & x \in \mathbb{R} \\ U(0,x) = U_0, \end{cases} \tag{5}$$

where W^μ, $F^\mu : \mathbb{R}^n \to \mathbb{R}^n$ are smooth functions that depend on a **parameter vector** $\mu = (\mu_1, \dots, \mu_k)$, $\mu_i \in [0, \mu_0]$, for $i = 1, \dots, k$. and $W^0(U) = U$.

Formulate an effective **approach** to establish L^1 estimates of the type:

$$\|U^\mu(t) - U(t)\|_{L^1} \leq C\, TV\{U_0\} \cdot t \cdot \|\mu\| \tag{6}$$

- U^μ is the entropy weak solution to (5) for $\mu \neq 0$ constructed by the front tracking method.
- $U(t) := \mathcal{S}_t U_0$, $\mathcal{S}$ is the Lipschitz Standard Riemann Semigroup associated with (5) for $\mu = 0$.
- $\|\mu\|$ is the magnitute of the parameter vector μ.

Error estimate

Let $\mathcal{S}$ be a Lipschitz continuous semigroup:

$$\mathcal{S} : \mathcal{D} \times [0, \infty) \mapsto \mathcal{D},$$

$$\|\mathcal{S}_t w(0) - w(t)\|_{L^1} \leq L \int_0^t \liminf_{h \to 0+} \frac{\|\mathcal{S}_h w(\tau) - w(\tau + h)\|_{L^1}}{h} d\tau, \quad (7)$$

where L is the Lipschitz constant of the semigroup and $w(\tau) \in \mathcal{D}$.
The above inequality appears extensively in the theory of front tracking method: e.g.
(i) the entropy weak solution by front tracking coincides with the trajectory of the semigroup $\mathcal{S}$ if the semigroup exists,
(ii) uniqueness within the class of viscosity solutions, etc....

References: Bressan et al.

Front-Tracking Method

For $\delta > 0$, let $U^{\delta,\mu}$ be the δ-approximate solution to

$$\begin{cases} \partial_t W^\mu(U) + \partial_x F^\mu(U) = 0 & \text{for} \quad \mu \neq 0 \\ U(0,x) = U_0, \end{cases}$$

(i) U_0^δ piecewise constant, $\|U_0^\delta - U_0\|_{L^1} < \delta$.
(ii) $U^{\delta,\mu}$ are globally defined piecewice constant functions with finite number of discontinuities.
(iii) The discontinuities are of three types:

- shock fronts,
- rarefaction fronts with strength less than δ,
- non-physical fronts with total strength $\sum |\alpha| < \delta$.

(iv) $U^{\delta,\mu} \to U^\mu$ in L^1_{loc} as $\delta \to 0+$.
References: Bressan, Dafermos, DiPerna, Holden–Risebro.

Approach

Apply the error estimate on $w = U^{\delta,\mu}$:

$$\|\mathcal{S}_t U_0^\delta - U^{\delta,\mu}(t)\|_{L^1} \leq L \int_0^t \liminf_{h\to 0+} \frac{\|\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)\|_{L^1}}{h} d\tau,$$

The aim is to estimate

$$\|\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)\|_{L^1} \tag{8}$$

which is equivalent to solving the Riemann problem of (5) when $\mu = 0$ for $\tau \leq t \leq \tau + h$ with data

$$(U_L, U_R) = \begin{cases} U^{\delta,\mu}(\tau, x) & x < \bar{x} \\ U^{\delta,\mu}(\tau, x) & x > \bar{x} \end{cases} \tag{9}$$

over each front of $U^{\delta,\mu}$ at time τ, i.e. find $\mathcal{S}_h(U_L, U_R)$. Then compare it with the same front of $U^{\delta,\mu}(\tau+h)$. We solve the Riemann problem at all non-interaction times of $U^{\delta,\mu}$.

If we can show:

$$\int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)|\, dx = \mathcal{O}(1) h(\|\mu\|\, |U_L - U_R| + \delta), \tag{10}$$

then summing over all fronts of $U^{\delta,\mu}(\tau)$,

$$\|\mathcal{S}_t U^{\delta,\mu}(0) - U^{\delta,\mu}(t)\|_{L^1} \leq \tag{11}$$

$$\leq L \int_0^t \sum_{\text{fronts } x=\bar{x}(\tau)} \frac{1}{h} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)|\, dx\, d\tau \tag{12}$$

If we can show:

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then summing over all fronts of $U^{\delta,\mu}(\tau)$,

$$\begin{aligned} \|\mathcal{S}_t U^{\delta,\mu}(0) - U^{\delta,\mu}(t)\|_{L^1} &\leq \\ &\leq L \int_0^t \sum_{\text{fronts } x=\bar{x}(\tau)} \frac{1}{h} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)|\, dx\, d\tau \\ &= \mathcal{O}(1) \left(\|\mu\| \int_0^t TV U^{\delta,\mu}(\tau)\, d\tau + \delta\, t \right) \end{aligned} \tag{11}$$

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$$\begin{aligned}
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&\qquad \leq L \int_0^t \sum_{\text{fronts } x=\bar{x}(\tau)} \frac{1}{h} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)|\, dx\, d\tau \\
&\qquad = \mathcal{O}(1) \left(\|\mu\| \int_0^t TV U^{\delta,\mu}(\tau)\, d\tau + \delta\, t \right) \\
&\qquad = \mathcal{O}(1)(\|\mu\|\, TV\{U_0\} + \delta) \cdot t
\end{aligned} \tag{11}$$

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then summing over all fronts of $U^{\delta,\mu}(\tau)$,

$$\begin{aligned} \|\mathcal{S}_t U^{\delta,\mu}(0) - U^{\delta,\mu}(t)\|_{L^1} \leq \\ &\leq L \int_0^t \sum_{\text{fronts } x=\bar{x}(\tau)} \frac{1}{h} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)|\, dx\, d\tau \\ &= \mathcal{O}(1) \left(\|\mu\| \int_0^t TV U^{\delta,\mu}(\tau)\, d\tau + \delta\, t \right) \\ &= \mathcal{O}(1)(\|\mu\|\, TV\{U_0\} + \delta) \cdot t \qquad (11) \end{aligned}$$

As $\delta \to 0+$, we obtain

$$\|U(t) - U^{\mu}(t)\|_{L^1} = \mathcal{O}(1)\, TV\{U_0\} \cdot t \|\mu\| \tag{12}$$

where $U := \mathcal{S}_t U_0$ is the entropy weak solution to (5) for $\mu = 0$.

Remarks

Note that $U := \mathcal{S}_t U_0$ is unique within the class of viscosity solutions. (Bressan et al). Thus, as $\mu \to 0$

$$U^\mu \to \mathcal{S}_t U_0 \qquad \text{in} \quad L^1.$$

• Temple: existence using that the nonlinear functional in Glimm's scheme depends on the properties of the equations at $\mu = 0$.

• Bianchini and Colombo: consider $\mathcal{S}^F$, $\mathcal{S}^G$ and show $\mathcal{S}^F$ is Lipschitz w.r.t. the C^0-norm of DF.

Isothermal Euler equations:

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho\, u) = 0 \\ &\partial_t(\rho\, u) + \partial_x(\rho\, u^2 + \kappa^2 \rho) = 0 \end{aligned} \tag{13}$$

where ρ is the density and u is the velocity of the fluid.

• Nishida [1968]: Existence of entropy solution for large initial data via the Glimm's scheme.

• Colombo-Risebro [1998]: Construction of the Standard Riemann Semigroup for large initial data. Existence, stability and uniqueness within viscosity solutions.

$\star$ Let $\mathcal{S}$ be the Lipschitz Standard Riemann Semigroup generated by Isothermal Euler Equations (13).

1. Isentropic Euler Equations:

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho\, u) = 0 \\ &\partial_t(\rho\, u) + \partial_x(\rho\, u^2 + p(\rho)) = 0 \end{aligned} \tag{14}$$

of a perfect polytropic fluid

$p(\rho) = \kappa^2 \rho^\gamma$, where $\gamma > 1$ is the adiabatic exponent.

Existence results: when $(\gamma - 1)\, TV\{U_0\} < N$
(i) Nishida-Smoller by Glimm's scheme, [1973]
(ii) Asakura by the front tracking method [2005].

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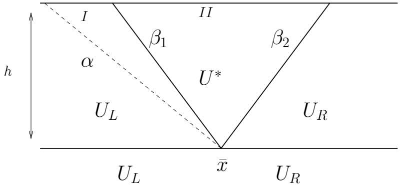
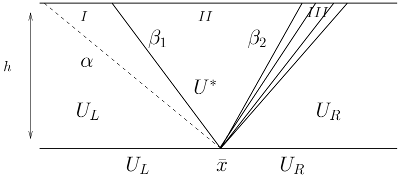
Theorem (G.-Q. Chen, Christoforou, Y. Zhang)

Suppose that $0 < \underline{\rho} \le \rho_0(x) \le \bar{\rho} < \infty$ and $(\gamma - 1)\, TV\{U_0\} < N$. Let $\mu = \frac{\gamma-1}{2}$ and U^μ be the entropy weak solution to (14) obtained by the front tracking method, then for every $t > 0$,

$$\|\mathcal{S}_t U_0 - U^\mu(t)\|_{L^1} = \mathcal{O}(1)\, TV\{U_0\} \cdot t\, (\gamma - 1) \tag{15}$$

Case 1: Shock Front of strength $\alpha = \dfrac{\rho_R}{\rho_L}$ and $\mu = \frac{1}{2}(\gamma - 1)$





$$\beta_1 = \alpha + \mathcal{O}(1)|\alpha - 1|\,(\gamma - 1) \qquad \beta_2 = 1 + \mathcal{O}(1)|\alpha - 1|\,(\gamma - 1).$$

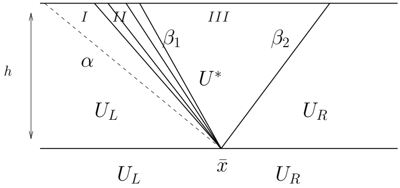
I: $|U_L - U_R| = |U_L - U_R|$ — length of I $= \mathcal{O}(1)\,h\,\mu$

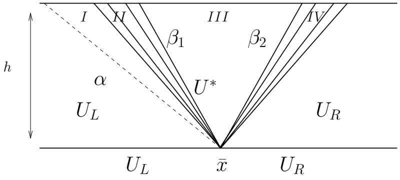
II: $|U^* - U_R| = \mathcal{O}(1)|U_L - U_R|\,\mu$ — length of II $= \mathcal{O}(1)\,h$

III: $|U(\xi) - U_R| = \mathcal{O}(1)|U_L - U_R|\,\mu$ — length of III $= \mathcal{O}(1)\,h\,|U_L - U_R|\mu$

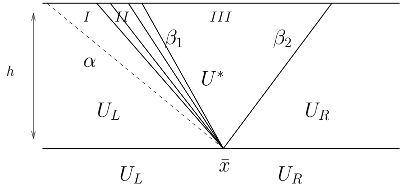
$$\frac{1}{h}\int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)|\,dx = \mathcal{O}(1)\,\mu\,|U^{\delta,\mu}(\tau,\bar{x}-) - U^{\delta,\mu}(\tau,\bar{x}+)|$$

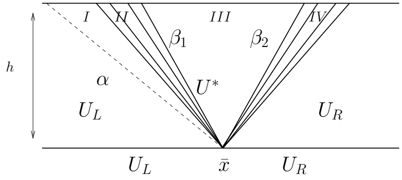
Case 2: Rarefaction Front



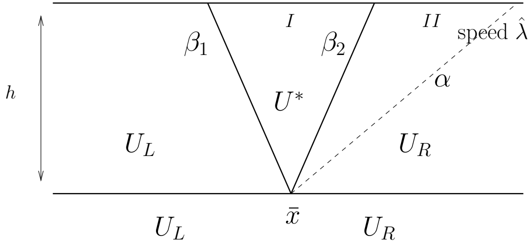


Case 2: Rarefaction Front





Case 3: Non-Physical Front



2. Relativistic Euler Equations for conservation of momentum:

$$\partial_t \left(\frac{(p+\rho\, c^2)}{c^2} \frac{u^2}{c^2-u^2} + \rho \right) + \partial_x \left((p+\rho\, c^2) \frac{u}{c^2-u^2} \right) = 0$$
$$\partial_t \left((p+\rho\, c^2) \frac{u}{c^2-u^2} \right) + \partial_x \left((p+\rho\, c^2) \frac{u^2}{c^2-u^2} + p \right) = 0, \tag{16}$$

of a perfect polytropic fluid

$$p(\rho) = \kappa^2 \rho^\gamma,$$

where $\gamma \geq 1$ is the adiabatic exponent and c is the speed of light.

Parameter vector: $\mu = (\gamma - 1, \frac{1}{c^2})$.

Existence results: by Glimm's scheme

(i) Smoller-Temple ($\gamma = 1$), for $TV\{U_0\}$ large, [1993]

(ii) J. Chen when $(\gamma - 1)\, TV\{U_0\} < N$, [1995]

Theorem (G.-Q. Chen, Christoforou, Y. Zhang)

Suppose that $0 < \underline{\rho} \leq \rho_0(x) \leq \bar{\rho} < \infty$ *and* $(\gamma - 1)\, TV\{U_0\} < N$. *Let* U^μ *be the entropy weak solution to* *Relativistic Euler Equations for conservation of momentum* (16) *for* $\gamma > 1$ *and* $c \geq c_0$ *constructed by the front tracking method, then for every* $t > 0$,

$$\|\mathcal{S}_t U_0 - U^\mu(t)\|_{L^1} = \mathcal{O}(1)\, TV\{U_0\} \cdot t \left((\gamma - 1) + \frac{1}{c^2} \right) \qquad (17)$$

for $\mu = (\gamma - 1, \frac{1}{c^2})$.

Proof.

1. Establish the front tracking method for $\gamma > 1$ and $c_0 < c < \infty$.
2. Due to the Lorenz invariance, employ the techniques of the previous theorem and solve the Riemann problem for each one of the three cases. □

3. Isentropic Relativistic Euler Equations of conservation laws of baryon number and momentum in special relativity:

$$\begin{aligned} &\partial_t \left(\frac{n}{\sqrt{1-u^2/c^2}} \right) + \partial_x \left(\frac{nu}{\sqrt{1-u^2/c^2}} \right) = 0 \\ &\partial_t \left(\frac{(\rho + p/c^2)u}{1-u^2/c^2} \right) + \partial_x \left(\frac{(\rho + p/c^2)u^2}{1-u^2/c^2} + p \right) = 0 \end{aligned} \tag{18}$$

For isentropic fluids, the proper number density of baryons n is

$$n = n(\rho) = n_0\, exp\Big(\int_1^\rho \frac{ds}{s + \frac{p(s)}{c^2}}\Big). \tag{19}$$

Theorem (G.-Q. Chen, Christoforou, Y. Zhang)

$$\|\mathcal{S}_t U_0 - U^\mu(t)\|_{L^1} = \mathcal{O}(1)\; TV\{U_0\} \cdot t \left((\gamma - 1) + \frac{1}{c^2} \right). \tag{20}$$

4. Non-Isentropic Euler equations

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho\, u) = 0 \\ &\partial_t(\rho\, u) + \partial_x(\rho\, u^2 + p) = 0 \\ &\partial_t(\rho(\tfrac{1}{2}u^2 + e)) + \partial_x(\rho\, u(\tfrac{1}{2}u^2 + e) + p\, u) = 0. \end{aligned} \tag{21}$$

ρ – density, u – velocity, p – pressure and e – internal energy.
T – temperature, S – entropy and $v = 1/\rho$ – specific volume.
Law of thermodynamics:

$$T\, dS = de + p\, dv.$$

Entropy condition:

$$(\rho\, S)_t + (\rho u S)_x \geq 0.$$

For a polytropic gas, i.e. $\varepsilon = \gamma - 1 > 0$, then

$$p = \kappa^2 \, e^{S/c_v} \rho^\gamma$$

and

$$e(\rho, S, \varepsilon) = \frac{1}{\varepsilon}\left(\left(\frac{e^{-S/R}}{\rho}\right)^{-\varepsilon} - 1\right)$$

Existence results: when $(\gamma - 1)\, TV\{U_0\} < N$
(i) T.-P. Liu [1977] and Temple [1981] by Glimm's scheme.
G.-Q. Chen–Wagner [2003]
(ii) Asakura by the front tracking method, preprint [2006]
Thus, as $\varepsilon \to 0$,

$$e_0(\rho, S) = \lim_{\varepsilon \to 0} e(\rho, S, \varepsilon) = \ln \rho + \frac{S}{R}$$

As $\varepsilon \to 0+$, non-isentropic Euler equations

$$\begin{aligned} &\partial_t\rho + \partial_x(\rho\, u) = 0 \\ &\partial_t(\rho\, u) + \partial_x(\rho\, u^2 + p) = 0 \\ &\partial_t(\rho(\tfrac{1}{2}u^2 + e)) + \partial_x(\rho\, u(\tfrac{1}{2}u^2 + e) + p\, u) = 0. \end{aligned} \tag{22}$$

$$\downarrow$$

$$\begin{aligned} &\partial_t\rho + \partial_x(\rho\, u) = 0 \\ &\partial_t(\rho\, u) + \partial_x(\rho\, u^2 + \kappa^2\,\rho) = 0 \\ &\partial_t(\rho(\tfrac{1}{2}u^2 + e_0)) + \partial_x(\rho\, u(\tfrac{1}{2}u^2 + e_0) + \kappa^2\,\rho\, u) = 0, \end{aligned} \tag{23}$$

with

$$(\rho\, S)_t + (\rho u S)_x \geq 0. \tag{24}$$

Non-Isentropic Euler equations

$$\begin{aligned}
&\partial_t\rho + \partial_x(\rho\, u) = 0\\
&\partial_t(\rho\, u) + \partial_x(\rho\, u^2 + p) = 0\\
&\partial_t(\rho(\tfrac{1}{2}u^2 + e)) + \partial_x(\rho\, u(\tfrac{1}{2}u^2 + e) + p\, u) = 0.
\end{aligned} \tag{25}$$

$$\downarrow$$

Isothermal Euler equations

$$\begin{aligned}
&\partial_t\rho + \partial_x(\rho\, u) = 0\\
&\partial_t(\rho\, u) + \partial_x(\rho\, u^2 + \kappa^2\, \rho) = 0,
\end{aligned} \tag{26}$$

with

$$(\rho(\frac{1}{2}u^2 + \ln\rho))_t + (\rho\, u(\frac{1}{2}u^2 + \ln\rho) + \kappa^2\rho\, u)_x \leq 0 \tag{27}$$

Theorem (G.-Q. Chen, Christoforou, Y. Zhang)

Suppose that $0 < \underline{\rho} \leq \rho_0(x) \leq \bar{\rho} < \infty$ *and* $(\gamma - 1)\, TV\{U_0\} < N$. *Let* $U^\varepsilon = (\rho_\varepsilon, \rho_\varepsilon\, u_\varepsilon, \rho_\varepsilon(\frac{1}{2}u_\varepsilon^2 + e_\varepsilon))^\top$ *be the entropy weak solution to* *Non-Isentropic Euler Equations* (21) *for* $\varepsilon > 0$ *constructed by the front-tracking method. Then, for every* $t > 0$,

$$\|\rho(t) - \rho_\varepsilon(t)\|_{L^1} + \|u(t) - u_\varepsilon(t)\|_{L^1} = \mathcal{O}(1)\, TV\{U_0\}\, t\, (\gamma - 1), \quad (28)$$

where $(\rho(t), u(t))$ *is the solution to* *Isothermal Euler Equations* (26) *generated by* $\mathcal{S}$.
As $\varepsilon \to 0$, *for every* $t > 0$,

$$\rho_\varepsilon(t) \to \rho(t), \qquad u_\varepsilon(t) \to u(t) \quad \text{in } L^1_{loc}.$$

Remarks:

- **For** $\varepsilon = 0$: The Standard Riemann Semigroup associated with the 3×3 limiting system: Colombo–Risebro for the Isothermal Euler equations.

$$
\begin{aligned}
&\partial_t \rho + \partial_x(\rho\, u) = 0\\
&\partial_t(\rho\, u) + \partial_x(\rho\, u^2 + \kappa^2\, \rho) = 0\\
&\partial_t(\rho(\tfrac{1}{2}u^2 + e_0)) + \partial_x(\rho\, u(\tfrac{1}{2}u^2 + e_0) + \kappa^2\, \rho\, u) = 0,
\end{aligned}
$$

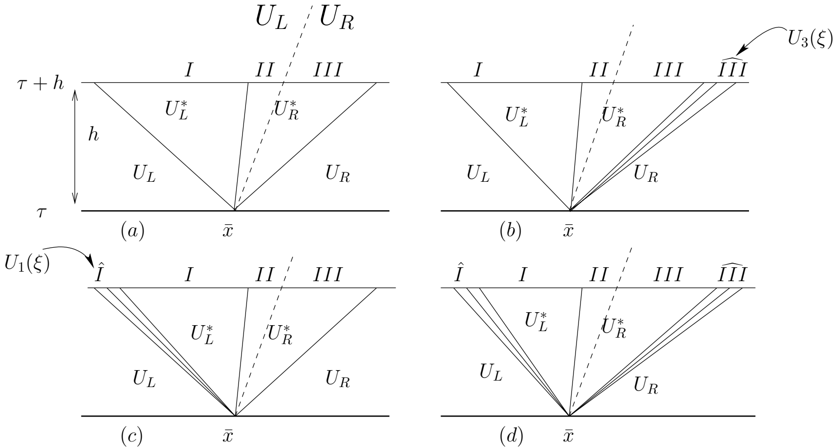
Remarks:

- **For** $\varepsilon = 0$**:** The Standard Riemann Semigroup associated with the 3×3 limiting system: Colombo–Risebro for the Isothermal Euler equations.
- **For** $\varepsilon > 0$**:** The front tracking method: Use Asakura's result.

Remarks:

- **For** $\varepsilon = 0$**:** The Standard Riemann Semigroup associated with the 3×3 limiting system: Colombo–Risebro for the Isothermal Euler equations.
- **For** $\varepsilon > 0$**:** The front tracking method: Use Asakura's result.
- **Cases:** Shock fronts, Rarefaction fronts, Non-Physical fronts and **also** 2- contact discontinuity fronts!!!

Case: Contact Discontinuity



5*. Compressible Euler Eqs with Mach Number

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0 \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + \tfrac{1}{\mathrm{M}^2} p) = 0 \qquad \mathrm{M} > 0 \\ &\partial_t(\rho E) + \partial_x((\rho E + p)u) = 0 \end{aligned} \tag{29}$$

with energy

$$E = \frac{p}{(\gamma - 1)\rho} + \mathrm{M}^2 \frac{u^2}{2}$$

and initial data in $BV(\mathbb{R})$:

$$\begin{cases} \rho|_{t=0} = \rho_0 + \mathrm{M}^2 \rho_2^{(0)}(x), & \rho_0 > 0 \text{ constant} \\ p|_{t=0} = p_0 + \mathrm{M}^2 p_2^{(0)}(x), & p_0 > 0 \text{ constant} \\ u|_{t=0} = \mathrm{M} u_1^{(0)}(x) \end{cases} \tag{30}$$

Denote the solution to (29)–(30) by $(\rho^{\mathrm{M}}, p^{\mathrm{M}}, u^{\mathrm{M}})$.

References: Majda, Klainerman-Majda, Metivier, Schochet.

$(\rho^{\mathrm{M}}, p^{\mathrm{M}}, u^{\mathrm{M}})$ has an asymptotic expansion:

$$\begin{aligned} \rho^{\mathrm{M}}(t,x) &= \rho_0 + {\scriptstyle\mathrm{M}}^2 \rho_2^{\mathrm{M}}(t,x) + O(1){\scriptstyle\mathrm{M}}^3, \\ p^{\mathrm{M}}(t,x) &= p_0 + {\scriptstyle\mathrm{M}}^2 p_2^{\mathrm{M}}(t,x) + O(1){\scriptstyle\mathrm{M}}^3, \\ u^{\mathrm{M}}(t,x) &= {\scriptstyle\mathrm{M}}\, u_1^{\mathrm{M}}(t,x) + O(1){\scriptstyle\mathrm{M}}^2, \end{aligned} \tag{31}$$

where $(\rho_2^{\mathrm{M}}, p_2^{\mathrm{M}}, u_1^{\mathrm{M}})$ satisfy the linear acoustic system:

$$\begin{aligned} \partial_t \rho_2 + \frac{\rho_0}{\mathrm{M}} \partial_x u_1 &= 0 \\ \partial_t p_2 + \frac{\gamma p_0}{\mathrm{M}} \partial_x u_1 &= 0 \\ \partial_t u_1 + \frac{1}{\mathrm{M}\rho_0} \partial_x p_2 &= 0 \end{aligned} \tag{32}$$

with the initial data

$$\rho_2\Big|_{t=0} = \rho_2^{(0)}(x) \qquad p_2\Big|_{t=0} = p_2^{(0)}(x) \qquad u_1\Big|_{t=0} = u_1^{(0)}(x). \tag{33}$$

Theorem (G.-Q. Chen, Christoforou, Y. Zhang: *Arch. Rat. Mech. An.*)

Suppose that $\rho_2^{(0)}, p_2^{(0)}, u_1^{(0)} \in BV(\mathbb{R}^1)$.
Then, there exists a constant $M_0 > 0$ *such that for* $M \in (0, M_0)$, *for every* $t \geq 0$,

$$
\begin{aligned}
||\rho^M(t) - \rho_0 - M^2\rho_2^M(t)||_{L^1} &= \mathcal{O}(1)\, TV\{(p_2^{(0)}, u_1^{(0)}, \rho_2^{(0)})\} \cdot t \cdot M^3,\\
||p^M(t) - p_0 - M^2 p_2^M(t)||_{L^1} &= \mathcal{O}(1)\, TV\{(p_2^{(0)}, u_1^{(0)}, \rho_2^{(0)})\} \cdot t \cdot M^3,\\
||u^M(t) - M u_1^M(t)||_{L^1} &= \mathcal{O}(1)\, TV\{(p_2^{(0)}, u_1^{(0)}, \rho_2^{(0)})\} \cdot t \cdot M^2,
\end{aligned}
$$

where (ρ_2^M, p_2^M, u_1^M) *is the unique weak solution to the* *linear acoustic* *system.*

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- $0 < \mathrm{M} < \mathrm{M}_0 \longrightarrow$ small data to compressible Euler

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- Define metric: for any $V = (\rho, p, u)$, $\widetilde{V} = (\tilde{\rho}, \tilde{p}, \tilde{u})$

$$d_{\mathrm{M}}(V, \widetilde{V}) = ||\rho - \tilde{\rho}||_{L^1} + ||p - \tilde{p}||_{L^1} + \mathrm{M}||u - \tilde{u}||_{L^1} \qquad (34)$$

so that the error formula becomes

$$d_{\mathrm{M}}(S_t^{\mathrm{M}} w(0), w(t)) \leq L \int_0^t \liminf_{h \to 0+} \frac{d_{\mathrm{M}}(S_h^{\mathrm{M}} w(\tau), w(\tau + h))}{h} d\tau$$

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- Do not need to employ the front tracking method! Approximate the solution to the linear acoustic limit by piecewise constant functions: $W^{\mathrm{M},n} = (\rho_2^{\mathrm{M},n}, p_2^{\mathrm{M},n}, u_1^{\mathrm{M},n}) \to W^{\mathrm{M}} = (\rho_2^{\mathrm{M}}, p_2^{\mathrm{M}}, u_1^{\mathrm{M}})$.

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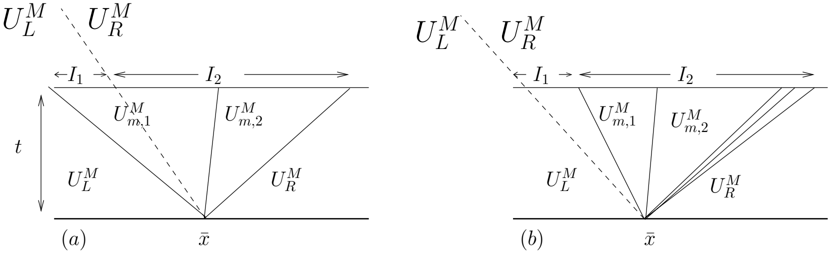
- Apply the error formula on

$$U^{\mathrm{M},n}(t,x) = (\rho_0 + \mathrm{M}^2 \rho_2^{\mathrm{M},n}(t,x), p_0 + \mathrm{M}^2 p_2^{\mathrm{M},n}(t,x), \mathrm{M} u_1^{\mathrm{M},n}(t,x))$$

Case of 1-shock: $U_L^{\mathrm{M},n} = (\rho_0 + \mathrm{M}^2 \rho_{2,L}^{\mathrm{M},n}, p_0 + \mathrm{M}^2 p_{2,L}^{\mathrm{M},n}, \mathrm{M} u_{1,L}^{\mathrm{M},n})$ and

$W_L^{\mathrm{M},n} = (\rho_{2,L}^{\mathrm{M},n}, p_{2,L}^{\mathrm{M},n}, u_{1,L}^{\mathrm{M},n}) \to |W_L^{\mathrm{M},n} - W_R^{\mathrm{M},n}| = \mathcal{O}(1)[u_1^*]$.



Length of	interval	$I_1 = \mathcal{O}(1)\,\mathrm{M}\,h$,	$I_2 = \mathcal{O}(1)\,\frac{1}{\mathrm{M}}\,h$
Difference:	in ρ is	$\mathcal{O}(1)[u_1^*]\,\mathrm{M}^2\,h$	$\mathcal{O}(1)[u_1^*]\mathrm{M}^4 h$
	in p is	$\mathcal{O}(1)[u_1^*]\,\mathrm{M}^2\,h$	$\mathcal{O}(1)[u_1^*]\,\mathrm{M}^4\,h$
	in u is	$\mathcal{O}(1)[u_1^*]\,\mathrm{M}\,h$	$\mathcal{O}(1)[u_1^*]\,\mathrm{M}^3\,h$

$$d_{\mathrm{M}}\left(S_h^{\mathrm{M}} U^{\mathrm{M},n}(\tau),\ U^{\mathrm{M},n}(\tau+h)\right) = \mathcal{O}(1)\,h\,TV\{W^{\mathrm{M},n}(\tau)\}\mathrm{M}^3,$$

We show

$$d_{\mathrm{M}}\left(S_h^{\mathrm{M}} U^{\mathrm{M},n}(\tau),\ U^{\mathrm{M},n}(\tau+h)\right) = \mathcal{O}(1)\, h\, TV\{W^{\mathrm{M},n}(\tau)\}\mathrm{M}^3,$$

and by the error estimate we get

$$d_{\mathrm{M}}(S_t^{\mathrm{M}} U^{\mathrm{M},n}(0), U^{\mathrm{M},n}(t)) = \mathcal{O}(1)\,\mathrm{M}^3\, TV\{(p_2^{(0)},\ u_1^{(0)},\ \rho_2^{(0)})\}\, t.$$

We show

$$d_{\mathrm{M}}\left(S_h^{\mathrm{M}} U^{\mathrm{M},n}(\tau),\ U^{\mathrm{M},n}(\tau+h)\right) = \mathcal{O}(1)\, h\, TV\{W^{\mathrm{M},n}(\tau)\}{\mathrm{M}}^3,$$

and by the error estimate we get

$$d_{\mathrm{M}}(S_t^{\mathrm{M}} U^{\mathrm{M},n}(0), U^{\mathrm{M},n}(t)) = \mathcal{O}(1)\, {\mathrm{M}}^3\, TV\{(p_2^{(0)},\ u_1^{(0)},\ \rho_2^{(0)})\}\, t.$$

As $n \to \infty$,

$$d_{\mathrm{M}}(S_t^{\mathrm{M}} U(0), U^{\mathrm{M}}(t)) = \mathcal{O}(1)\, {\mathrm{M}}^3\, TV\{(p_2^{(0)},\ u_1^{(0)},\ \rho_2^{(0)})\}\, t \qquad (35)$$

where

$$S_t^{\mathrm{M}} U(0) = (\rho^{\mathrm{M}}(t,x), p^{\mathrm{M}}(t,x), u^{\mathrm{M}}(t,x))$$

$$U^{\mathrm{M}} = (\rho_0 + {\mathrm{M}}^2 \rho_2^{\mathrm{M}}(t,x), p_0 + {\mathrm{M}}^2 p_2^{\mathrm{M}}(t,x), {\mathrm{M}} u_1^{\mathrm{M}}(t,x))$$

Publications:

• G.-Q. Chen, C. Christoforou and Y. Zhang, *Dependence of Entropy Solutions in the Large for the Euler Equations on Nonlinear Flux Functions*, *Indiana University Mathematics Journal*, **56** (2007), (5) 2535–2568.

• G.-Q. Chen, C. Christoforou and Y. Zhang, *L^1 estimates of entropy solutions to the Euler equations with respect to the adiabatic exponent and Mach number*, *Archive for Rational Mechanics and Analysis*, to appear.