Energy

$$\partial_t v_j + \partial_i (v_i v_j) + \partial_j p = 0$$

$$v_j \partial_t v_j + v_j (\partial_i v_i) v_j + v_i (\partial_i v_j) v_j + $$
$$+ v_j \partial_j p = 0$$

$$\partial_t \frac{|v|^2}{2} + v \cdot \nabla \left(\frac{|v|^2}{2} + p \right) = 0$$

$$\partial_t \frac{|v|^2}{2} + \operatorname{div} \left[\left(\frac{|v|^2}{2} + p \right) v \right] = 0$$

$$\frac{d}{dt} \int_{\mathbb{R}^n} \frac{|v|^2}{2} (x,t)\, dx = 0$$

Dissipating means

$$\frac{d}{dt} \int_{\mathbb{R}^n} \frac{|v|^2}{2} (x,t)\, dx \leq 0$$

and $<$ SOMEWHERE

JOINT WORK WITH
LÁSZLÓ SZÉKELYHIDI JR.
(UNIV. BONN)

p-system
isentropic gas dynamics in
Eulerian coordinates n space dimens.

$$
\begin{cases}
\partial_t \rho + \operatorname{div}(\rho v) = 0 \\
\partial_t (\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla [p(\rho)] = 0 \\
\rho(0,\cdot) = \rho^0 \\
v(0,\cdot) = v^0
\end{cases}
$$

($p' > 0$; typical
$p(\rho) = \kappa \rho^\gamma$)

Entropy solutions (or admissible solutions) := distributional solutions satisfying the energy (entropy) inequality

Internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$

$$p(r) = r^2 \varepsilon'(r)$$

Entropy inequality

$$\partial_t \left[\rho \varepsilon(\rho) + \rho \frac{|v|^2}{2} \right] + \operatorname{div} \left[\left(\rho \varepsilon(\rho) + \rho \frac{|v|^2}{2} + p(\rho) \right) v \right] \leq 0$$

(which indeed should be understood taking into account the initial condition:

$$\int_{\mathbb{R}^n \times \mathbb{R}^+} \left[\left(\rho \varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) \partial_t \psi + \left(\rho \varepsilon(\rho) + \rho \frac{|v|^2}{2} + p(\rho) \right) v \cdot \nabla_x \psi \right] + \int_{\mathbb{R}^n} \left(\rho^0 \varepsilon(\rho^0) + \rho^0 \frac{|v^0|^2}{2} \right) \psi(\cdot, 0) \geq 0$$

$$\forall \psi \in C^\infty_c(\mathbb{R} \times \mathbb{R}^n), \ \psi \geq 0.$$

THEOREM (2008)

Let $n \geq 2$. Then, for any given function p, there exist bounded initial data (ρ^0, v^0) with $\rho^0 \geq c > 0$ for which there are infinitely many bounded admissible solutions (ρ, v) with $\rho \geq c > 0$.

Suggested by Elling.

Remark

The solutions constructed satisfy the energy equality. Therefore:

COROLLARY

The same result holds for the full Euler system.

STARTING POINT:

Incompressible Euler

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0 \\ \operatorname{div} v = 0 \end{cases}$$

THEOREM (Scheffer 1993) $n = 2$

There exists a weak solution of (IE) which is compactly supported in space and time and nontrivial.

Shnirelman 1997

Different proof.

Remark

The solutions of Scheffer and Shnirelman do not belong to the energy space.

THEOREM (D-S 2007)

There exist nontrivial <u>bounded</u> weak solutions of (IE) that are compactly supported in space and time.

- Short, elementary proof
- Surprising application of a (by now) well-known technique in differential inclusions

↪ connections with the C^1 isometric embedding of Riem. manifolds

THEOREM (Shnirelman 2000)

$n = 3$ There are weak solutions of (IE) that dissipate energy.

Important:

Connection to the Kolmogorov theory of Turbulence

CONJECTURE (Onsager 1949)

$n = 3$ $v \in C^{0,\alpha}$ solution of (IE)

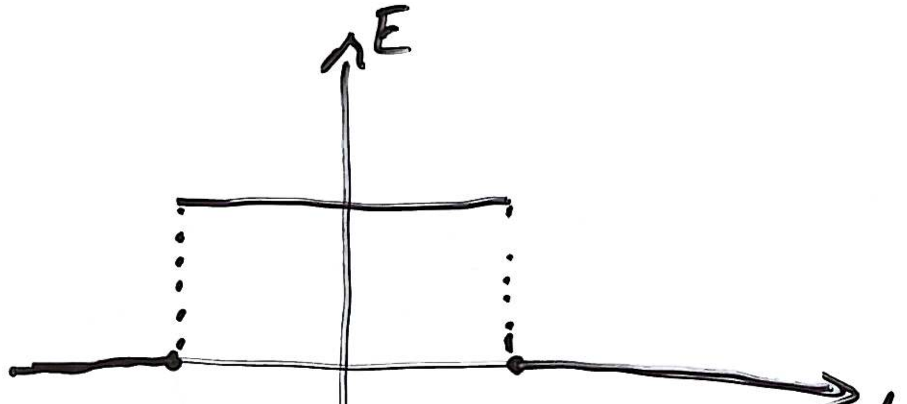
- $\alpha \geq \frac{1}{3} \Rightarrow$ Conservation of energy
- $\alpha < \frac{1}{3}$ $\exists$ sol. dissipating energy

Constantin - E - Titi 1994

$\alpha \geqslant \frac{1}{3} \Rightarrow$ Energy conservation

Shnirelman's Theorem is a trivial corollary of our construction (in any $n \geq 2$ and with bounded solutions).

<u>Rem</u> In fact the solutions might display a more surprising behavior, like the <u>istantaneous loss of energy</u>.



Q: Is there a condition that might restore uniqueness of (IE)?

- Global energy inequality (Leray type see Navier-Stokes)
- Strong global energy inequality (i.e. strong L^2 continuity)
- Local energy inequality * (i.e. $\partial_t \frac{|v|^2}{2} + \mathrm{div}\left(\left(\frac{|v|^2}{2} + p\right) v\right) \leq 0$)
- equalities and all the possible comb. of the requirements.

* proposed in the literature by Duchon - Robert.

THEOREM (D-S. 2008)
None of the previous requirements restores uniqueness for (IE).

Remark

In fact the local energy inequality is an "entropy condition"
It can be derived by a formal vanishing viscosity approximation

NAVIER STOKES → EULER

"Formal" because it is not known whether weak (Leray) solutions of NS do satisfy it

Remark

The THEOREM for compressible Euler is an easy corollary of the construction for incompressible Euler.

IMPORTANT

The behaviour of the solutions is quite wild. Classical arguments DO NOT APPLY.

FIRST PAPER:

Differential inclusion
Convex integration $\Big\} \to$ Euler

Clean approach
Many aspects of Euler fit nicely into this framework

SECOND PAPER:

Bridging together some tools of DI with typical issues in evolutionary PDE.

Some new tools for DI also.

MAIN IDEAS IN THE FIRST PAPER:

- Reformulation of Euler (in Tartar's spirit)
- Tartar's wave analysis
- Convex integration
  Baire category arguments

Müller - Šverák for CI

Cellina
Bressan
Dacorogna - Marcellini for Baire
Kirchheim

Step 1

$$\begin{cases} \partial_t v + \operatorname{div} u + \nabla q = 0 \\ \operatorname{div} v = 0 \end{cases} \qquad (LPDE)$$

$$u = v \otimes v - \frac{|v|^2}{n} \mathrm{Id} \qquad (AC)$$

$$q = p + \frac{|v|^2}{n}$$ ⟵ fake constraint

$\vdash$

$$Z = (x,t) \qquad U = \begin{pmatrix} u + q\mathrm{Id} & v \\ v & 0 \end{pmatrix}$$

$$\operatorname{div}_Z U = 0 \qquad (LPDE)$$

$$U = \begin{pmatrix} v \otimes v - \frac{|v|^2}{n} \mathrm{Id} & v \\ v & 0 \end{pmatrix} \qquad (AC)$$

Step 2 Plane wave analysis

Wave cone Λ :=

State A (in this case $\in \mathbb{R}^{(n+1)\times(n+1)}$)

s.t. $\exists\ \xi$ (direction of oscillation)

for which

$$U(z) = A\, h(z \cdot \xi)$$

is a solution $\forall h : \mathbb{R} \to \mathbb{R}$

Remark

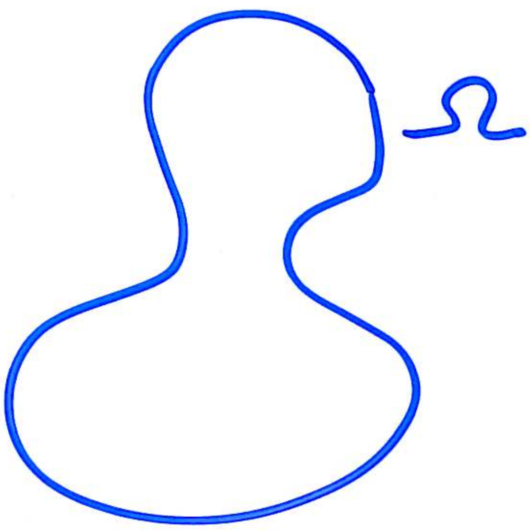
Λ is very large in the case at hand

Step 3

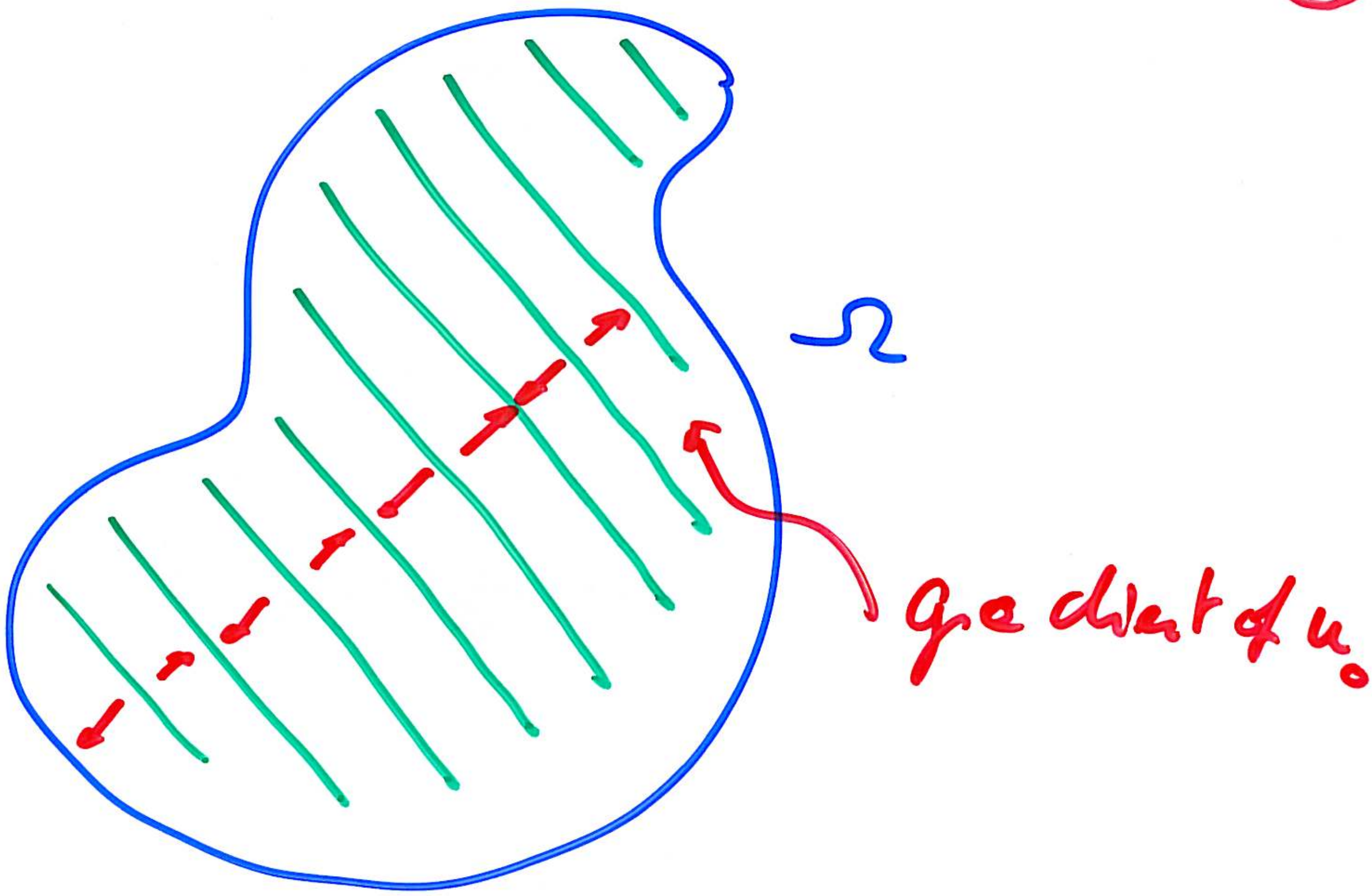
Building solutions adding waves

TOY EXAMPLE

$$|Du| = 1 \quad \left(\longrightarrow \begin{array}{ll} \nabla \times v = 0 & \text{(LPDE)} \\ + \ |v| = 1 & \text{(AC)} \end{array} \right)$$



$$u\big|_{\partial\Omega} = 0$$



$$|\nabla u_0| = 1$$

But $\quad u_0\big|_{\partial\Omega} \neq 0$

$$\hat{u}_0 = (1-\varepsilon)\, u_0\, \varphi$$

cut off

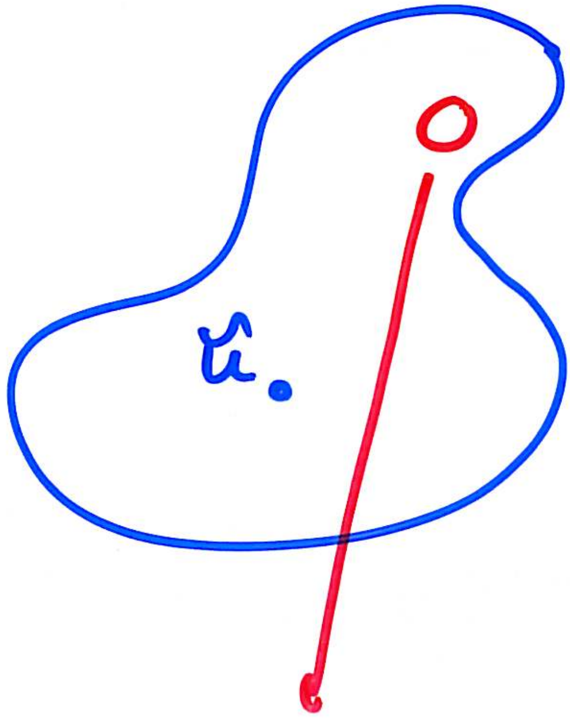
$$\nabla \hat{u}_0 = (1-\varepsilon)\,\varphi\, \nabla u_0$$

ALMOST 1, say in L^2

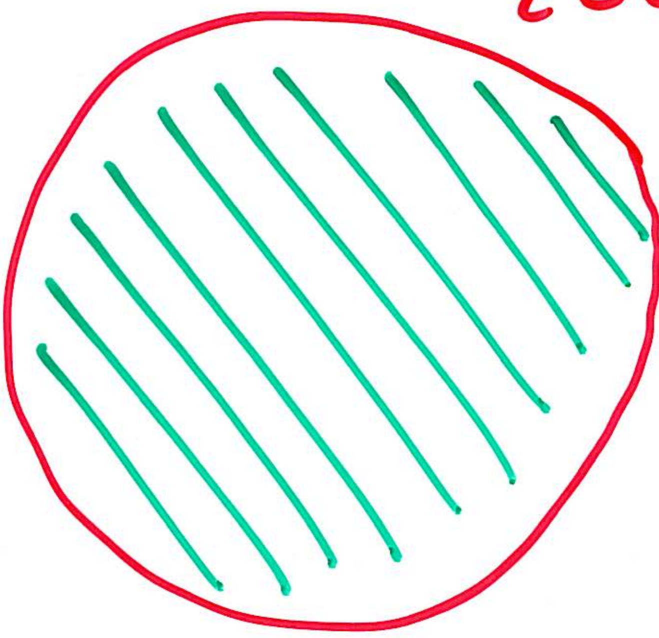
$$+ \underbrace{u_0}_{\text{Small if many oscillations!}}\, \underbrace{\nabla\varphi\,(1-\varepsilon)}_{\text{bounded}}$$

Uniformly small $\left(< \frac{\varepsilon}{2}\right)$

$$|\nabla \hat{u}_0| < 1$$



ZOOM:



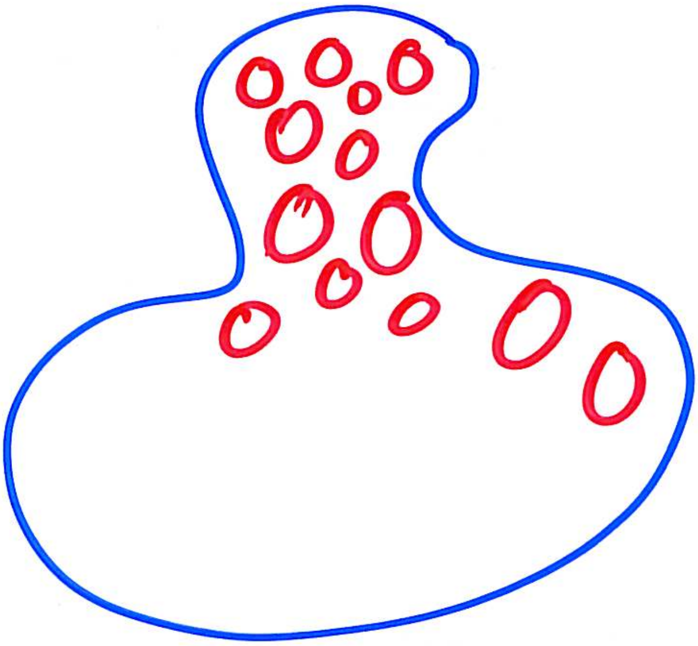
$\nabla \tilde{u}_0 \sim$ constant

add a ∇w "wave" to get $\nabla \check{u}_0$ closer to $1 \cdot 1 = 1$

Cut off the wave:

$\check{u}_0 + w\bar{\varphi}$

$\bar{\varphi}$ supported in the circle



Covering argument:

$\hat{u}_0 \longrightarrow \hat{u}_1$ with

- $|\nabla \hat{u}_1| < 1$
- $\| |\nabla \hat{u}_1| - 1 \|_{L^2}$ much smaller than $\| |\nabla \hat{u}_0| - 1 \|_{L^2}$

Step 4 Iteration

$|\nabla \tilde{u}_i| \longrightarrow 1$ in L^2

IF $\nabla \tilde{u}_i \longrightarrow \nabla u$ STRONGLY

THEN $|\nabla u| = 1$

Rem. The strong convergence is not at all obvious

CONVEX INTEGRATION (BAIRE) PROVIDES IT.

Main idea in CI : separation of scales.

Remark

In our formulation (Euler)

- there are many waves
- it is difficult to find suitable potentials
- apparently time does not play a special role (but in fact it does and it is possible to implement
  - entropy inequalities
  - strong continuity in time)

De Lellis, Székelyhidi

- The Euler equations as a differential inclusion
- On admissibility criteria for weak solutions of the Euler equations.