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A SPECTRAL VISCOSITY METHOD BASED ON HERMITE FUNCTIONS FOR NONLINEAR CONSERVATION LAWS*

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Abstract. We consider the approximation by a spectral method of the solution of the Cauchy problem for a scalar conservation law in one dimension posed in the whole real line. We analyze a spectral viscosity method in which the orthogonal basis considered is the one of Hermite functions. We prove the convergence of the approximate solution to the unique entropy solution of the problem by using compensated compactness arguments.

Key words. hyperbolic conservation laws, Hermite functions, spectral methods, spectral viscosity, compensated compactness

AMS subject classifications. 33C45, 65M70, 35L65

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1. Introduction.

We consider the Cauchy problem for a genuinely nonlinear scalar conservation law in one space dimension posed in the whole real line:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $f \in C^1$ is a smooth nonlinear function and $\varphi \in L^\infty(\mathbb{R})$. It is well known that, in general, there are no classical solutions to this problem. Moreover, weak solutions are not unique. To isolate the physically relevant solution, one has to impose the entropy condition

$$\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \leq 0 \tag{1.2}$$

in the sense of distributions, for all entropy pairs (U, F), with $U \in C^2$ convex and $F'(u) = U'(u)f'(u)$; see [7], [11].

Our aim is to approximate the unique entropy solution of (1.1), that is, the only weak solution verifying (1.2), by means of a spectral viscosity method based on Hermite functions.

The idea of a spectral method [8] is to approximate the solution of a PDE by a truncated series of the form $\sum_{k=0}^{N} \tilde{u}_k(t)\phi_k(x)$, where $\{\phi_k\}$ is an orthogonal basis of some Hilbert space. When applied to a hyperbolic conservation law, the spectral method must be supplied with an appropriate amount of artificial viscosity to avoid the possible instabilities that can appear due to the nonsmoothness of the exact solution. The so-called spectral viscosity method was first introduced in [21] and then developed in [15], [14], and [12] among others.

On the other hand, numerical methods such as finite differences, finite elements, or Fourier spectral methods always assume a finite domain of computation. When

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dealing with a problem posed in an unbounded domain, the usual techniques to avoid this difficulty are the following:

(i) choose a large bounded computational domain and impose artificial boundary conditions;

(ii) assume the solution is periodic;

(iii) make a change of variables transforming the original domain into a bounded one.

Our approach respects the unboundedness of the domain considering as orthogonal basis for the spectral method the one consisting of Hermite functions.

The use of Hermite functions for the approximation of solutions of partial differential equations posed in the whole real line was proposed by Funaro and Kavian [6] and developed in [16], [13] for the case of viscous scalar conservation laws, in [5] for the Fokker–Planck equation, and in [9] for the Dirac equation.

One of the advantages of spectral methods is that they enjoy spectral accuracy; this is, if the function to be approximated is very regular, the approximation converges faster than any negative power of N. They are, therefore, very appropriate in the case of elliptic and parabolic equations thanks to the regularization properties of these operators. However, solutions of hyperbolic nonlinear problems may develop discontinuities in finite time, as mentioned before, leading to at most first order accurate spectral approximations. In [18], a postprocessing technique has been developed to enhance the convergence rate of the approximations, at least in points not too close to the shock.

The paper is organized as follows. In section 2 we describe the elements involved in the definition of our spectral method and recall the basic properties that will be needed in what follows. In section 3 we describe the method. Section 4 is devoted to finding a priori estimates for the approximate solutions that will be used in section 5 to prove the convergence of the method by compensated compactness arguments, supplemented by an L^∞ uniform boundedness assumption on the numerical solution. Finally, in section 6, we present some numerical experiments

2. Preliminaries. In this section we will describe the elements that take part in our spectral method as well as the basic properties that will be used in the proof of convergence.

As we have mentioned, we will consider the Hermite functions as an orthogonal basis. These functions are defined as

$$h_k(x) = e^{-x^2} H_k(x), \quad k = 0, 1, 2 \dots,$$

where H_k is the Hermite polynomial of degree k and leading coefficient 2^k, the solution of the second order differential equation

$$y'' - 2xy' + 2ky = 0. \tag{2.1}$$

From the properties of Hermite polynomials [20], it follows that

$$h_k'(x) = -h_{k+1}(x), \quad k = 0, 1, 2 \dots. \tag{2.2}$$

Hermite functions satisfy an orthogonality relation with respect to the weight function $w(x) = e^{x^2}$:

$$\int_{-\infty}^{\infty} h_n(x) h_m(x)\, w(x) dx = \delta_{m,n} 2^n n! \sqrt{\pi} \quad \forall\, n, m = 0, 1, 2, \dots. \tag{2.3}$$

Hence, they form an orthogonal basis of the weighted L^2 space

$$L^2_w = \left\{ \varphi \colon \mathbb{R} \to \mathbb{R} \ \text{ measurable } \ \left| \ \int_{-\infty}^{\infty} \right| \varphi(x)|^2 \, w(x) dx < \infty \right\}.$$

We shall denote the norm in this space by

$$\|\varphi\|_{0,w} = \left(\int_{-\infty}^{\infty} |\varphi(x)|^2 \, w(x) dx \right)^{1/2}.$$

The weighted Sobolev spaces associated to L^2_w are

$$H^k_w = \left\{ \varphi \colon \mathbb{R} \to \mathbb{R} \quad | \quad \varphi, \varphi', \dots, \varphi^{(k)} \in L^2_w \right\}, \quad k = 0, 1, 2, \dots.$$

In [6] one can find a characterization of the functions of H^1_w that we recover in the following proposition, as well as some inequalities regarding the norms of functions in this space.

PROPOSITION 1. *Let* $\varphi \in L^2_w$. $\varphi \in H^1_w$ *if and only if* $w^{1/2}\varphi \in H^1(\mathbb{R})$ *and* $x\varphi \in L^2_w$. *Besides,*

$$\|(w^{1/2}\varphi)'\|_{L^2} \le \|\varphi'\|_{0,w}\,, \tag{2.4}$$

$$\|\varphi\|_{0,w} \le \|\varphi'\|_{0,w}\,, \tag{2.5}$$

$$\|x\varphi\|_{0,w} \le \|\varphi'\|_{0,w}\,. \tag{2.6}$$

Inequality (2.5), analogous to Poincare's one in bounded domains, implies that $\|\varphi^{(k)}\|_{0,w}$ is equivalent to the usual norm $\sum_{j=0}^{k} \|\varphi^{(j)}\|_{0,w}$ on H^k_w. Therefore, we define the norm in H^k_w as

$$\|\varphi\|_{k,w} = \|\varphi^{(k)}\|_{0,w}.$$

For noninteger $s > 0$, H^s_w is defined by interpolation, and, for $s < 0$, H^s_w is the dual of H^{-s}_w.

Since Hermite functions form an orthogonal basis of L^2_w, any $\varphi \in L^2_w$ can be expressed in a unique way as a Fourier–Hermite series of the form

$$\varphi(x) = \sum_{k=0}^{\infty} \widehat{\varphi}_k h_k(x), \qquad \text{where} \qquad \widehat{\varphi}_k = \frac{1}{\sqrt{\pi}\, 2^k \, k!} \int_{-\infty}^{\infty} \varphi(x) h_k(x) w(x) dx.$$

If we denote by V_N the subspace of L^2_w generated by the first $N+1$ Hermite functions, $h_0, h_1, \dots, h_N$, given $\varphi \in L^2_w$, its best approximation in the L^2_w norm by functions of V_N is the orthogonal or spectral projection, that is,

$$\pi_N \varphi(x) = \sum_{k=0}^{N} \widehat{\varphi}_k h_k(x).$$

From the definition of the operator π_N, one has

$$\|\pi_N \varphi\|_{0,w} \le \|\varphi\|_{0,w} \quad \forall \varphi \in L^2_w.$$

Besides, the following estimate for the weighted Sobolev norms of the error of the approximation of φ by $\pi_N\varphi$ is given in [6].

PROPOSITION 2. *Let $0 \le \mu \le \sigma$. There exists a positive constant C, independent of N, such that, for all $\varphi \in H_w^\sigma$,*

$$\|\varphi - \pi_N \varphi\|_{\mu,w} \le C N^{\frac{\mu-\sigma}{2}} \|\varphi\|_{\sigma,w} . \tag{2.7}$$

In practice, the spectral projection $\pi_N\varphi$ is of limited value since it is difficult to compute accurately. There is another way of approximating a continuous function $\varphi \in L_w^2$ by an element of V_N. Let $z_0, z_1, \ldots, z_N$ be the zeros of H_{N+1}, which are all real and distinct. Given $\varphi \in C(\mathbb{R}) \cap L_w^2$, its Nth pseudospectral projection is the unique function $I_N\varphi \in V_N$ such that

$$I_N\varphi(z_j) = \varphi(z_j), \quad j = 0, 1, \ldots, N.$$

If $I_N\varphi(x) = \sum_{k=0}^N \tilde{\varphi}_k h_k(x)$, the kth coefficient $\tilde{\varphi}_k$ is the result of approximating the exact Fourier–Hermite coefficient $\widehat{\varphi}_k$ by means of a Gaussian quadrature formula. More precisely,

$$\tilde{\varphi}_k = \sum_{j=0}^N w_{k,j}\varphi(z_j), \qquad \text{where } w_{k,j} = \frac{2^{N-k} N! h_k(z_j)}{(N+1)k!(h_N(z_j))^2}. \tag{2.8}$$

In [1], the following estimate for the weighted Sobolev norms of the error of the pseudospectral projection is given.

PROPOSITION 3. *Let $\sigma \ge 1$ and $0 \le \mu \le \sigma$. There exists a positive constant C, independent of N, such that, for all $\varphi \in H_w^\sigma$,*

$$\|\varphi - I_N\varphi\|_{\mu,w} \le C N^{\frac{1}{6}+\frac{\mu-\sigma}{2}} \|\varphi\|_{\sigma,w}. \tag{2.9}$$

Finally, we introduce a differential operator which will play the role of the Laplacian in the usual viscosity spectral methods. Given $\varphi \in H_w^2$,

$$\mathcal{L}\varphi(x) = -e^{-x^2}\left(e^{x^2}\varphi'(x)\right)' = -(\varphi''(x) + 2x\varphi'(x)).$$

From (2.1) and the definition of Hermite functions it follows that the eigenfunctions of $\mathcal{L}$ are the Hermite functions h_k with eigenvalues $\lambda_k = 2(k+1)$; that is,

$$\mathcal{L}h_k(x) = 2(k+1)h_k(x), \quad k = 0, 1, 2, \ldots.$$

In [3] it is proved that $\mathcal{L}$ is a positive definite self-adjoint operator with compact inverse in L_w^2 that generates an analytic semigroup of contractions $\{e^{-t\mathcal{L}}\}$.

3. The Hermite spectral and pseudospectral viscosity methods. It is well known that the entropy solution of (1.1) can be obtained as the limit when ϵ tends to 0 of the solution of the parabolic problem obtained when one introduces in the right-hand side of (1.1) a vanishing viscosity term of the form ϵu_{xx}. Based on this fact, Tadmor defined the spectral viscosity approximation of the entropy solution u as a truncated Fourier series that is the solution of the approximated problem

$$\begin{cases} \dfrac{\partial u_N}{\partial t} + P_N \dfrac{\partial f(u_N)}{\partial x} = \epsilon_N \dfrac{\partial}{\partial x}\left(Q_{m_N}\dfrac{\partial u_N}{\partial x}\right), & x \in I, t > 0, \\ u_N(x,0) = P_N u(x,0), & x \in I, \end{cases}$$

where P_N stands for the Fourier spectral or pseudospectral projection. Since the exponentials e^{ikx} are eigenfunctions of the Laplacian, the term on the right-hand side can be easily implemented. However, when one takes the Hermite functions as an orthogonal basis, the Laplacian is not an appropriate way to introduce the artificial viscosity in the numerical scheme.

By using the properties of the operator $\mathcal{L}$ as well as compensated compactness arguments similar to those to be used in the proof of convergence in section 5, one can prove the following result about the entropy solution of (1.1).

THEOREM 1. *Let $f \in C^1$ be a nonlinear function such that $f'(0) = 0$, $\varphi \in L^2_w \cap L^\infty(\mathbb{R})$ and u_ϵ the solution of*

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} + \dfrac{\partial f(u_\epsilon)}{\partial x} + \epsilon \mathcal{L} u_\epsilon = 0, & x \in \mathbb{R},\ t > 0, \\ u_\epsilon(x,0) = \varphi(x), & x \in \mathbb{R}. \end{cases}$$

Then, for any $p \geq 1$ and any $\Omega \subset \mathbb{R} \times (0,\infty)$ open and bounded, u_ϵ converges in $L^p(\Omega)$ to the unique entropy solution of (1.1) *when ϵ tends to* 0.

By taking this into account, we define the spectral (or pseudospectral) viscosity approximation to the unique entropy solution of (1.1) as $u_N(x,t) = \sum_{k=0}^N \widetilde{u}_k(t)\, h_k(x)$ such that

$$\begin{cases} \dfrac{\partial u_N}{\partial t} + \dfrac{\partial}{\partial x}\left(P_{N-1} f(u_N)\right) + \epsilon_N \mathcal{L}(Q_{m_N} u_N) = 0, & x \in \mathbb{R},\ t \in (0,T), \\ u_N(x,0) = P_N \varphi(x), & x \in \mathbb{R}. \end{cases}$$

P_N stands for π_N in the spectral viscosity approximation and for I_N in the pseudospectral one. Q_{m_N} is a viscosity operator which modifies only the high modes of the Fourier–Hermite expansion, that is,

$$Q_{m_N}\left(\sum_{k=0}^N \widehat{\varphi}_k\, h_k(x)\right) = \sum_{k=0}^N \hat{q}_k\, \widehat{\varphi}_k\, h_k(x), \tag{3.1}$$

where

$$\begin{cases} \hat{q}_k = 0 & \text{if } k \leq m_N, \\ 1 - \dfrac{m_N}{k} \leq \hat{q}_k \leq 1 & \text{if } k > m_N, \end{cases} \tag{3.2}$$

$m_N < N$ being a positive integer that tends to ∞ with N. Finally, ϵ_N is a positive parameter depending on N that will tend to 0 as N tends to infinity.

We will give conditions on m_N and ϵ_N to ensure the convergence of u_N to the unique entropy solution of (1.1).

4. A priori estimates.

In this section we will find two kinds of a priori estimates, following the lines of [14]. The first kind of estimate is related to the viscosity operator Q_{m_N}, while the second one is concerned with the approximate solution u_N.

LEMMA 1. *Let R be a linear operator defined in V_N in the following way:*

$$R\left(\sum_{k=0}^N \widehat{\varphi}_k\, h_k(x)\right) = \sum_{k=0}^N \hat{r}_k\, \widehat{\varphi}_k\, h_k(x),$$

where $\hat r_0,\dots,\hat r_N$ are real numbers. Then, for all $\varphi\in V_N$,

$$\|R\varphi\|_{1,w}^2 \le 2\left(\sum_{k=0}^N \hat r_k^2(k+1)\right)\|\varphi\|_{0,w}^2 .$$

Proof. Let $\varphi(x)=\sum_{k=0}^N \widehat\varphi_k h_k(x)$. From (2.2), $(R\varphi)'(x)=-\sum_{k=0}^N \hat r_k\,\widehat\varphi_k\,h_{k+1}(x)$, so that, by taking norms and recalling (2.3), one has

$$\begin{aligned}\|R\varphi\|_{1,w}^2 &= \sum_{k=0}^N \hat r_k^2\,\widehat\varphi_k^2\,2^{k+1}(k+1)!\,\sqrt\pi\\ &\le 2\left(\sum_{k=0}^N \hat r_k^2(k+1)\right)\left(\sum_{k=0}^N \widehat\varphi_k^2\,2^k\,k!\,\sqrt\pi\right)\\ &= 2\left(\sum_{k=0}^N \hat r_k^2\,(k+1)\right)\|\varphi\|_{0,w}{}^2. \qquad\square\end{aligned}$$

By using this result we can now bound the norm of the derivative of a function of V_N by the sum of the H^1_w norms of its high frequencies modified by Q_{m_N} and the L^2_w norm of the function itself, multiplied by a constant that grows with N.

LEMMA 2. *Let Q_{m_N} be defined as in* (3.1), (3.2). *Then there exist positive constants C_1 and C_2, independent of N, such that*

$$\|\varphi\|_{1,w}^2 \le C_1\,\|Q_{m_N}\varphi\|_{1,w}^2 + C_2 m_N^2\,\|\varphi\|_{0,w}^2 \quad \forall\varphi\in V_N.$$

Proof. Let $\varphi(x)=\sum_{k=0}^N \widehat\varphi_k h_k(x)$, and let $R_{m_N}=I-Q_{m_N}$, where I is the identity operator. Then

$$\|\varphi\|_{1,w}^2 \le C\left(\|Q_{m_N}\varphi\|_{1,w}^2 + \|R_{m_N}\varphi\|_{1,w}^2\right).$$

We split φ in dyadic parts $\varphi(x)=\sum_{k=0}^{m_N}\widehat\varphi_k\,h_k(x)+\sum_{j=1}^J\varphi^j(x)$, where

$$\varphi^j(x)=\sum_{k>2^{j-1}m_N}^{2^j m_N}\widehat\varphi_k\,h_k(x), \qquad j=1,\dots,J.$$

Here $J=[\log_2\frac{N}{m_N}]+1$ and $\widehat\varphi_k=0$ for $k=N+1,\dots,2^J m_N$.

From the orthogonality relation (2.3), one has

$$\|R_{m_N}\varphi\|_{1,w}^2 = \left\|R_{m_N}\sum_{k=0}^{m_N}\widehat\varphi_k\,h_k\right\|_{1,w}^2 + \sum_{j=1}^J\left\|R_{m_N}\varphi^j\right\|_{1,w}^2. \tag{4.1}$$

We bound each summand by using the result obtained in Lemma 1. Since $\hat q_k=0$ for $k\le m_N$,

$$\begin{aligned}\left\|R_{m_N}\sum_{k=0}^{m_N}\widehat\varphi_k\,h_k\right\|_{1,w}^2 &\le 2\left(\sum_{k=0}^{m_N}(1-\hat q_k)^2(k+1)\right)\left\|\sum_{k=0}^{m_N}\widehat\varphi_k\,h_k\right\|_{0,w}^2\\ &\le C m_N^2\left\|\sum_{k=0}^{m_N}\widehat\varphi_k\,h_k\right\|_{0,w}^2.\end{aligned}$$

For the terms in the summatory, since $\hat{q}_k \geq 1 - \frac{m_N}{k}$, we get, for any $j = 1, \dots, J$,

$$\begin{aligned}
\left\|R_{m_N}\varphi^j\right\|_{1,w}^2 &\leq C\left(\sum_{k>2^{j-1}m_N}^{2^j m_N}(1-\hat{q}_k)^2(k+1)\right)\left\|\varphi^j\right\|_{0,w}^2\\
&\leq Cm_N^2\sum_{k>2^{j-1}m_N}^{2^j m_N}\frac{1}{k}\left\|\varphi^j\right\|_{0,w}^2\\
&\leq Cm_N^2\frac{1}{2^{j-1}m_N}(2^j m_N - 2^{j-1}m_N)\left\|\varphi^j\right\|_{0,w}^2\\
&= Cm_N^2\left\|\varphi^j\right\|_{0,w}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|R_{m_N}\varphi\right\|_{1,w}^2 &\leq Cm_N^2\left(\left\|\sum_{k=0}^{m_N}\widehat{\varphi}_k\, h_k\right\|_{0,w}^2 + \sum_{j=1}^{J}\left\|\varphi^j\right\|_{0,w}^2\right)\\
&\leq Cm_N^2\left\|\varphi\right\|_{0,w}^2,
\end{aligned}$$

and by substituting in (4.1) we get the desired result. ∎

Now we will find some a priori estimates for the solution of the approximate problem u_N. We will use the following notation for norms in $\mathbb{R}\times[0,T]$, where $T>0$:

$$\|\varphi\|_{k,w,T}^2 = \int_0^T \|\varphi(\cdot,t)\|_{k,w}^2\, dt \quad \forall\varphi\in L^2(\mathbb{R}, H_w^k).$$

There are some differences between the spectral and the pseudospectral cases; therefore we will distinguish both cases. We start with the spectral approximation.

LEMMA 3. *Let $f\in C^2(\mathbb{R})$ be such that $f'(0)=0$, $\varphi\in L_w^2$, $T>0$, $m_N=[O(N^\beta)]$, $\epsilon_N = O(N^{-\theta})$, with $0<2\beta<\theta<1/2$, Q_{m_N} given by* (3.1), (3.2), *and $u_N\colon[0,T]\to V_N$ the solution of*

$$\begin{cases} \dfrac{\partial u_N}{\partial t} + \dfrac{\partial}{\partial x}\left(\pi_{N-1}f(u_N)\right) + \epsilon_N\mathcal{L}(Q_{m_N}u_N) = 0, & x\in\mathbb{R},\ t\in(0,T),\\ u_N(x,0) = \pi_N\varphi(x), & x\in\mathbb{R}. \end{cases} \tag{4.2}$$

Let us assume that there exists a positive constant c, independent of N, such that

$$\|(1+|x|)u_N\|_{L^\infty(\mathbb{R}\times(0,T))}\leq c. \tag{4.3}$$

Then there exist positive constants k and $C(T)$, independent of N, such that

$$\|u_N(\cdot,t)\|_{0,w}\leq e^{kt}\|u_N(\cdot,0)\|_{0,w}\quad\forall t\in(0,T), \tag{4.4}$$

$$\|Q_{m_N}u_N\|_{1,w,T}\leq\frac{C(T)}{\sqrt{\epsilon_N}}, \tag{4.5}$$

$$\|u_N\|_{1,w,T}\leq\frac{C(T)}{\sqrt{\epsilon_N}}. \tag{4.6}$$

Proof. We multiply (4.2) by u_N and integrate in space with respect to the weight $w(x)$ to yield

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\partial u_N}{\partial t} u_N\, w\, dx + \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\pi_{N-1} f(u_N)\right) u_N\, w\, dx \\ - \epsilon_N \int_{-\infty}^{\infty} e^{-x^2} \frac{\partial}{\partial x}\left(e^{x^2} \frac{\partial}{\partial x}(Q_{m_N} u_N)\right) u_N\, w\, dx = 0.\end{aligned} \tag{4.7}$$

By taking into account that $\frac{d}{dx}\pi_{N-1} = \pi_N \frac{d}{dx}$ and the orthogonality of Hermite functions, we get

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\pi_{N-1} f(u_N)\right) u_N\, w\, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(f(u_N)\right) u_N\, w\, dx.$$

Let F be a primitive of $uf'(u)$ such that $F(0) = 0$. Then by integrating by parts above

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(f(u_N)\right) u_N\, w\, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(F(u_N)\right) w\, dx = -\int_{-\infty}^{\infty} 2x\, F(u_N)\, w\, dx.$$

From the mean value theorem, for each (x,t) there exists η_N between 0 and u_N such that

$$F(u_N) = F'(\eta_N)\, u_N = \eta_N\, f'(\eta_N) u_N.$$

By using again the mean value theorem, this time with f', for each (x,t) there exists ξ_N between 0 and η_N such that

$$F(u_N) = \eta_N\, \left(f'(\eta_N) - f'(0)\right)\, u_N = \eta_N\, f''(\xi_N)\, \eta_N\, u_N = f''(\xi_N)\, \eta_N^2\, u_N$$

and hence

$$\begin{aligned}\left|\int_{-\infty}^{\infty} 2x\, F(u_N)\, w\, dx\right| &= \left|\int_{-\infty}^{\infty} 2x\, \eta_N^2\, f''(\xi_N)\, u_N\, w\, dx\right| \\ &\le \left\|2\, |x|\, u_N\right\|_{L^\infty(\mathbb{R}\times(0,T))} \sup_{|\xi|\le c} |f''(\xi)|\, \|\eta_N(\cdot,t)\|_{0,w}^2 \\ &\le k\, \|u_N(\cdot,t)\|_{0,w}^2\,.\end{aligned} \tag{4.8}$$

In the third integral in (4.7) we integrate by parts to yield

$$-\epsilon_N \int_{-\infty}^{\infty} e^{-x^2} \frac{\partial}{\partial x}\left(e^{x^2} \frac{\partial}{\partial x}\left(Q_{m_N} u_N\right)\right) u_N\, w\, dx = \epsilon_N \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(Q_{m_N} u_N\right) \frac{\partial u_N}{\partial x}\, w\, dx.$$

Let $u_N(x,t) = \sum_{k=0}^{N} \widetilde{u}_k(t) h_k(x)$. From the definition of Q_{m_N} and the orthogonality of Hermite functions,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(Q_{m_N} u_N\right) \frac{\partial u_N}{\partial x}\, w\, dx &= \sum_{k=0}^{N} \widehat{q}_k\, \left(\widetilde{u}_k(t)\right)^2 \|h_{k+1}\|_{0,w}^2 \\ &\ge \sum_{k=0}^{N} \widehat{q}_k^2\, \left(\widetilde{u}_k(t)\right)^2 \|h_{k+1}\|_{0,w}^2 \\ &= \left\|\frac{\partial}{\partial x}(Q_{m_N} u_N)(\cdot,t)\right\|_{0,w}^2.\end{aligned} \tag{4.9}$$

By substituting (4.8) and (4.9) in (4.7), we get

$$\frac{1}{2}\frac{d}{dt}\|u_N(\cdot,t)\|_{0,w}^2+\epsilon_N\left\|\frac{\partial}{\partial x}(Q_{m_N}u_N)(\cdot,t)\right\|_{0,w}^2\le k\,\|u_N(\cdot,t)\|_{0,w}^2\,. \tag{4.10}$$

In particular, $\frac{d}{dt}\|u_N(\cdot,t)\|_{0,w}^2\le 2k\|u_N(\cdot,t)\|_{0,w}^2$; hence

$$\|u_N(\cdot,t)\|_{0,w}^2\le e^{2kt}\,\|u_N(\cdot,0)\|_{0,w}^2\,,$$

and (4.4) follows. Besides, by integrating (4.10) in time between 0 and T,

$$\begin{aligned}\|u_N(\cdot,T)\|_{0,w}^2+2\epsilon_N\left\|\frac{\partial}{\partial x}(Q_{m_N}u_N)\right\|_{0,w,T}^2&\le\|u_N(\cdot,0)\|_{0,w}^2+2k\int_0^T\|u_N(\cdot,t)\|_{0,w}^2\,dt\\&\le C(T)\,\|u_N(\cdot,0)\|_{0,w}^2\,.\end{aligned}\tag{4.11}$$

Since $u_N(x,0)=\pi_N\varphi(x)$ and $\varphi\in L_w^2$, we know that $\|u_N(\cdot,t)\|_{0,w}\le\|\varphi\|_{0,w}$. Therefore, from (4.11) one deduces the estimate (4.5). Finally, by Lemma 2 one can obtain

$$\begin{aligned}\left\|\frac{\partial u_N}{\partial x}\right\|_{0,w,T}^2&\le C\left\|\frac{\partial}{\partial x}(Q_{m_N}u_N)\right\|_{0,w,T}^2+Cm_N^2\int_0^T\|u_N(\cdot,t)\|_{0,w}^2\,dt\\&\le\frac{C(T)}{\epsilon_N}+C(T)m_N^2\\&\le C(T)\frac{1+\epsilon_N m_N^2}{\epsilon_N}.\end{aligned}$$

The inequality $2\beta<\theta$ implies that $\epsilon_N m_N^2=o(1)$ and, hence, the estimate (4.6) is verified. ∎

In the case of the pseudospectral approximation, more restrictive conditions must be imposed due to the fact that the approximation properties of I_N are not as good as those of π_N.

LEMMA 4. *Let $f\in C^2(\mathbb{R})$ be such that $f'(0)=0$, $\varphi\in H_w^1$, $T>0$, $m_N=[O(N^\beta]$, $\epsilon_N=O(N^{-\theta})$, with $0<2\beta<\theta<1/3$, Q_{m_N} given by* (3.1), (3.2), *and $u_N\colon[0,T]\to V_N$ the solution of*

$$\begin{cases}\dfrac{\partial u_N}{\partial t}+\dfrac{\partial}{\partial x}\left(I_{N-1}f(u_N)\right)+\epsilon_N\mathcal{L}(Q_{m_N}u_N)=0, & x\in\mathbb{R},\ t\in(0,T),\\ u_N(x,0)=I_N\varphi(x), & x\in\mathbb{R}.\end{cases}\tag{4.12}$$

Let us assume that there exists a positive constant c, independent of N, such that

$$\|(1+|x|)u_N\|_{L^\infty(\mathbb{R}\times(0,T))}\le c. \tag{4.13}$$

Then there exist positive constants k and $C(T)$, independent of N, such that

$$\|u_N(\cdot,t)\|_{0,w}\le e^{kt}\,\|u_N(\cdot,0)\|_{0,w}\quad\forall\,t\in(0,T), \tag{4.14}$$

$$\|Q_{m_N}u_N\|_{1,w,T}\le\frac{C(T)}{\sqrt{\epsilon_N}}, \tag{4.15}$$

$$\|u_N\|_{1,w,T}\le\frac{C(T)}{\sqrt{\epsilon_N}}. \tag{4.16}$$

Proof. Since the proof of these results is very similar to the one of Lemma 3, we will detail only the steps that present some difference. By multiplying (4.12) by u_N and integrating in space with respect to w, we get

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial u_N}{\partial t} u_N\, w\, dx + \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(I_{N-1} f(u_N)\right) u_N\, w\, dx \\ - \epsilon_N \int_{-\infty}^{\infty} e^{-x^2} \frac{\partial}{\partial x}\left(e^{x^2} \frac{\partial}{\partial x}(Q_{m_N} u_N)\right) u_N\, w\, dx = 0. \end{aligned} \tag{4.17}$$

While the first and third integrals are rewritten and bounded as in Lemma 3, the second integral needs a different analysis, since $\frac{d}{dx} I_{N-1} \neq I_N \frac{d}{dx}$. We decompose this integral as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(I_{N-1} f(u_N)\right) u_N\, w\, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(f(u_N)\right) u_N\, w\, dx \\ - \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left((I - I_{N-1}) f(u_N)\right) u_N\, w\, dx. \end{aligned}$$

The first summand is bounded as for u_N:

$$\left| \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(f(u_N)\right) u_N\, w\, dx \right| \leq C \left\| u_N(\cdot, t) \right\|_{0,w}^2 . \tag{4.18}$$

In the second term, we integrate by parts:

$$- \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left((I - I_{N-1}) f(u_N)\right) u_N\, w\, dx = \int_{-\infty}^{\infty} (I - I_{N-1}) f(u_N) \left(\frac{\partial u_N}{\partial x} + 2\, x\, u_N\right) w\, dx.$$

By the Cauchy–Schwartz inequality, followed by (2.9) and (2.6), one can deduce that

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left((I - I_{N-1}) f(u_N)\right) u_N\, w\, dx \right| \\ &\quad \leq \left\| (I - I_{N-1}) f(u_N(\cdot, t)) \right\|_{0,w} \left\| 2\, x\, u_N(\cdot, t) + \frac{\partial u_N}{\partial x}(\cdot, t) \right\|_{0,w} \\ &\quad \leq C\, N^{-1/3} \left\| \frac{\partial}{\partial x}\left(f(u_N(\cdot, t))\right) \right\|_{0,w} \left\| \frac{\partial u_N}{\partial x}(\cdot, t) \right\|_{0,w} \\ &\quad \leq C\, N^{-1/3} \left\| \frac{\partial u_N}{\partial x}(\cdot, t) \right\|_{0,w}^2, \end{aligned}$$

where one has to take into account that $|f'(u_N)| \leq C$ due to the uniform boundedness assumed in (4.13).

Finally, we apply the estimate in Lemma 2 and obtain

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left((I - I_{N-1}) f(u_N)\right) u_N\, w\, dx \right| \\ &\quad \leq C\, N^{-1/3} \left(\left\| \frac{\partial}{\partial x}(Q_{m_N} u_N)(\cdot, t) \right\|_{0,w}^2 + m_N^2 \left\| u_N(\cdot, t) \right\|_{0,w}^2 \right). \end{aligned} \tag{4.19}$$

By substituting (4.18) and (4.19) in (4.17) we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\| u_N(\cdot, t) \right\|_{0,w}^2 + \left(\epsilon_N - C\, N^{-1/3}\right) \left\| \frac{\partial}{\partial x}(Q_{m_N} u_N)(\cdot, t) \right\|_{0,w}^2 \\ \leq C \left(1 + N^{-1/3} m_N^2\right) \left\| u_N(\cdot, t) \right\|_{0,w} . \end{aligned}$$

Since $\epsilon_N = O(N^{-\theta})$ with $\theta < 1/3$ and $m_N = [O(N^\beta)]$ with $2\beta < \theta$, $\epsilon_N - CN^{-1/3} = O(\epsilon_N)$ and $N^{-1/3}m_N^2 = o(1)$, so that

$$\frac{1}{2}\frac{d}{dt}\,\|u_N(\cdot,t)\|_{0,w}^2 + C\,\epsilon_N\Big\|\frac{\partial}{\partial x}(Q_{m_N}u_N)(\cdot,t)\Big\|_{0,w}^2 \leq C\,\|u_N(\cdot,t)\|_{0,w}^2\,.$$

By arguing as in Lemma 3, we get (4.14) as well as

$$\epsilon_N\Big\|\frac{\partial}{\partial x}(Q_{m_N}u_N)\Big\|_{0,w,T} \leq C(T)\,\|u_N(\cdot,0)\|_{0,w}^2\,.$$

By recalling that $\varphi \in H_w^1$, we use (2.9) to deduce that

$$\|u_N(\cdot,0)\|_{0,w}^2 = \|I_N\varphi\|_{0,w}^2 \leq \|\varphi\|_{0,w}^2 + C\,N^{-1/3}\,\|\varphi\|_{1,w}^2 \leq C\,\|\varphi\|_{1,w}^2\,,$$

and inequalities (4.15) and (4.16) are satisfied. ☐

The uniform boundedness hypothesis on $\{u_N\}$ is common in spectral viscosity approximations as can be seen in [21], [15], [14], although in some cases it is deduced from the approximate equation itself, as in [2]. In (4.3) and (4.13) an L^∞ bound for xu_N is also imposed. The factor x is related to the weight $2x = w'w^{-1}$. Another consequence of the presence of the weight is the more involved arguments needed to deduce the a priori estimates of Lemmas 3 and 4, compared with the case of Fourier or Legendre approximations, where the basis functions are orthogonal in the unweighted L^2 space.

5. Convergence to the unique entropy solution.

In the proof of convergence of u_N to the unique entropy solution of (1.1), we will make use of compensated compactness arguments [4], [10], [19], in which the estimates just proved in the previous section will be determinant. Since both the spectral and the pseudospectral cases are analogous, we will detail only the proof of convergence of the first one and state the result for the pseudospectral case under the more restrictive conditions mentioned earlier.

The proof that, for any entropy pair (U,F) associated to (1.1), $\{\frac{\partial u_N}{\partial t} + \frac{\partial f(u_N)}{\partial x}\}$ and $\{\frac{\partial U(u_N)}{\partial t} + \frac{\partial F(u_N)}{\partial x}\}$ satisfy the hypothesis of Murat's lemma will be split into several lemmas.

LEMMA 5. *Under the hypothesis of Lemma 3 the sequence $\{\epsilon_N\mathcal{L}(Q_{m_N}u_N)\}_{N\in\mathbb{N}}$ tends to 0 in $H_{\mathrm{loc}}^{-1}(\mathbb{R}\times(0,T))$.*

Proof. From the definition of $\mathcal{L}$,

$$\mathcal{L}(Q_{m_N}u_N) = 2Q_{m_N}u_N - \frac{\partial}{\partial x}(2xQ_{m_N}u_N) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(Q_{m_N}u_N)\right).$$

Let $K \subset \mathbb{R}\times(0,T)$ be compact. Since $0 \leq \widehat{q}_k \leq 1$ for $k = 0,\dots,N$, and by taking into account the estimate (4.4),

$$\|2Q_{m_N}u_N\|_{L^2(K)} \leq 2\|Q_{m_N}u_N\|_{0,w,T} \leq 2\|u_N\|_{0,w,T} \leq C(T)\|\varphi\|_{0,w}.$$

By using now property (2.6) and estimate (4.5), we get

$$\|2xQ_{m_N}u_N\|_{L^2(K)} \leq 2\|xQ_{m_N}u_N\|_{0,w,T} \leq 2\|Q_{m_N}u_N\|_{1,w,T} \leq \frac{C(T)}{\sqrt{\epsilon_N}}.$$

Finally, by applying again estimate (4.5) we can deduce that

$$\left\|\frac{\partial}{\partial x}(Q_{m_N}u_N)\right\|_{L^2(K)} \le \|Q_{m_N}u_N\|_{1,w,T} \le \frac{C(T)}{\sqrt{\epsilon_N}}.$$

Hence, $\epsilon_N \mathcal{L}(Q_{m_N}u_N)$ is the sum of a function tending to 0 in $L^2(K)$ and the derivatives of two functions also tending to 0 in $L^2(K)$, K being any compact subset, and, therefore, $\epsilon_N \mathcal{L}(Q_{m_N}u_N)$ tends to 0 in $H^{-1}_{\text{loc}}(\mathbb{R}\times(0,T))$. □

LEMMA 6. *Under the same hypothesis of Lemma* 3, $\{\frac{\partial}{\partial x}((I-\pi_{N-1})f(u_N))\}_{N\in\mathbb{N}}$ *tends to* 0 *in* $H^{-1}_{\text{loc}}(\mathbb{R}\times(0,T))$.

Proof. Let $K \subset \mathbb{R}\times(0,T)$ be compact. From (2.7) we have

$$\|(I-\pi_{N-1})f(u_N)\|_{L^2(K)} \le \|(I-\pi_{N-1})f(u_N)\|_{0,w,T} \le CN^{-1/2}\|f(u_N)\|_{1,w,T}.$$

Without loss of generality, we can suppose that $f(0)=0$. The uniform boundedness of u_N together with the continuity of f' allows us to deduce that if $C=\max_{|\xi|\le c}|f'(\xi)|$, then $|f(u_N)|\le C|u_N|$, so that, by (4.6), we obtain

$$\|(I-\pi_{N-1})f(u_N)\|_{L^2(K)} \le CN^{-1/2}\|u_N\|_{1,w,T} \le \frac{C(T)}{\sqrt{N\epsilon_N}},$$

and $\frac{\partial}{\partial x}((I-\pi_{N-1})f(u_N))$ is the derivative of a function that tends to 0 in $L^2_{\text{loc}}(\mathbb{R}\times(0,T))$, since $\epsilon_N = O(N^{-\theta})$ with $\theta<1/2$. Therefore, the result follows. □

LEMMA 7. *Let* $U\in C^2(\mathbb{R})$ *be an entropy function of* (1.1)*, and assume that the hypotheses of Lemma* 3 *are satisfied. Then* $\epsilon_N U'(u_N)\mathcal{L}(Q_{m_N}u_N)$ *can be written as the sum of two terms, one of them tending to* 0 *in* $H^{-1}_{\text{loc}}(\mathbb{R}\times(0,T))$ *and the other one bounded in* $L^1(\mathbb{R}\times(0,T))$.

Proof. From the definition of $\mathcal{L}$, we can write

$$\epsilon_N U'(u_N)\mathcal{L}(Q_{m_N}u_N) = -\epsilon_N U'(u_N)\left(\frac{\partial^2}{\partial x^2}(Q_{m_N}u_N)+2x\frac{\partial}{\partial x}(Q_{m_N}u_N)\right) = I+II,$$

where

$$I = \epsilon_N\left(2U'(u_N)Q_{m_N}u_N - \frac{\partial}{\partial x}\left(2xU'(u_N)Q_{m_N}u_N\right) - \frac{\partial}{\partial x}\left(U'(u_N)\frac{\partial}{\partial x}(Q_{m_N}u_N)\right)\right),$$

$$II = \epsilon_N U''(u_N)\frac{\partial u_N}{\partial x}\left(2xQ_{m_N}u_N + \frac{\partial}{\partial x}(Q_{m_N}u_N)\right).$$

U' is continuous and u_N uniformly bounded in L^∞ so that there exists a constant $C>0$, independent of N, such that $|U'(u_N)|\le C$. By arguing as in Lemma 5, $2U'(u_N)Q_{m_N}u_N$, $2xU'(u_N)Q_{m_N}u_N$, and $U'(u_N)\frac{\partial}{\partial x}(Q_{m_N}u_N)$ tend to 0 in $L^2_{\text{loc}}(\mathbb{R}\times(0,T))$, and hence I tends to 0 in $H^{-1}_{\text{loc}}(\mathbb{R}\times(0,T))$.

On the other hand, U'' is also continuous; therefore, by the Cauchy–Schwartz inequality, followed by property (2.6) and estimates (4.5) and (4.6), we get

$$\begin{aligned}\|II\|_{L^1(\mathbb{R}\times(0,T))} &= \epsilon_N\left\|U''(u_N)\frac{\partial u_N}{\partial x}\left(2xQ_{m_N}u_N+\frac{\partial}{\partial x}(Q_{m_N}u_N)\right)\right\|_{L^1(\mathbb{R}\times(0,T))}\\ &\le C\epsilon_N\left\|\frac{\partial u_N}{\partial x}\right\|_{L^2(\mathbb{R}\times(0,T))}\left\|2xQ_{m_N}u_N+\frac{\partial}{\partial x}(Q_{m_N}u_N)\right\|_{L^2(\mathbb{R}\times(0,T))}\\ &\le C\epsilon_N\|u_N\|_{1,w,T}\|Q_{m_N}u_N\|_{1,w,T}\\ &\le C(T);\end{aligned}$$

that is, II is bounded in $L^1(\mathbb{R}\times(0,T))$. □

LEMMA 8. *Let $U \in C^2(\mathbb{R})$ be an entropy function of* (1.1)*, and assume that the hypotheses of Lemma 3 are satisfied. Then $U'(u_N)\frac{\partial}{\partial x}((I-\pi_{N-1})f(u_N))$ can be written as the sum of two terms, one of them tending to 0 in $H^{-1}_{\text{loc}}(\mathbb{R}\times(0,T))$ and the other one bounded in $L^1(\mathbb{R}\times(0,T))$.*

Proof. We write $U'(u_N)\frac{\partial}{\partial x}((I-\pi_{N-1})f(u_N))$ as a difference in the following way:

$$U'(u_N)\frac{\partial}{\partial x}\left((I-\pi_{N-1})f(u_N)\right) = \frac{\partial}{\partial x}\left(U'(u_N)(I-\pi_{N-1})f(u_N)\right) - U''(u_N)\frac{\partial u_N}{\partial x}\left((I-\pi_{N-1})f(u_N)\right).$$

The first term is the derivative of a function that tends to 0 in $L^2(\mathbb{R}\times(0,T))$ since, by arguing as in Lemma 6,

$$\|U'(u_N)(I-\pi_{N-1})f(u_N)\|_{L^2(\mathbb{R}\times(0,T))} \le C\|(I-\pi_{N-1})f(u_N)\|_{0,w,T} \le \frac{C(T)}{\sqrt{N\epsilon_N}}.$$

Therefore, $\frac{\partial}{\partial x}\left(U'(u_N)(I-\pi_{N-1})f(u_N)\right)$ tends to 0 in $H^{-1}_{\text{loc}}(\mathbb{R}\times(0,T))$.

To bound the L^1 norm of the second term we use the Cauchy–Schwartz inequality, followed by (2.7) and the estimates obtained in Lemma 3, yielding

$$\begin{aligned}
&\left\|U''(u_N)\frac{\partial u_N}{\partial x}\left((I-\pi_{N-1})f(u_N)\right)\right\|_{L^1(\mathbb{R}\times(0,T))} \\
&\le C\left\|\frac{\partial u_N}{\partial x}\right\|_{L^2(\mathbb{R}\times(0,T))}\left\|(I-\pi_{N-1})f(u_N)\right\|_{L^2(\mathbb{R}\times(0,T))} \\
&\le C\|u_N\|_{1,w,T}N^{-1/2}\|u_N\|_{1,w,T} \\
&\le \frac{C(T)}{N^{1/2}\epsilon_N}.
\end{aligned}$$

By recalling that $\epsilon_N = O(N^{-\theta})$ with $\theta < 1/2$, the result follows. ☐

We are now ready to prove the convergence of u_N, by making use of Murat's and Tartar's lemmas.

THEOREM 2. *Let $f \in C^2(\mathbb{R})$ be a nonlinear function such that $f'(0)=0$, $\varphi \in L^2_w \cap L^\infty(\mathbb{R})$, $m_N = [O(N^\beta)]$, $\epsilon_N = O(N^{-\theta})$, with $0 < 2\beta < \theta < 1/2$, and assume that, for a given $T>0$, the solution u_N of* (4.2) *verifies the uniform bound* (4.3)*. Then $\{u_N\}$ converges in $L^p(\Omega)$ to the unique entropy solution of* (1.1) *for any $\Omega \subset \mathbb{R}\times[0,T]$ open and bounded and any $p \ge 1$.*

Proof. The uniform boundedness of $\{u_N\}$ in $L^\infty(\mathbb{R}\times(0,T))$ ensures that there exists a subsequence that we will still denote by $\{u_N\}$, which converges in the weak-* topology of L^∞. Let u be its limit. We will prove that u is the unique entropy solution of (1.1), and we will also show that the whole sequence tends to u in $L^p(\Omega)$ for every $p \ge 1$ and every $\Omega \subset \mathbb{R}\times(0,T)$ open and bounded.

Let (U,F) be an entropy pair associated to (1.1). We first prove that $\frac{\partial u_N}{\partial t}+\frac{\partial f(u_N)}{\partial x}$ and $\frac{\partial U(u_N)}{\partial t}+\frac{\partial F(u_N)}{\partial x}$ are in a compact set of $H^{-1}_{\text{loc}}(\mathbb{R}\times(0,T))$. From (4.2), we can write

$$\begin{aligned}
\frac{\partial u_N}{\partial t}+\frac{\partial f(u_N)}{\partial x} &= \frac{\partial u_N}{\partial t}+\frac{\partial}{\partial x}\left(\pi_{N-1}f(u_N)\right)+\frac{\partial}{\partial x}\left((I-\pi_{N-1})f(u_N)\right) \\
&= -\epsilon_N\mathcal{L}(Q_{m_N}u_N)+\frac{\partial}{\partial x}\left((I-\pi_{N-1})f(u_N)\right).
\end{aligned} \tag{5.1}$$

We have shown in Lemmas 5 and 6 that both summands in the right-hand side tend to 0 in $H^{-1}_{\text{loc}}(\mathbb{R}\times(0,T))$, and therefore $\frac{\partial u_N}{\partial t}+\frac{\partial f(u_N)}{\partial x}$ is in a compact set of this space.

On the other hand, since $F'(u) = f'(u)U'(u)$,

$$\frac{\partial U(u_N)}{\partial t} + \frac{\partial F(u_N)}{\partial x} = U'(u_N)\frac{\partial u_N}{\partial t} + F'(u_N)\frac{\partial u_N}{\partial x}$$
$$= -\epsilon_N U'(u_N)\mathcal{L}(Q_{m_N} u_N) + U'(u_N)\frac{\partial}{\partial x}\left((I - \pi_{N-1})f(u_N)\right).$$

From Lemmas 7 and 8 it follows that $\frac{\partial U(u_N)}{\partial t} + \frac{\partial F(u_N)}{\partial x}$ is the sum of two terms, one of them being in a compact set of $H^{-1}_{\text{loc}}(\mathbb{R} \times (0,T))$ and the other one being bounded in the set of measures $\mathcal{M}(\mathbb{R}\times(0,T))$. Besides, $\frac{\partial U(u_N)}{\partial t} + \frac{\partial F(u_N)}{\partial x}$ is in $W^{-1,p}_{\text{loc}}(\mathbb{R}\times(0,T))$ for any $p > 2$, since U and F are continuous and u_N uniformly bounded in $L^\infty(\mathbb{R}\times(0,T))$. Therefore, by Murat's lemma, $\frac{\partial U(u_N)}{\partial t} + \frac{\partial F(u_N)}{\partial x}$ is in a compact set of $H^{-1}_{\text{loc}}(\mathbb{R}\times(0,T))$.

We apply now Tartar's div-culr lemma [22] to conclude that u is in fact a weak solution of (1.1). Let us now turn to the analysis of the entropy condition.

$$\frac{\partial U(u_N)}{\partial t} + \frac{\partial F(u_N)}{\partial x} = \epsilon_N e^{-x^2}\frac{\partial}{\partial x}\left(e^{x^2}U'(u_N)\frac{\partial}{\partial x}(Q_{m_N}u_N)\right)$$
$$-\epsilon_N U''(u_N)\frac{\partial u_N}{\partial x}\frac{\partial}{\partial x}(Q_{m_N}u_N)$$
$$+\frac{\partial}{\partial x}\left(U'(u_N)(I - \pi_{N-1})f(u_N)\right)$$
$$-U''(u_N)\frac{\partial u_N}{\partial x}\left((I - \pi_{N-1})f(u_N)\right).$$

We multiply this equality by a nonnegative function test $\phi \in C^1_0(\mathbb{R} \times (0,T))$ and integrate by parts to obtain

$$\int_0^\infty \int_{-\infty}^\infty \left(U(u_N)\frac{\partial \phi}{\partial t} + F(u_N)\frac{\partial \phi}{\partial x}\right) dx\, dt$$
$$= \epsilon_N \int_0^\infty \int_{-\infty}^\infty U'(u_N)\frac{\partial}{\partial x}(Q_{m_N}u_N)\left(-2x\phi + \frac{\partial \phi}{\partial x}\right) dx\, dt$$
$$+\epsilon_N \int_0^\infty \int_{-\infty}^\infty U''(u_N)\frac{\partial u_N}{\partial x}\frac{\partial}{\partial x}(Q_{m_N}u_N)\phi\, dx\, dt$$
$$+ \int_0^\infty \int_{-\infty}^\infty U'(u_N)\left((I - \pi_{N-1})f(u_N)\right)\frac{\partial \phi}{\partial x}\, dx\, dt$$
$$+ \int_0^\infty \int_{-\infty}^\infty U''(u_N)\frac{\partial u_N}{\partial x}\left((I - \pi_{N-1})f(u_N)\right)\phi\, dx\, dt.$$

Estimates (4.5) and (4.6) and property (2.7) imply that the first, third, and fourth integrals on the right-hand side tend to 0 when N tends to infinity. Besides, if $R_{m_N} = I - Q_{m_N}$, then

$$\frac{\partial u_N}{\partial x}\frac{\partial}{\partial x}(Q_{m_N}u_N) = \left(\frac{\partial u_N}{\partial x}\right)^2 - \frac{\partial u_N}{\partial x}\frac{\partial}{\partial x}(R_{m_N}u_N),$$

and, by taking into account the estimate obtained in the proof of Lemma 2 for $\|R_{m_N}\varphi\|_{1,w}$ and the bound (4.6), we have

$$\left\|\frac{\partial u_N}{\partial x}\frac{\partial}{\partial x}(R_{m_N}u_N)\right\|_{L^1(\mathbb{R}\times(0,T))} \le \left\|\frac{\partial u_N}{\partial x}\right\|_{0,w,T}\left\|\frac{\partial}{\partial x}(R_{m_N}u_N)\right\|_{0,w,T}$$
$$\le C(T)m_N\epsilon_N^{-1/2}\left\|\varphi\right\|_{0,w}.$$

By taking this into account, we split the second integral in the right-hand side above into two terms, and we deduce that

$$\lim_{N\to\infty} \epsilon_N \int_0^\infty \int_{-\infty}^\infty U''(u_N) \frac{\partial u_N}{\partial x} \frac{\partial}{\partial x}(R_{m_N} u_N)\phi \, dx \, dt = 0,$$

while

$$\epsilon_N \int_0^\infty \int_{-\infty}^\infty U''(u_N) \left(\frac{\partial u_N}{\partial x}\right)^2 \phi \geq 0,$$

thanks to the convexity of U. We have proved, therefore, that u verifies the entropy condition. The uniqueness of the entropy solution allows us to assure that not only a subsequence but the whole sequence $\{u_N\}$ converges to u and, by recalling results about Young measures, the convergence in $L^p(\Omega)$ for any $p \geq 1$ and any $\Omega \subset \mathbb{R}\times(0,T)$ open and bounded is also fulfilled. ∎

The same convergence result is obtained for the pseudspectral viscosity approximation, under the more restrictive conditions of Lemma 4.

THEOREM 3. *Let $f \in C^2(\mathbb{R})$ be a nonlinear function such that $f'(0) = 0$, $\varphi \in H_w^1$, $m_N = [O(N^\beta)]$, $\epsilon_N = O(N^{-\theta})$, with $0 < 2\beta < \theta < 1/3$, and assume that, given $T > 0$, the solution u_N of* (4.12) *verifies* (4.13). *Then $\{u_N\}$ converges in $L^p(\Omega)$ to the unique entropy solution of* (1.1) *for any $\Omega \subset \mathbb{R}\times[0,T]$ open and bounded and any $p \geq 1$.*

6. Numerical experiments. In this section we present the numerical results obtained when applying our spectral viscosity method to Burgers' equation with initial condition $u(x, 0) = h_0(x)$, that is, to the Cauchy problem

$$\text{(6.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{u^2}{2}\right) = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = e^{-x^2}, & x \in \mathbb{R}. \end{cases}$$

The solution to this problem presents a shock at time $T^* = (e/2)^{1/2} \sim 1.1658$. All of the numerical results presented in this section have been recorded at time $t = 1.5 > T^*$.

Although we have proved the convergence of the spectral and pseudospectral viscosity methods, the implementation of the spectral one is expensive since the Fourier–Hermite coefficients defined as an integral must be approximated sufficiently accurately. Because of this, we will consider only the pseudospectral case. Hence our approximation $u_N(x,t) = \sum_{k=0}^N \widetilde{u}_k(t)h_k(x)$ will be the solution of

$$\text{(6.2)} \qquad \begin{cases} \dfrac{\partial u_N}{\partial t} + \dfrac{1}{2}\dfrac{\partial}{\partial x}\left(I_{N-1}u_N^2\right) + \epsilon_N \mathcal{L}\left(Q_{m_N} u_N\right) & = 0, \\ u_N(x,0) = I_N\varphi(x) = e^{-x^2}. \end{cases}$$

As stated in (2.8), the coefficients of the pseudospectral projection are obtained by a Gaussian quadrature formula, whose weights have been computed for different values of N by using *Mathematica*® and stored, so that there is no need to compute them each time the method is applied. To obtain the values of u_N^2 at the nodes z_j, we have used Clenshaw's formula [17] that gives an efficient and stable algorithm to compute the values of sums of the form $\sum_{k=0}^N \hat{\varphi}_k h_k(x)$. Besides, from (2.2), one has $(\sum_{k=0}^{N-1} \hat{\varphi}_k h_k(x))' = -\sum_{k=1}^N \hat{\varphi}_{k-1} h_k(x)$. Therfore, the coefficients of $\frac{\partial}{\partial x}(I_{N-1}u_N^2)$ are a nonlinear combination of those of u_N.

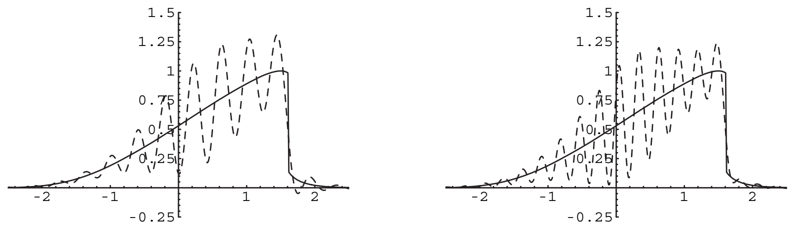

FIG. 6.1. *Graphs of the exact solution of Burgers' equation (solid line) and of the pseudospectral approximation without viscosity (dashed line), with* $N = 129$ *and* $N = 257$.

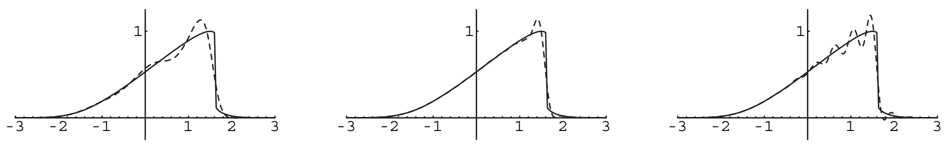

FIG. 6.2. *Graphs of the exact solution of Burgers' equation (solid line) and the viscosity pseudospectral approximation (dashed line) at time* $t = 1.5$, *for* $N = 257$, $\epsilon_N = 0.5N^{-0.33}$, *and* $m_N = [5N^{0.16}]$, *by taking* $\widehat{q}_k = \widehat{q}_k^1$, $\widehat{q}_k = \widehat{q}_k^2$, *and* $\widehat{q}_k = \widehat{q}_k^3$, *respectively, for* $k > m_N$.

On the other hand, from the definition of Q_{m_N} and the fact that h_k is an eigenfunction of $\mathcal{L}$, the coefficients of the viscosity term are obtained by multiplying those of u_N by $2(k+1)\hat{q}_k$.

Hence, the coefficients $\widetilde{u}_k(t)$, $k = 0, \dots, N$, are the solution of a nonlinear system of ordinary differential equations that has been solved by using a fourth order Runge–Kutta method with an adaptive time step.

In Figure 6.1 we show the result of approximating u by a Hermite pseudospectral method without viscosity ($\epsilon_N = 0$) for $N = 129$ and $N = 257$. The appearance of instabilities prevents the convergence of the approximation.

The viscosity introduced by the numerical scheme (6.2) depends on the parameters ϵ_N and m_N and the operator Q_{m_N}. Convergence is ensured if

$$\epsilon_N = O(N^{-\theta}), \quad 0 < \theta < 1/3, \tag{6.3}$$

$$m_N = [O(N^{\beta})], \quad 0 < \beta < \theta/2, \tag{6.4}$$

$$1 - \frac{m_N}{k} \le \hat{q}_k \le 1, \quad m_N < k \le N. \tag{6.5}$$

The bigger the coefficients $\hat{q}_k$ and ϵ_N are, the stronger the viscosity introduced is, while high values of m_N mean less viscosity since fewer coefficients are present in the viscosity term. Too much viscosity yields a worse resolution of the shock, but too little viscosity allows more oscillations that can lead to lack of convergence. By taking this into account, we have chosen $\epsilon_N = 0.5N^{-0.33}$ and $m_N = [5N^{0.16}]$, so that conditions (6.3) and (6.4) are satisfied but viscosity is not too strong.

In Figure 6.2 we show the results obtained, for $N = 257$ and $t = 1.5$, with different operators Q_{m_N} that verify in any case that $\hat{q}_N = 1$.

The first figure corresponds to taking

$$\widehat{q}_k^1 = \frac{N}{N - m_N}\left(1 - \frac{m_N}{k}\right) \quad \text{for} \quad k > m_N$$

which satisfies (6.5). The approximate solution presents few oscillations, but the shock is not very well approximated.

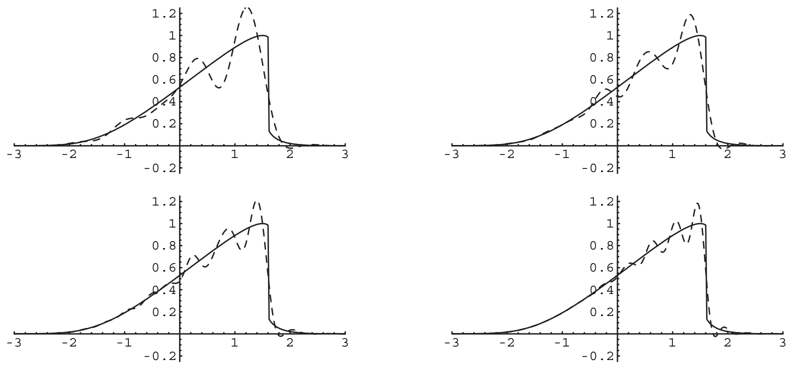

FIG. 6.3. *Graphs of the exact solution to Burgers' equation (solid line) and of the pseudospectral viscosity approximation (dashed line), by taking* $\widehat{q}_k = \widehat{q}_k^3$*,* $\epsilon_N = 0.5N^{-0.33}$*, and* $m_N = [5N^{0.16}]$*, for* $N = 33$*,* $N = 65$*,* $N = 129$*, and* $N = 257$*.*

In the second figure we present the solution obtained when

$$\widehat{q}_k^2 = \frac{k - m_N}{N - m_N} \quad \text{for } k > m_N,$$

which corresponds to coefficients laying on the straight line that joins the points $(m_N, 0)$ and $(N, 1)$, in a similar fashion as in one of the examples presented in [15]. The resulting approximation is more accurate near the shock, and there are very few oscillations.

Finally, we have taken

$$\widehat{q}_k^3 = \exp\left(-\left(\frac{k - N}{k - m_N}\right)^2\right) \quad \text{for} \quad k > m_N,$$

since in [15] the author suggests that coefficients that can be written as $\hat{q}_k = q(k/N)$, with $q \in C^\infty$, may lead to better results. This approximation is more accurate near the discontinuity, at the expense of more oscillations all over the domain.

In the three cases, the results can be slightly improved by properly choosing ϵ_N and m_N, but the third viscosity operator considered gives the best resolution of the shock.

The approximations obtained with this last operator for different values of N are shown in Figure 6.3. The improvement in the resolution of the shock when N increases is clear. However, the overshoot near the discontinuity does not decrease, and oscillations are present all along the domain. These are the main features of the so-called Gibbs phenomenon that appears when a nonregular function is approximated by its spectral or pseudospectral projection. Associated to this Gibbs phenomenon there is also a poor pointwise convergence rate even at points far away from the discontinuity.

In [18], a filter has been developed to enhance the convergence rate of the spectral projection of nonsmooth functions of L^2_w. Given $\varphi \in L^2_w$, the new approximation is defined as

$$\mathcal{F}^{\theta,p}[\pi_N \varphi](x) = \pi_p F^{N,\theta,x}(x),$$

where $F^{N,\theta,x}$ is a localization of the Nth spectral projection of φ:

$$F^{N,\theta,x}(y) = \pi_N \varphi(y) \rho\left(\frac{x - y}{\theta}\right).$$

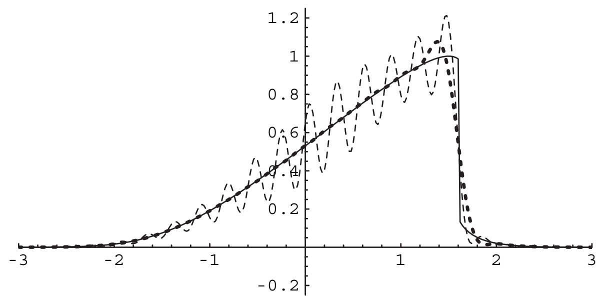

FIG. 6.4. *Graphs of the solution of Burgers' equation (solid line) and its pseudospectral viscosity approximation before filtering (dashed line) and after filtering (dotted line).*

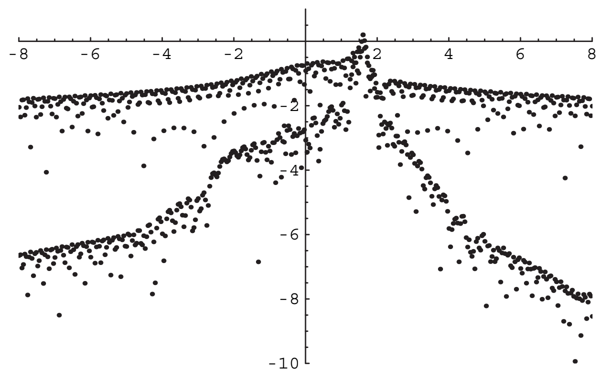

FIG. 6.5. *Logarithms of the errors multiplied by $w^{1/2}$ for the pseudospectral viscosity approximation of the solution of Burgers' equation before filtering and after filtering.*

ρ is an even smooth function with compact support, θ is a positive number, and p is an integer that will typically be taken as $p = [O(N^\alpha)]$.

If φ is piecewise C^∞ and has a jump discontinuity at $x_0 \in \mathbb{R}$, it is proven in [18] that for $\rho \in C^m$, $\alpha = \frac{1}{2} - \frac{1}{3m+1}$, and $|x - x_0| > \theta$ one has

$$e^{x^2/2}\left|\varphi(x) - \mathcal{F}^{\theta,p}[\pi_N\varphi](x)\right| = O(N^\gamma), \qquad \text{with } \gamma = -\frac{m}{4} + \frac{m}{3m+1}.$$

Spectral convergence is thus recovered, away from the discontinuity, by choosing $\rho \in C^\infty$ and $p = [O(\sqrt{N})]$.

When this filtering procedure is applied to the pseudospectral viscosity approximation, smaller artificial viscosity can be allowed. Although more oscillations will appear, the shock will be better resolved, and it is the task of the filter to diminish the oscillatory behavior and improve the convergence rate away from the discontinuity.

From the definition of $\tilde{q}_k^3$, increasing m_N makes almost no difference in the amount of viscosity introduced. Therefore we have modified only the parameter ϵ_N.

In Figure 6.4 we present the pseudospectral approximation u_N, with $\tilde{q}_k = \tilde{q}_k^3$, $m_N = [5N^{0.16}]$, and $\epsilon_N = 0.05N^{0.33}$, as well as the filtered approximation $\mathcal{F}^{\theta,p}[u_N]$ with parameters $p = [5\sqrt{N}]$, $\rho(x) = \exp(-\frac{5x^2}{1-x^2})\chi_{[-1,1]}(x)$, and $\theta = 0.9$, for $N = 257$. Figure 6.5 corresponds to the graphs of the logarithms of

$$e^{x^2/2}|u(x,t) - u_{257}(x,t)| \quad \text{and} \quad e^{x^2/2}|u(x,t) - \mathcal{F}^{\theta,p}[u_{257}](x,t)|.$$

The postprocessed solution gives a clear improvement in the convergence rate at points not too close to the discontinuity. However, the resolution of the shock is quite poor, and a better postprocessing technique should be developed in order to retain the good properties of the viscosity approximation near discontinuities.

REFERENCES

[1] J. Aguirre and J. Rivas, *Hermite pseudospectral approximations. An error estimate*, J. Math. Anal. Appl., 304 (2005), pp. 189–197.

[2] G. Q. Chen, Q. Du, and E. Tadmor, *Spectral viscosity approximations to multidimensional scalar conservation laws*, Math. Comp., 61 (1993), pp. 629–643.

[3] M. Escobedo and O. Kavian, *Variational problems related to self-similar solutions of the heat equation*, Nonlinear Anal., 11 (1987), pp. 1103–1133.

[4] L. C. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations*, American Mathematical Society, Providence, RI, 1988.

[5] J. C. Fok, B.-Y. Guo, and T. Tang, *Combined Hermite spectral-finite difference method for the Fokker-Planck equation*, Math. Comp., 71 (2002), pp. 1497–1528.

[6] D. Funaro and O. Kavian, *Approximation of some diffusion evolution equations in unbounded domains by Hermite functions*, Math. Comp., 57 (1991), pp. 597–619.

[7] E. Godlewski and P. A. Raviart, *Hyperbolic Systems of Conservation Laws*, Societé de Mathématiques Appliquées et Industrielles, Paris, 1991.

[8] D. Gottlieb and S. A. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications*, SIAM-CBMS, Philadelphia, 1977.

[9] B.-Y. Guo, J. Shen, and C. L. Xu, *Spectral and pseudospectral approximations using Hermite functions: Application to the Dirac equation*, Adv. Comput. Math., 19 (2003), pp. 35–55.

[10] L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Springer–Verlag, Berlin, Heidelberg, 1997.

[11] R. J. Leveque, *Numerical methods for conservation laws*, in Lectures in Mathematics ETH Zürich, Birkhäuser-Verlag, Basel, 1992.

[12] H. Ma, *Chebyshev–Legendre spectral viscosity method for nonlinear conservation laws*, SIAM J. Numer. Anal., 35 (1998), pp. 869–892.

[13] H. Ma, W. Sun, and T. Tang, *Hermite spectral methods with a time-dependent scaling for parabolic equations in unbounded domains*, SIAM J. Numer. Anal., 43 (2005), pp. 58–75.

[14] Y. Maday, S. M. Ould Kaber, and E. Tadmor, *Legendre pseudospectral viscosity method for nonlinear conservation laws*, SIAM J. Numer. Anal., 30 (1993), pp. 321–342.

[15] Y. Maday and E. Tadmor, *Analysis of the spectral vanishing viscosity method for periodic conservation laws*, SIAM J. Numer. Anal., 26 (1989), pp. 854–870.

[16] M. B. Pérez Pinilla, *Un método espectral basado en funciones de Hermite para la resolución numérica de algunas ecuaciones del calor semilineales*, Ph.D. thesis, Universidad del País Vasco, Bilbao, 1997.

[17] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C*, Cambridge University Press, Cambridge, 1992.

[18] J. Rivas, *Filtered Hermite Approximations*, preprint.

[19] D. Serre, *Systèmes de Lois de Conservation* II: *Structures Géométriques, Oscillations et Problèmes Mixtes*, Diderot Editeur, Paris, 1996.

[20] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society, Providence, RI, 1939.

[21] E. Tadmor, *Convergence of spectral methods for nonlinear conservation laws*, SIAM J. Numer. Anal., 26 (1989), pp. 30–44.

[22] L. Tartar, *The compensated compactness method applied to systems of conservation laws*, NATO Sci. Ser. C Math. Phys. Sci., 111 (1983), pp. 263–285.