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RESEARCH ARTICLE

Asymptotic Galerkin convergence and dynamical system results for the 3-D spectrally-hyperviscous Navier–Stokes equations on bounded domains

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Abstract
The spectrally-hyperviscous Navier–Stokes equations (SHNSE) represent a subgrid-scale model of turbulence for which previous studies were limited to periodic-box domains. Then in Avrin (The 3-D spectrally-hyperviscous Navier–Stokes equations on bounded domains with zero boundary conditions. arXiv:1908.11005) the SHNSE was adapted to general bounded domains with zero boundary conditions. Here we extend to this new setting the convergence and dynamical-system results in Avrin (J Dyn Differ Equ 20(2):479–518, 2008) and Avrin and Xiao (J Differ Equ 247(10):2778–2798, 2009), obtaining clear and straightforward Galerkin-convergence estimates, and in the case of decaying turbulence new convergence results featuring asymptotic decay rates in time. In extending the attractor-dimension results in Avrin (2008) our new degrees-of-freedom estimates stay strictly within the Landau–Lifschitz estimates (Landau and Lifshitz in Fluid mechanics, Addison-Wesley, Reading, 1959) for most computationally-relevant parameter values and exhibit a reduction in the number of degrees of freedom in calculations. The foundational properties of our bounded-domain setting also allow us to adapt the quadratic-form machinery of Temam (in: Brézis, Lions (eds) Nonlinear partial differential equations and their applications, Pitman, Boston, 1985; Browder (ed) Nonlinear functional analysis and its applications, American Mathematical Society, Providence, 1986) to carry over the main inertial-manifold results of Avrin (2008).

Keywords Spectral hyperviscosity · Zero boundary conditions · Galerkin convergence · Stability · Degrees of freedom · Inertial manifolds

Mathematics Subject Classification 35A35 · 35B40 · 35B41 · 35B42 · 35Q35 · 76F02 · 93D20

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1 Introduction

The spectrally-hyperviscous Navier–Stokes equations (SHNSE) introduce terms $\mu A_\varphi u$ that apply hyperviscosity concentrated on the high frequencies to the NSE of viscous incompressible homogeneous flow:

$$\begin{aligned} &u_t + \mu A_\varphi u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = g, \\ &\nabla \cdot u = 0. \end{aligned} \tag{1.1}$$

Here $u = (u_1, u_2, u_3)$ is the fluid velocity, $g = (g_1, g_2, g_3)$ is the external force, and p is the pressure, with $u_i = u(x, t)$, $g_i = g_i(x, t)$, $i = 1, 2, 3$, and $p = p(x, t)$ where $x \in \Omega$, a domain in $\mathbb{R}^3$. As previously studied on periodic boxes the system (1.1) becomes the hyperviscous NSE if $A_\varphi = B^\alpha \equiv (-\Delta)^\alpha$ for integers $\alpha \geqslant 2$, and if P'_{E_j} is the projection onto the jth eigenspace E'_j of B, then for $P'_m \equiv E'_1 \oplus \cdots \oplus E'_m$ and $Q'_m = I - P'_m$, the basic assumption on the operators A_φ as in [2,6,7] to obtain the SHNSE is that $A_\varphi \geqslant Q'_m B^\alpha$ in the sense of quadratic forms, i.e., $\int_\Omega v A_\varphi v \, dx \geqslant \int_\Omega v Q'_m B^\alpha v \, dx$ for all smooth v.

The SHNSE system (1.1) is a subgrid-scale model of turbulence, adding to the NSE an extra dissipative term (e.g. an approximation to the subgrid-scale tensor) to simulate the dynamic effect of frequency scales too small to be resolved in computations. With its roots in spectral-eddy viscosity as first developed in [37] (also see, e.g. [11,14, 32]), spectral hyperviscosity in application to the NSE was discussed in [11,32] and advocated in [26,27]. The resulting SHNSE system, studied theoretically in [2,6,7,26, 27] on periodic-box domains, combines the subgrid-scale modeling and regularity of the hyperviscous NSE (see, e.g. [1,8–10,14,33]) with the spectral accuracy philosophy of spectral vanishing viscosity (see, e.g. [32,34,44,46,47]). See, e.g. [7,11,27] for more discussion and references for the SHNSE and related models.

The hyperviscous NSE and SHNSE systems had not received effective treatment on realistic domains until the results of [5]. Adaptation to the general bounded domain case with zero (no-slip) boundary conditions requires solving several technical issues. Additional boundary conditions need to be specified for the higher-order operators $B^\alpha = (-\Delta)^\alpha$ to be well-posed, but the extra conditions must not overdetermine the NSE system and must stay consistent with its physics. As discussed in [5] for example, when $\alpha = 2$ standard choices such as $u = \Delta u = 0$ on $\Gamma = \partial\Omega$, or $u = \partial u / \partial \mathbf{n} = 0$ on Γ (as discussed in [41]) are mathematically well-defined but are either entirely unphysical or at best have extremely limited applicability.

All of these and related issues are resolved in [5]. Let P be the Leray projection onto the divergence-free vectors and let $A = -P\Delta$ be the Stokes operator. Let $0 < \lambda_1 < \lambda_2 < \cdots$ represent the eigenvalues of A with corresponding eigenspaces $E_1, E_2, \ldots$ and projections P_{E_j}. Let P_m be the projection onto $E_1 \oplus \cdots \oplus E_m$, and let $Q_m = I - P_m$. We apply P to both sides of the first equation in (1.1) and replace A_φ with operators A_φ satisfying $A_\varphi \geqslant Q_m A^\alpha$. This general assumption is sufficient for most of our results, but to simplify technical details in Sect. 2 we will assume the explicit computational form $A_\varphi = \sum_{j=m_0+1}^{m} d_j (\lambda_j)^\alpha P_{E_j} + Q_m A^\alpha$ where for $0 < m_0 \leqslant m$ the $\{d_j\}_{j=m_0+1}^{m}$ are such that $0 < d_j \uparrow 1$. Note in any case that all eigenspaces E_k for

$k \geqslant m$ are under the full hyperviscous influence of the operator A^α. We then have as justified and developed in [5] the following formulation of the SHNSE for general bounded domains:

$$\begin{aligned} &\frac{d}{dt}u + \mu A_\varphi u + \nu A u + P(u\cdot\nabla)u = f,\\ &u(x,0) = u_0(x). \end{aligned} \tag{1.2}$$

If P_n is the projection onto $E_1 \oplus \cdots \oplus E_n$, then for $f_n \equiv P_n f$ the Galerkin approximations to (1.2) are

$$\begin{aligned} &\frac{d}{dt}u_n + \mu A_\varphi u_n + P_n P(u_n\cdot\nabla)u_n = f_n,\\ &u_n(x,0) = P_n u_0(x) \equiv u_{n,0}(x). \end{aligned} \tag{1.3}$$

We review for completeness the derivation of (1.2) in Sect. 6, and in particular we derive and apply the identity $A^2 u = P(-\Delta)^2$. We now extend to (1.2) the qualitative theory developed for (1.1) in [2,3,6].

First we extend the Galerkin-convergence results of [6]. For $L \equiv \sup_{t\geqslant 0}\|f(t)\|_2$ we will use the standard a priori estimate

$$\|v(t)\|_2^2 \leqslant \|u_0\|_2^2 + \left(\frac{L}{\nu\lambda_1}\right)^2 \equiv U_L^2 \tag{1.4}$$

which we derive for completeness in Sect. 2. Here $\|v\|_2 \equiv \left(\int_\Omega v\cdot v\,dx\right)^{1/2}$ where the components of the vector v in $\mathbb{R}^3$ are in $L^2(\Omega)$, the set of measurable square-integrable functions. We recall that m is the spectral cutoff such that $A_\varphi \geqslant Q_m A^\alpha$ as discussed prior to (1.2), and for Galerkin approximates defined by (1.3) we assume that $n \geqslant m$ to insure that eigenspaces E_j for $m \leqslant j \leqslant n$ are under the full hyperviscous influence of A_φ. This is not a severe restriction since we are interested in the behavior of solutions of (1.3) as $n \to \infty$.

Theorem 1.1 *Let u and u_n be solutions of (1.2) and (1.3), let U_L be as in (1.4) and suppose that $m \leqslant k \leqslant n$ and $0 \leqslant t \leqslant T$. Assuming that k is large enough so that $(8/\mu)K_1 U_L^2 \leqslant (\mu/2)\lambda_{k+1}^{3/2}$ where K_1 is a generic constant, we have for $w = u - u_n$ and $F_n = f - f_n$,*

$$\begin{aligned} \left\|A^{1/2}Q_k w(t)\right\|_2^2 \leqslant{} & \left\|A^{1/2}Q_k w(0)\right\|_2^2 e^{-\nu\lambda_{k+1}t}\\ & + \int_0^t \left(C_0\left\|P_k A^{1/2} w\right\|_2^2 + C_n(u)\right)e^{-\nu\lambda_{k+1}(t-s)}\,ds \end{aligned} \tag{1.5}$$

and

$$\left\|A^{1/2}P_k w(t)\right\|_2^2 \leqslant \left[\|A^{1/2}w_0\|_2^2 + \int_0^T C_n(u)\,ds\right] e^{8K_1U_L^2 t/\mu} \tag{1.6}$$

for a constant C_0 and a function $C_n(u)$. We have from (1.5), (1.6) that $\|A^{1/2}(u(t) - u_n(t))\|_2^2 \to 0$ uniformly on $[0, T]$ as $n \to \infty$.

Specifically, $C_0 = (8/\mu)K_1 U_L^2$ and $C_n(u) = \frac{4}{\mu}\|Q_n A^{-1/2} P(u \cdot \nabla) u\|_2^2 + \frac{4}{\nu}\|F_n\|_2^2$. Proving Theorem 1.1 we will show that $\|A^{-1/2} P(u \cdot \nabla) u\|_2$ is uniformly bounded on $[0, T]$, hence from (1.6) and the dominated convergence theorem, $\|A^{1/2} P_k w(t)\|_2 \to 0$ uniformly, which in turn implies that $\|A^{1/2} Q_k w(t)\|_2 \to 0$ uniformly by (1.5).

Theorem 1.1 shows that for large enough k the convergence of the high-frequency Galerkin modes depends linearly on the convergence of the low-frequency modes and on known or estimatable parameters. The estimates (1.5) and (1.6) are much more direct than those in [6] and no longer require a complex series of nested integral formulas.

The case of decaying turbulence is defined as in [3,6] by the condition

$$f \in L^2([0, \infty); H). \tag{1.7}$$

Assuming (1.7) we obtain the next result, which shows that solutions of (1.2) can be approximated uniformly in time with finite-dimensional Galerkin solutions.

Theorem 1.2 *Under the conditions of Theorem 1.1 assume also that (1.7) holds, then $\|A^{1/2} P_k w(t)\|_2^2$ is integrable on $(0, \infty)$ and for C_0 and C_n as in Theorem 1.1 we have for all $t \geqslant 0$,*

$$\left\| P_k A^{1/2} w(t) \right\|_2^2 \leqslant \left[\|A^{1/2} w_0\|_2^2 + \int_0^\infty C_n(u)\, ds \right] e^{2C_0 U_0 \lambda_k^{1/2} \lambda_1^{-1}/\nu} \tag{1.8}$$

where $U_0 = \|u_0\|_2^2 + \frac{1}{\nu\lambda_1}\int_0^\infty \|f\|_2^2 ds$. Thus from (1.5) we have that $\|A^{1/2} Q_k w(t)\|_2^2$ is also integrable on $(0, \infty)$ and hence $\|A^{1/2}(u(t) - u_n(t))\|_2^2 \to 0$ uniformly for all $t \geqslant 0$ as $n \to \infty$.

We now add a new asymptotic convergence result assuming (1.7) which estimates the decay in time to zero of $\|A^{1/2}(u(t) - u_n(t))\|_2$ as well as its convergence to zero in n.

Theorem 1.3 *Let f satisfy (1.7), let u be the solution of (1.2), and let $\{u_n\}$ be the solutions of (1.3). Then for $w = u - u_n$, and F_n and C_n as in Theorem 1.1 there exists for all n a $t_1 \geqslant 0$ such that for all $t \geqslant t_0 \geqslant t_1$,*

$$\left\| A^{1/2} w(t) \right\|_2^2 \leqslant \left\| A^{1/2} w(t_0) \right\|_2^2 e^{-(\nu/2)\lambda_1 (t - t_0)} + \int_{t_0}^t C_n(u)\, e^{-(\nu/2)\lambda_1 (t-s)} ds. \tag{1.9}$$

We will in particular show that

$$\int_{t_0}^t C_n(u)\, e^{-(\nu/2)\lambda_1(t-s)} ds \leqslant \frac{8}{\mu\nu\lambda_1} K_3 U_2^3 \sup_{t \geqslant t_0} \|u(t)\|_2^{5/2}$$

for all $t \geqslant t_0$ where U_2 is a bound on $\|Au(t)\|_2^2$ (see (2.31) below). The right-hand side decays faster than the convergence of $\|u(t)\|_2^2$ to zero as $t_0 \to \infty$ and thus (1.9) gives a decay for $\|A^{1/2}u(t)\|_2$ that is faster than can be obtained from the known decay of $\|u(t)\|_2$ and interpolation. Theorem 1.3 adapts techniques used to develop asymptotic stability results in [3]. These we extend to (1.2) in the following, which establishes Lyapunov stability and exponential asymptotic stability for solutions of (1.2).

Theorem 1.4 *Let f satisfy (1.7), let u and v be two solutions of (1.2) with fixed forcing data f and initial data u_0 and v_0. Then there exists a $t_1 \geqslant 0$ such that for all $t \geqslant t_1$,*

$$\left\| A^{1/2}(u(t) - v(t)) \right\|_2^2 \leqslant \|u(t_1) - v(t_1)\|_2^2 \, e^{-(\nu/2)\lambda_1(t-t_1)}, \tag{1.10}$$

and given $\epsilon > 0$ there exists a $\Delta > 0$ such that if $\|A^{1/2}(u_0 - v_0)\|_2 < \Delta$, we have $\|A^{1/2}(u(t) - v(t))\|_2 < \epsilon$ for all $t \in [0, t_1]$.

To establish Theorem 1.4 the techniques of [3] need modification to handle the case of positive m_0, where we recall that in (1.2), $A_\varphi = \sum_{j=m_0+1}^{m} d_j(\lambda_j)^\alpha P_{E_j} + Q_m A^\alpha$; additional modifications give improved estimates on the size of t_1.

The proofs of Theorems 1.1, 1.2, and 1.3 use techniques similar to those used in the proofs of determining-mode results, which we now exhibit for (1.2).

Theorem 1.5 *Let u and v be two solutions of (1.2) with initial data u_0 and forcing data f and g, respectively. Suppose that $k \geqslant m$ is such that $(6/\mu)K_1U_L^2 \leqslant (\mu/2)\lambda_{k+1}^{3/2}$. Then the first k modes are determining modes for (1.2) in the sense that for each $t_0 \geqslant 0$ we have*

$$\begin{aligned} &\|A^{1/2}Q_k(u-v)(t)\|_2^2 \\ &\quad\leqslant \|A^{1/2}Q_k(u-v)(t_0)\|_2^2 \, e^{-\nu\lambda_{k+1}(t-t_0)} \\ &\qquad + \int_{t_0}^{t} \left(\frac{6}{\mu} K_1 U_L^2 \lambda_k^{3/2} \|P_k(u-v)(s)\|_2^2 + \frac{4}{\nu} \|f(s) - g(s)\|_2^2 \right) e^{-\nu\lambda_{k+1}(t-s)} ds. \end{aligned} \tag{1.11}$$

Determining-mode results for the 2-D NSE (see, e.g. [17,21,22,29–31,45] and the references therein) highlight the largely finite-dimensional character of solutions, showing that the lower frequencies control the behaviour of the high frequencies in the sense that if $\|P_k(u-v)(s)\|_2 \to 0$ and $\|f(s) - g(s)\|_2 \to 0$ as $s \to \infty$ then $\|A^{1/2}Q_k(u-v)(s)\|_2 \to 0$ as $s \to \infty$. This is shown for solutions of (1.2) in (1.11) by choosing t_0 large enough on the right-hand side.

We now discuss attractor results for (1.2), and assume in standard fashion that f is time-independent. The identity $A^2u = P(-\Delta)^2$ discussed above and in Sect. 6 will allow the basic set-up in [2] to be adapted for (1.2). This will build on some of the elements of the "CFT" framework [15,16,18,48,50] and on the methodology based on the generalized Lieb–Thirring inequalities developed in [48–50]. Related attractor results can be found in [12,13]. In our discussion $0 < \lambda_1 < \lambda_2 < \cdots$ will always represent the eigenvalues of A.

Kolmogorov's mean rate of dissipation of energy in turbulent flow (see [35], and e.g. [22,50] for further discussion) is defined as

$$\epsilon = \lambda_1^{3/2}\nu \limsup_{T\to\infty} \frac{1}{T} \int_0^T \|A^{1/2}u\|_2^2 \, ds \equiv \lambda_1^{3/2}\nu E_T. \tag{1.12}$$

The Kolmogorov length scale is $l_\epsilon = (\nu^3/\epsilon)^{1/4}$, and $1/l_\epsilon$ is an asymptotic estimate of the size of the inertial range consisting of the dynamically-active modes; in cases of interest we expect this to be large. E_T is the time-averaged total enstrophy, and as in [2] (also see, e.g. [22,50]) we set $l_0 = \lambda_1^{-1/2}$ to represent characteristic macroscopic length. Let $\dim_H \mathcal{A}$ and $\dim_F \mathcal{A}$ denote the Hausdorff and fractal dimensions of the attractor $\mathcal{A}$ for the system (1.2). The Landau–Lifschitz estimates [39] propose the upper bound $\dim_H \mathcal{A} \leqslant \dim_F \mathcal{A} \leqslant [l_0/l_\epsilon]^3$ on the number of degrees of freedom in turbulent flow; these estimates will serve as a benchmark for our new attractor estimates.

Theorem 1.6 *For a constant $\gamma_\alpha = \gamma_\alpha(\alpha)$ on the order of unity we have for the case $\lambda_m^{\alpha-1} \geqslant \nu/\mu$,*

$$\dim_H \mathcal{A} \leqslant \dim_F \mathcal{A} \leqslant \gamma_\alpha \left(\frac{\lambda_m}{\lambda_1}\right)^{3(\alpha-1)/(2\alpha)} \left[\frac{l_0}{l_\epsilon}\right]^{3/\alpha} \tag{1.13}$$

and for the complementary case $\lambda_m^{\alpha-1} \leqslant \nu/\mu$ we have for a similar constant K_α,

$$\dim_H \mathcal{A} \leqslant \dim_F \mathcal{A} \leqslant K_\alpha \left(\frac{\nu}{\mu}\right)^{3/(2\alpha)} |\Omega|^{(\alpha-1)/\alpha} \left[\frac{l_0}{l_\epsilon}\right]^{3/\alpha}, \tag{1.14}$$

where $|\Omega|$ is the size of the domain Ω.

The power $3/\alpha$ on l_0/l_ϵ was also derived in [51] for the hyperviscous NSE with $\alpha = 2$ on periodic domains. Here modifications are needed to handle positive m and m_0 along with adjustments for the bounded-domain setting. The choice of whether to use (1.13) or (1.14) depends on the size of the coefficient μ according to the prescriptions in [26], and this in turn largely depends on the choice of α as discussed in the conclusion. Note the estimate (1.13) only depends on m, and since $\lambda_m \sim c\lambda_1 m^{2/3}$ (see, e.g. [22,50]) where c depends on the shape but not the size of Ω, thus the estimate (1.13) is scale-invariant.

We will compare the results of Theorem 1.6 with the attractor estimates found in [2] in the concluding section. The NS-α model is another SGS model for which there are attractor estimates available [20], and these will also be compared in the conclusion with the estimates in Theorem 1.6. The NS-α model is also referred to as the Camassa–Holm or LANS-α model; see, e.g. [19,20,28,36,43,52] for an overview and references.

For $\alpha = 2$ we have by the remarks in the conclusion that $\nu/\mu \leqslant 1$ according to the prescriptions in [26]. The criterion $\lambda_m^{\alpha-1} \geqslant \nu/\mu$ for (1.13) is thus clearly satisfied, and to compare with the Landau–Lifschitz estimates we set $\lambda_m/\lambda_1 \leqslant (l_0/l_\epsilon)^p$ for $p \in [1, 2]$. Substituting into (1.13) we obtain

$$\dim_H \mathcal{A} \leqslant \dim_F \mathcal{A} \leqslant \gamma_\alpha \left[\frac{l_0}{l_\epsilon}\right]^{3p(\alpha-1)+6/(2\alpha)}. \tag{1.15}$$

Given presently computational values of m, it is safe to assume the upper bound $\lambda_m \leqslant 1/l_\epsilon$. Accordingly we set $p = 1$ and (1.15) becomes $\dim_H A \leqslant \dim_F A \leqslant$

$\gamma_\alpha[l_0/l_\epsilon]^{(3\alpha+3)/(2\alpha)}$ which for $\alpha = 2$ is $\dim_H A \leqslant \dim_F A \leqslant \gamma_\alpha[l_0/l_\epsilon]^{9/4}$. Since γ_α is on the order of unity it is safe to expect that $\gamma_\alpha \leqslant [l_0/l_\epsilon]^{1/4}$, from which we obtain that $\dim_H A \leqslant \dim_F A \leqslant [l_0/l_\epsilon]^{5/2}$. This estimate is not only within but definitely less than the Landau–Lifschitz estimates. We thus have direct evidence of the potential of (1.2) to reduce the number of degrees of freedom needed for calculations.

If we set $m = m_0$ and $(\lambda_m/\lambda_1) \leqslant (l_0/l_\epsilon)^{3/2}$, then from (1.15) and $p = 3/2$ we obtain the estimate $\dim_H A \leqslant \dim_F A \leqslant \gamma_\alpha[l_0/l_\epsilon]^{21/8}$. Again assuming that $\gamma_\alpha \leqslant [l_0/l_\epsilon]^{1/4}$, we have $\dim_H A \leqslant \dim_F A \leqslant [l_0/l_\epsilon]^{23/8}$, which is slightly less than the Landau–Lifschitz estimates. Here m is well beyond the limits of present-day computations; of theoretical interest is that we stay within the Landau–Lifschitz estimates even though current computational techniques cannot resolve the difference between the NSE and SHNSE systems.

For $\alpha = 8$ our discussion in the conclusion shows that (1.14) is satisfied according to the prescriptions in [26]. We estimate E_T in (1.12) as in [2] through the standard energy inequality (see (2.6) below), from which it follows in standard fashion for $L \equiv \|f\|_2$ that

$$E_T \leqslant \frac{L^2}{\nu^2\lambda_1}. \tag{1.16}$$

We rewrite the right-hand side of (1.14) as $\left(K_\alpha^{(2\alpha/3)}(\nu/\mu)|\Omega|^{2(\alpha-1)/3}\right)^{3/(2\alpha)}[l_0/l_\epsilon]^{3/\alpha} \equiv (\omega_\Lambda)^{3/(2\alpha)}[l_0/l_\epsilon]^{3/\alpha}$ and set $\omega_\Lambda \leqslant (l_0/l_\epsilon)^p$; to be within the Landau–Lifschitz estimates we need $3p/(2\alpha)+3/\alpha = 3$ or $p = 14$ in the case $\alpha = 8$. By the definitions of l_0/l_ϵ and E_T and assuming that (1.16) is a reasonable estimate for E_T we obtain $K_8^{8/21}(\nu/\mu)^{1/14}|\Omega|^{1/3} \leqslant L^{1/2}/(\lambda_1^{3/8}\nu^{1/2})$. Applying the relationship $1/\lambda_1 \sim |\Omega|^{2/3}$ (see, e.g. [22,50]) and collecting terms we have

$$(K_8|\Omega|^{1/3})^7\nu^8 \lesssim L^7\mu. \tag{1.17}$$

Since K_8 is on the order of unity the term $(K_8|\Omega|^{1/3})^7$ is roughly balanced off by the term L^7. Thus we stay within the Landau–Lifschitz estimates provided that $\nu^8 \sim \mu$. For $\nu = 10^{-6}$ this is met provided that $\mu \sim 10^{-48}$, which will clearly satisfy the parameter ranges appropriate for this case as discussed in the conclusion.

We now extend results of [2] to show that the system (1.2) possesses an inertial manifold in the case $A_\varphi = Q_m A^\alpha$. We recall the following definition from [23,24]; also see [50]:

Definition 1.7 An *inertial manifold* $\mathcal{M}$ for (1.2) is a finite-dimensional manifold satisfying:

(i) $\mathcal{M}$ is Lipschitz.
(ii) $\mathcal{M}$ is positively invariant for the semigroup, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geqslant 0$.
(iii) $\mathcal{M}$ attracts exponentially all the orbits of (1.2).

Here $S(t)$ is the mapping $S(t)u_0 = u(t)$ for each $u_0 \in H$. We have that S is well-defined for (1.2) for all $t \geqslant 0$ by the global-existence results of Sect. 2. We will use the identity $A^2u = P(-\Delta)^2$ to adapt the arguments in [2], and exploit a unique spectral-gap property similar to that used in [2] to establish the following result.

Theorem 1.8 *Let A_φ satisfy $P_m A_\varphi = 0$ and $Q_m A_\varphi \geqslant Q_m A^\alpha$ with $\alpha \geqslant 2$. Then for m large enough the system (1.2) has an inertial manifold $\mathcal{M}$ of dimension m.*

We have as in [2] that $\mathcal{M}$ is a graph over $P_m H$. For large enough m this says that trajectories on $\mathcal{M}$ are controlled by finite-dimensional NSE dynamics. We note that an inertial manifold for a model related to the NS-α model was established in [36]. At the end of Sect. 5 we will compare estimates on the size of the inertial manifold in the case $\alpha = 5/2$ with the estimates in Theorem 1.5.

In Sect. 2 we will establish preliminary results, including the absorbing-set estimates needed to assert the existence of an attractor for (1.2), and sketch the proof of Theorem 1.8. Theorems 1.1, 1.2, and 1.5 will be proven in Sect. 3, and Theorems 1.3 and 1.4 will be proven in Sect. 4. We will prove Theorem 1.6 in Sect. 5 and make concluding remarks in Sect. 7.

2 Preliminaries

We define the standard Sobolev spaces $W^{m,p}(\Omega)$ as follows: for integers p_i let $D_i^{q_i} = \partial^{q_i}/\partial x_i^{q_i}$, $i = 1, 2, 3$, and let $D^q = D_1^{q_1} D_2^{q_2} D_3^{q_3}$ where $q = q_1 + q_2 + q_3$. Then $W^{m,p}(\Omega)$ for $1 \leqslant p < \infty$ is the set of all $v \in L^p(\Omega)$ such that the distributional derivatives $D^q v$ exist and satisfy $D^q v \in L^p(\Omega)$ for all q with $0 \leqslant q \leqslant m$, and the norm on $W^{m,p}(\Omega)$ can be expressed as $\|v\|_{m,p} = \left(\sum_{0 \leqslant q \leqslant m} \int_\Omega |D^q v|^p dx\right)^{1/p}$. When $p = 2$ we use the standard notation $W^{m,2}(\Omega) = H^m(\Omega)$. If $D(B^{\theta/2})$ denotes the domain of $B^{\theta/2}$ where the operator $B = -\triangle$ is equipped with zero boundary conditions on $\Gamma = \partial\Omega$ then we have the standard embedding $D(B^{\theta/2}) \subset H^\theta(\Omega)$ (see, e.g. [25]) and thus we can express the Sobolev inequalities on Ω in terms of the operator $B = -\triangle$, equipped with zero boundary conditions on $\Gamma = \partial\Omega$,

$$\|v\|_q \leqslant M_1 \|B^{m_1/2} v\|_2^{1-\theta} \, \|B^{m_2/2} v\|_2^\theta \tag{2.1}$$

where $q \leqslant 6/(3 - 2[(1-\theta)m_1 + \theta m_2])$ and $M_1 = M_1(\theta, q, m_1, m_2, \Omega)$. By [25, Proposition 1.4], $D(A^\gamma)$ is continuously embedded into $H \cap H^{2\gamma}(\Omega)$ for any $\gamma \geqslant 0$, and thus we have for a constant $M_0 = M_0(\theta, p, q, \Omega)$ and for q as above that for all $v \in D(A^{\theta/2})$,

$$\|v\|_q \leqslant M_1 \|B^{m_1/2} v\|_2^{1-\theta} \, \|B^{m_2/2} v\|_2^\theta \leqslant M_0 \|A^{m_1/2} v\|_2^{1-\theta} \, \|A^{m_2/2} v\|_2^\theta. \tag{2.2}$$

For the semigroup $\exp(-tA)$ we have the decay estimate

$$\|e^{-tA} v\|_2 \leqslant \|v\|_2 e^{-\lambda_1 t}. \tag{2.3}$$

Like the standard NSE, (1.2) and (1.3) satisfy energy inequalities, which we derive as follows: let $v = u$ or $v = u_n$, then taking the inner product of both sides of (1.6) or (1.7) with v we have

$$\frac{1}{2} \frac{d}{dt} \|v\|_2^2 + \nu \|A^{1/2} v\|_2^2 + \mu \|A_m v\|_2^2 = (f, v) \tag{2.4}$$

where we recall that since $\nabla \cdot v = 0$, $(P(v \cdot \nabla)v, v) = -((\nabla \cdot v)v, v) = 0$, and where in the case of $v = u_n$ we have $(P_n P(u_n \cdot \nabla)u_n, u_n) = (P(u_n \cdot \nabla)u_n, P_n u_n) = (P(u_n \cdot \nabla)u_n, u_n) = 0$. We have also assumed that A_φ is as in the remarks preceding (1.2) with $(A_\varphi v, v) \geqslant (A_m v, v)$ for suitably smooth vectors v.

From our assumptions on A_φ, it also follows straightforwardly that for nonnegative θ we have $(A_\varphi v, A^\theta v) \geqslant (Q_m A^{\alpha+\theta} v, v)$ and by Poincaré's inequality we have in turn $(Q_m A^{\alpha+\theta} v, v) \geqslant \lambda_m^{\alpha-2}(Q_m A^{2+\theta} v, v)$. Given that m is of reasonably significant size and α is an integer with $\alpha \geqslant 2$, it is safe to assume that $\lambda_m^{\alpha-2} \geqslant 1$. Hence we have for nonnegative θ, $(A_\varphi v, A^\theta v) \geqslant (Q_m A^{2+\theta} v, v)$ so for what follows we can without loss of generality assume that $A_\varphi = Q_m A^2$. Accordingly applying the Cauchy–Schwartz and Young inequalities in standard fashion to (f, v) in (2.4) and multiplying by 2 we have our basic energy inequality

$$\frac{d}{dt}\|v\|_2^2 + \nu\|A^{1/2}v\|_2^2 + 2\mu\|Q_m A v\| \leqslant \frac{1}{\nu\lambda_1}\|f\|_2^2 \tag{2.5}$$

where we note that by Poincaré's inequality $\|A^{-1/2}f\|_2 \leqslant \lambda_1^{-1/2}\|f\|_2$; note that (2.5) reduces to the standard NSE energy inequality when $\mu = 0$. Integrating both sides of (2.5) we have for $v_0 = v(x, 0)$,

$$\|v\|_2^2 + \nu\int_0^T \|A^{1/2}v\|_2^2\,ds + 2\mu\int_0^T \|Av\|_2^2\,ds \leqslant \|u_0\|_2^2 + \frac{1}{\nu\lambda_1}\int_0^T \|f\|_2^2\,ds. \tag{2.6}$$

In the case that (1.7) holds we have from (2.6),

$$\|v\|_2^2 + \nu\int_0^\infty \|A^{1/2}v\|_2^2\,ds + 2\mu\int_0^\infty \|Av\|_2^2\,ds \leqslant \|u_0\|_2^2 + \frac{1}{\nu\lambda_1}\int_0^\infty \|f\|_2^2\,ds \tag{2.7}$$

holds for all $t \geqslant 0$. Meanwhile, discarding the term $2\mu\|Q_m A v\|$ in (2.5) and again using Poincaré's inequality we obtain

$$\frac{d}{dt}\|v\|_2^2 + \nu\lambda_1\|v\|_2^2 \leqslant \frac{1}{\nu\lambda_1}\|f\|_2^2 \tag{2.8}$$

so that with L defined as above by $L \equiv \sup_{t\geqslant 0}\|f(t)\|_2$ we have

$$\frac{d}{dt}\|v\|_2^2 + \nu\lambda_1\|v\|_2^2 \leqslant \frac{L^2}{\nu\lambda_1}. \tag{2.9}$$

Solving the differential inequality (2.9) we have

$$\|v(t)\|_2^2 \leqslant \|v_0\|_2^2 e^{-\nu\lambda_1 t} + \int_0^t \left(\frac{L^2}{\nu\lambda_1}\right) e^{-\nu\lambda_1(t-s)}\,ds \tag{2.10}$$

or, since $L^2/(\nu\lambda_1)$ is a constant and $\|v_0\|_2 \leqslant \|u_0\|_2$,

$$\|v(t)\|_2^2 \leqslant \|u_0\|_2^2 e^{-\nu\lambda_1 t} + \left(\frac{L}{\nu\lambda_1}\right)^2. \tag{2.11}$$

Thus, we have the standard a priori estimate

$$\|v(t)\|_2^2 \leqslant \|u_0\|_2^2 + \left(\frac{L}{\nu\lambda_1}\right)^2 \equiv U_L^2. \tag{2.12}$$

Next we bootstrap (2.12) to obtain an H^1-bound, from which global regularity is readily obtained by standard methods. We will then bootstrap our H^1-bound to obtain an H^2-bound. Both of these bounds will be used to show the existence of a global attractor. Standard methods can be used to obtain these bounds, so readers not needing to see the estimates are invited to skip below past (2.34) for the proof of Theorem 1.8. At the same time, the proofs are slightly non-standard and are our first example of techniques using spectral decomposition that are related to the central arguments used in establishing determining-mode results. We first focus on $v = u_n$; taking the inner product of both sides of (1.3) with $Q_k A u_n$ for $m \leqslant k < n$ we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|A^{1/2}Q_k u_n\right\|_2^2 + \mu\left\|Q_k A^{3/2} u_n\right\|_2^2 \\ &\qquad + \nu\left\|Q_k A u_n\right\|_2^2 + \left(P_n P(u_n\cdot\nabla)u_n,\, Q_k A u_n\right) = (f_n,\, Q_k A u_n) \end{aligned} \tag{2.13}$$

where we note that $(v, Q_k A u_n) = (Q_k v, Q_k A u_n) = (A^{1/2} Q_k v, Q_k A^{1/2} u_n) = (A^{1/2} Q_k v, A^{1/2} Q_k u_n)$ for any $v \in D(A^{1/2})$.

Now by the Cauchy–Schwartz and Young inequalities

$$|(f_n, Q_k A u_n)| \leqslant \frac{\nu}{2}\|Q_k A u_n\|_2^2 + \frac{1}{2\nu}\|f\|_2^2 \tag{2.14}$$

where we also use the fact that $\|P_n v\|_2 \leqslant \|v\|_2$ for any $v \in H$. Since powers of A commute with each of the P_n, we have

$$\left(P_n P(u_n\cdot\nabla)u_n,\, Q_k A u_n\right) = \left(P_n A^{-1/2} P(u_n\cdot\nabla)u_n,\, Q_k A^{3/2} u_n\right)$$

so by the Cauchy–Schwartz and Young inequalities

$$\begin{aligned} &\left|\left(P_n P(u_n\cdot\nabla)u_n,\, Q_k A u_n\right)\right| \\ &\qquad \leqslant \frac{1}{2\mu}\left\|A^{-1/2}P(u_n\cdot\nabla)u_n\right\|_2^2 + \frac{\mu}{2}\left\|Q_k A^{3/2} u_n\right\|_2^2. \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15) with (2.13) and multiplying by 2 we have

$$\begin{aligned}\frac{d}{dt}\left\|A^{1/2}Q_k u_n\right\|_2^2 + \mu\left\|Q_k A^{3/2}u_n\right\|_2^2 + \nu\left\|Q_k A u_n\right\|_2^2 \\ \leqslant \frac{1}{\mu}\left\|A^{-1/2}P(u_n\cdot\nabla)u_n\right\|_2^2 + \frac{1}{\nu}\|f\|_2^2.\end{aligned} \tag{2.16}$$

Now $A^{-1/2}P(u_n\cdot\nabla)u_n = A^{-1/2}P\operatorname{div}(u_n\otimes u_n)$ for the appropriate tensor product $\otimes$ while $A^{-1/2}P\operatorname{div} \equiv T_{\mathrm{b}}$ is a bounded operator on H (see, e.g. [25, Lemma 1.3]); thus using (2.2) there is a constant M_1 such that for appropriately smooth vectors v and w,

$$\begin{aligned}\left\|A^{-1/2}P(v\cdot\nabla)w\right\|_2^2 &\leqslant \|T_{\mathrm{b}}\|_2^2\,\|v\otimes w\|_2^2 \\ &\leqslant \|T_{\mathrm{b}}\|_2^2\,\|v\|_2^2\,\|w\|_\infty^2 \leqslant M_1\|T_{\mathrm{b}}\|_2^2\,\|v\|_2^2\,\|A^{3/4}w\|_2^2.\end{aligned} \tag{2.17}$$

Now, $\|A^{3/4}u_n\|_2^2 = \|P_k A^{3/4}u_n\|_2^2 + \|Q_k A^{3/4}u_n\|_2^2$; combining this with (2.16) and (2.17) with $v = w = u_n$, setting $K_1 = M_1\|T_{\mathrm{b}}\|_2^2$, and using Poincaré's inequality on the left-hand side of (2.16) we have

$$\begin{aligned}\frac{d}{dt}\left\|A^{1/2}Q_k u_n\right\|_2^2 + \mu\lambda_{k+1}^{3/2}\left\|Q_k A^{3/4}u_n\right\|_2^2 + \nu\lambda_{k+1}\left\|Q_k A^{1/2}u_n\right\|_2^2 \\ \leqslant \frac{1}{\mu}K_1\|u_n\|_2^2\left(\left\|P_k A^{3/4}u_n\right\|_2^2 + \left\|Q_k A^{3/4}u_n\right\|_2^2\right) + \frac{1}{\nu}\|f\|_2^2.\end{aligned} \tag{2.18}$$

Meanwhile $\|P_k A^{3/4}u_n\|_2^2 = \|A^{3/4}P_k u_n\|_2^2 \leqslant \lambda_k^{3/2}\|P_k u_n\|_2^2 \leqslant \lambda_k^{3/2}\|u_n\|_2^2$; combining this with (2.12) and (2.18) we have

$$\begin{aligned}\frac{d}{dt}\left\|A^{1/2}Q_k u_n\right\|_2^2 + \mu\lambda_{k+1}^{3/2}\left\|Q_k A^{3/4}u_n\right\|_2^2 + \nu\lambda_{k+1}\left\|Q_k A^{1/2}u_n\right\|_2^2 \\ \leqslant \frac{1}{\mu}K_1 U_L^2\left\|Q_k A^{3/4}u_n\right\|_2^2 + \frac{1}{\mu}K_1\lambda_k^{3/2}U_L^2 + \frac{1}{\nu}L^2.\end{aligned} \tag{2.19}$$

Now choose k (and hence n) large enough so that $K_1U_L^2/\mu \leqslant (\mu/2)\lambda_{k+1}^{3/2}$, then subtracting from both sides of (2.19) and neglecting the term $(\mu/2)\lambda_{k+1}^{3/2}\|Q_k A^{3/4}u_n\|_2^2$ on the left-hand side we have

$$\frac{d}{dt}\left\|A^{1/2}Q_k u_n\right\|_2^2 + \frac{\nu}{2}\lambda_{k+1}\left\|A^{1/2}Q_k u_n\right\|_2^2 \leqslant \frac{1}{\mu}K_1\lambda_k^{3/2}U_L^2 + \frac{1}{\nu}L^2. \tag{2.20}$$

Integrating both sides of (2.20) we obtain for $d = \nu/2$,

$$\begin{aligned}&\left\|A^{1/2}Q_k u_n\right\|_2^2 \\ &\quad\leqslant \left\|A^{1/2}Q_k u_{n,0}\right\|_2^2 e^{-d\lambda_{k+1}t} \\ &\qquad + \int_0^T\left(\frac{1}{\mu}K_1\lambda_k^{3/2}U_L^2 + \frac{1}{\nu}L^2\right)e^{-d\lambda_{k+1}(t-s)}\,ds\end{aligned} \tag{2.21}$$

from which we obtain similarly to getting (2.12) from (2.10),

$$\left\|A^{1/2}Q_k u_n(t)\right\|_2^2 \leqslant \left\|A^{1/2}u_0\right\|_2^2 + \frac{1}{d\lambda_{k+1}}\left(\frac{1}{\mu}\,K_1\lambda_k^{3/2}U_L^2 + \frac{1}{\nu}\,L^2\right) \tag{2.22}$$

where we have used the fact that $\|A^{1/2}Q_k u_{n,0}\|_2^2 = \|Q_k P_n A^{1/2}u_0\|_2^2 \leqslant \|A^{1/2}u_0\|_2^2$. But $\|A^{1/2}P_k u_n\|_2^2 \leqslant \lambda_k\|P_k u_n\|_2^2 \leqslant \lambda_k\|u_n\|_2^2 \leqslant \lambda_k U_L^2$ so, since $\|A^{1/2}u_n\|_2^2 = \|A^{1/2}P_k u_n\|_2^2 + \|A^{1/2}Q_k u_n\|_2^2$, we have, combining with (2.22),

$$\begin{aligned}&\left\|A^{1/2}u_n(t)\right\|_2^2\\&\quad\leqslant \lambda_k U_L^2 + \left\|A^{1/2}u_0\right\|_2^2 + \frac{1}{d\lambda_{k+1}^2}\left(\frac{1}{\mu}\,K_1\lambda_k^{3/2}U_L^2 + \frac{1}{\nu}\,L^2\right) \equiv U_1^2.\end{aligned} \tag{2.23}$$

We thus obtain a uniform bound on $\|A^{1/2}u_n(t)\|_2$ for k large enough and $n \geqslant k$. The restriction on n which requires that $n > m$ is not severe as outlined in the remarks preceding Theorem 1.1. Replacing u_n by u and using similar (and in fact slightly simpler) arguments we obtain

$$\begin{aligned}&\left\|A^{1/2}u(t)\right\|_2^2\\&\quad\leqslant \lambda_k U_L^2 + \left\|A^{1/2}u_0\right\|_2^2 + \frac{1}{d\lambda_{k+1}}\left(\frac{1}{\mu}\,K_1\lambda_k^{3/2}U_L^2 + \frac{1}{\nu}\,L^2\right) \equiv U_1^2.\end{aligned} \tag{2.24}$$

With (2.23) and (2.24) and using (2.2) we obtain global H^1-bounds for (1.2) and (1.3). Refining our arguments further using (2.11) and (2.21) we have for $v = u$ or $v = u_n$,

$$\begin{aligned}&\left\|A^{1/2}v(t)\right\|_2^2\\&\quad\leqslant \lambda_k\|u_0\|_2^2\,e^{-\nu\lambda_1 t} + \left\|A^{1/2}u_0\right\|_2^2 e^{-d\lambda_{k+1}t}\\&\qquad\quad + \lambda_k\left(\frac{L}{\nu\lambda_1}\right)^2 + \frac{1}{d\lambda_{k+1}}\left(\frac{1}{\mu}\,K_1\lambda_k^{3/2}U_L^2 + \frac{1}{\nu}\,L^2\right).\end{aligned} \tag{2.25}$$

Next we take the inner product of both sides of (1.3) with $Q_k A^2 u_n$ for $m \leqslant k < n$ then we have similarly to the derivation of (2.13) that

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}\,\|AQ_k u_n\|_2^2 + \mu\|Q_k A^2 u_n\|_2^2\\&\quad + \nu\left\|Q_k A u_n^{3/2}\right\|_2^2 + \left(P_n P(u_n\cdot\nabla)\,u_n,\, Q_k A^2 u_n\right) = (f_n,\, Q_k A^2 u_n)\end{aligned} \tag{2.26}$$

from which we obtain similarly to (2.16) using calculations similar to (2.14) and (2.15),

$$\begin{aligned}&\frac{d}{dt}\,\|AQ_k u_n\|_2^2 + \mu\|Q_k A^2 u_n\|_2^2 + 2\nu\left\|Q_k A^{3/2}u_n\right\|_2^2\\&\qquad\qquad \leqslant \frac{2}{\mu}\,\|(u_n\cdot\nabla)\,u_n\|_2^2 + \frac{2}{\mu}\,\|f\|_2^2\end{aligned} \tag{2.27}$$

where we have used the fact that P and P_n are projections. We have from (2.2) that there exists a constant M_2 that $\|(u_n \cdot \nabla) u_n\|_2^2 \leqslant \|u_n\|_\infty^2 \|\nabla u_n\|_2^2 \leqslant M_2 \|Au_n\|_2^2 \|A^{1/2} u_n\|_2^2$ while $\|Au_n\|_2^2 = \|P_k A u_n\|_2^2 + \|Q_k A u_n\|_2^2 \leqslant \lambda_k^2 \|u_n\|_2^2 + \|Q_k A u_n\|_2^2$; combining this with (2.12), (2.23) and (2.27), neglecting the term $\mu \|Q_k A^2 u_n\|_2^2$ on the left-hand side, and using Poincaré's inequality again we have

$$\begin{aligned} &\frac{d}{dt} \|A Q_k u_n\|_2^2 + 2\nu\lambda_{k+1} \|A Q_k u_n\|_2^2 \\ &\qquad \leqslant \frac{2}{\mu} M_2 \lambda_k^2 U_L^2 U_1 + \frac{2}{\mu} M_2 \|Q_k A u_n\|_2^2 U_1 + \frac{2}{\mu} \|f\|_2^2. \end{aligned} \tag{2.28}$$

Now choose k (and hence n) large enough so that $(2/\mu) M_2 U_1 \leqslant \nu\lambda_{k+1}$ is satisfied as well, then subtracting from both sides of (2.28) we obtain

$$\frac{d}{dt} \|A Q_k u_n\|_2^2 + \nu\lambda_{k+1} \|A Q_k u_n\|_2^2 \leqslant \frac{2}{\mu} M_2 \lambda_k^2 U_L^2 U_1 + \frac{2}{\mu} L^2 \tag{2.29}$$

from which, proceeding as in the development of (2.21) and (2.22), we have

$$\|A Q_k u_n(t)\|_2^2 \leqslant \|Au_0\|_2^2 e^{-\nu\lambda_{k+1} t} + \frac{1}{\nu\lambda_{k+1}} \left(\frac{2}{\mu} M_2 \lambda_k^2 U_L^2 U_1 + \frac{2}{\mu} L^2 \right) \tag{2.30}$$

and then proceeding as in (2.25) we obtain for $v = u$ or $v = u_n$,

$$\begin{aligned} &\|Av(t)\|_2^2 \\ &\quad \leqslant \lambda_k^2 \|u_0\|_2^2 e^{-\nu\lambda_1 t} + \|Au_0\|_2^2 e^{-\nu\lambda_{k+1} t} \\ &\qquad\qquad + \lambda_k^2 \left(\frac{L}{\nu\lambda_1} \right)^2 + \frac{1}{\nu\lambda_{k+1}} \left(\frac{2}{\mu} M_2 \lambda_k^2 U_L^2 U_1 + \frac{2}{\mu} L^2 \right) \\ &\quad \equiv U_2^2. \end{aligned} \tag{2.31}$$

From (2.11), (2.25), and (2.31) we obtain absorbing-ball behavior of (1.2) and (1.3); in particular we have

$$\limsup_{t \to \infty} \|v(t)\|_2^2 \leqslant \left(\frac{L}{\nu\lambda_1} \right)^2 \equiv \rho_0, \tag{2.32}$$

$$\begin{aligned} &\limsup_{t \to \infty} \left\| A^{1/2} v(t) \right\|_2^2 \\ &\quad \leqslant \lambda_k \left(\frac{L}{\nu\lambda_1} \right)^2 + \frac{1}{d\lambda_{k+1}} \left(\frac{1}{\mu} K_1 \lambda_k^{3/2} U_L^2 + \frac{1}{\nu} L^2 \right) \equiv \rho_1, \end{aligned} \tag{2.33}$$

and

$$\limsup_{t\to\infty} \|Av(t)\|_2^2 \leqslant \lambda_k^2\left(\frac{L}{\nu\lambda_1}\right)^2 + \frac{1}{\nu\lambda_{k+1}}\left(\frac{2}{\mu}M_2\lambda_k^2U_L^2U_1 + \frac{2}{\mu}L^2\right) \equiv \rho_2. \quad (2.34)$$

We thus have from (2.32)–(2.34) that the balls of radius ρ_j about zero are absorbing sets in $D(A^{j/2})$ equipped with the norms $\|A^{j/2}v(t)\|_2$, $j = 0, 1, 2$. From (2.2) without loss of generality we can take the spaces $D(A^{j/2})$ defined with these norms as our Sobolev spaces of order 0, 1, 2, respectively. As in, e.g. [2] we thus have a globally regular solution for (1.2) that has a compact attractor of finite Hausdorff and fractal dimension.

For the proof of Theorem 1.8 we will as was done in [2] use [50, Theorem VIII.3.2], and as in [2] we refer to this result as Theorem GFST; it generalizes the conditions of the main theorems of [23,24] and applies to systems of the form

$$\begin{aligned} &\frac{du}{dt} + A_1u + R(u) = f, \\ &u(0) = u_0 \end{aligned} \quad (2.35)$$

for a linear operator A_1 with dense domain in a Hilbert space H, and R a bounded map from $D(A_1^\beta)$ into $D(A_1^{\beta-\gamma})$ for β, γ non-negative constants to be determined below. Theorem GFST requires the following conditions:

(1) For every $u_0 \in D(A_1^\beta)$, (2.35) has a unique solution $u \in C(\mathbb{R}^+; D(A_1^\beta)) \cap L^2((0, T)); D(A_1^{\beta+\gamma})$ and the mapping $S(t)\colon u_0 \to u(t)$ is continuous from $D(A_1^\beta)$ into itself.
(2) $S(t)$ possesses an absorbing set $\mathcal{B}_0$ in $D(A_1^\beta)$ which is positively invariant, i.e., $S(t)\mathcal{B}_0 \subset \mathcal{B}_0$ for all $t \geqslant 0$, and the ω-limit set of $\mathcal{B}_0$, denoted $\mathcal{A}$, is the maximal attractor for $S(\cdot)$ in $D(A_1^\beta)$.
(3) For some $\beta \geqslant 0$ and $\gamma \geqslant 0$ as in (5) below,

$$\left\|A_1^{\beta-\gamma}R(u) - A_1^{\beta-\gamma}R(v)\right\|_2 \leqslant C_M\left\|A_1^\beta(u - v)\right\|_2 \quad (2.36)$$

for all $u, v \in D(A^\beta)$, $\|A_1^\beta u\|_2 \leqslant M$, $\|A_1^\beta v\|_2 \leqslant M$.
(4) There exists a $\rho > 0$ such that the ball of radius $\rho/2$ centered at 0 in $D(A_1^\beta)$ is absorbing for (1.6).
(5) Let $\{\lambda_N^1\}$ be the eigenvalues of A_1, then there exists a function $K_{m_0} = K_{m_0}(N)$ such that for $N \geqslant m_0$,

$$\lambda_{N+1}^1 - \lambda_N^1 \geqslant K_{m_0}(N)\big((\lambda_{N+1}^1)^\gamma + (\lambda_N^1)^\gamma\big) \quad (2.37)$$

where $K_{m_0}(N) \to \infty$ as $N \to \infty$.

To see that the system (1.2) meets the conditions (1)–(5) required of (2.35), we note that (1) follows from (2.24) (or Theorem 1.1) and standard arguments, while (2) and

(4) follow from (2.32)–(2.34). For (3) and (5) we fit (1.2) into the format of (2.35) by setting $A_1 = \nu A + \mu A_\varphi$ as in [2] only now A is the Stokes operator and A_φ is as in (1.2), and we as in [2] take $\gamma = 1/2$ and $\beta \geqslant 5/(4\alpha) - 1/2$ in the case $3/2 \leqslant \alpha < 5/2$, while for $\alpha \geqslant 5/2$ we take $\beta = 0$. Taking $R(u) = P(u \cdot \nabla) u$ we have that (2.36) follows using (2.2) and arguments similar to those above used for the proof of Theorem 1.2 to handle the Leray projection P. Meanwhile, for (2.37) we take advantage as in [2] of a unique spectral-gap property inherent in the assumption that A_φ satisfies $P_m A_\varphi = 0$ and $Q_m A_\varphi \geqslant Q_m A^\alpha$; we take $N = m$ and note as in [2] that $\lambda^1_{m+1} \geqslant \mu \lambda^\alpha_{m+1}$ so that

$$\begin{aligned}\lambda^1_{N+1} - \lambda^1_N &= \left((\lambda^1_{m+1})^{1/2} - (\lambda^1_m)^{1/2}\right)\left((\lambda^1_{m+1})^{1/2} + (\lambda^1_m)^{1/2}\right) \\ &\geqslant \left((\mu\lambda^\alpha_{m+1})^{1/2} - (\nu\lambda_m)^{1/2}\right)\left((\lambda^1_{m+1})^{1/2} + (\lambda^1_m)^{1/2}\right).\end{aligned} \tag{2.38}$$

As noted, e.g. in [22, Section II.6], the eigenvalues of the Stokes operator have qualitatively the same asymptotic growth behavior as the eigenvalues of $B = -\Delta$; in particular there is a constant c such that $\lambda_n \sim c\lambda_1 n^{2/3}$ and we have as in [2, Section 4] that $\mu^{1/2}\lambda^{\alpha/2}_{m+1} - \nu^{1/2}\lambda^{1/2}_m \geqslant (1/2)\mu^{1/2}(c\lambda_1)^{\alpha/2} m^{\alpha/3}$ provided that $m^{(\alpha-1)/3} \geqslant 2(c\lambda_1)^{-(\alpha-1)/2}(\nu/\mu)^{1/2}$. The latter inequality is satisfied by choosing m_0 large enough (and hence m since $m \geqslant m_0$). Then by the preceding remarks (2.38) is satisfied and hence (5) follows; this completes our discussion of the proof of Theorem 1.8.

3 Proof of Theorems 1.1, 1.2, and 1.5

We first prove Theorem 1.1. Let $w = u - u_n$ then subtracting (1.3) from (1.2) we obtain the following equations for w:

$$w_t + \nu A w + \mu A_\varphi w + P_n P(u_n \cdot \nabla) w + P_n P(w \cdot \nabla) u = F_n + Q_n P(u \cdot \nabla) u \tag{3.1}$$

where $F_n = f - f_n$. As in the remarks preceding (2.5) without loss of generality we can also assume that $A_\varphi = Q_m A^2$. Taking the inner product of both sides of (3.1) with $Q_k A w$ for $m \leqslant k \leqslant n$ and proceeding similarly to the development of (2.16) we obtain

$$\begin{aligned}&\frac{d}{dt}\|A^{1/2} Q_k w\|_2^2 + \mu \|Q_k A^{3/2} w\|_2^2 + \nu \|Q_k A w\|_2^2 \\ &\quad\leqslant \frac{4}{\mu}\|A^{-1/2} P(u_n \cdot \nabla) w\|_2^2 + \frac{4}{\mu}\|A^{-1/2} P(w \cdot \nabla) u\|_2^2 \\ &\qquad + \frac{4}{\mu}\|Q_n A^{-1/2} P(u \cdot \nabla) u\|_2^2 + \frac{4}{\nu}\|F_n\|_2^2.\end{aligned} \tag{3.2}$$

We have, using (2.12), (2.17), and remarks following (2.17), $\|A^{-1/2} P(u_n \cdot \nabla) w\|_2^2 \leqslant K_1 \|u_n\|_2^2 \|A^{3/4} w\|_2^2 \leqslant K_1 U_L^2 \|A^{3/4} w\|_2^2$ and $\|A^{-1/2} P(w \cdot \nabla) u\|_2^2 \leqslant K_1 \|u\|_2^2 \|A^{3/4} w\|_2^2 \leqslant K_1 U_L^2 \|A^{3/4} w\|_2^2$. Combining this with (2.2) and Poincaré's inequality applied to the left-hand side of (3.2), together with $\|A^{3/4}w\|_2^2 = \|P_k A^{3/4}w\|_2^2 + \|Q_k A^{3/4}w\|_2^2$ we have from (3.2),

$$\begin{aligned}\frac{d}{dt}\left\|A^{1/2}Q_k w\right\|_2^2 + \mu\lambda_{k+1}^{3/2}\left\|Q_k A^{3/4}w\right\|_2^2 + \nu\lambda_{k+1}\left\|Q_k A^{1/2}w\right\|_2^2 \\ \leqslant \frac{8}{\mu}K_1 U_L^2\left(\left\|P_k A^{3/4}w\right\|_2^2 + \left\|Q_k A^{3/4}w\right\|_2^2\right) \\ + \frac{4}{\mu}\left\|Q_n A^{-1/2}P(u\cdot\nabla)u\right\|_2^2 + \frac{4}{\nu}\|F_n\|_2^2.\end{aligned} \tag{3.3}$$

We choose k large enough so that $(8/\mu)K_1U_L^2 \leqslant (\mu/2)\lambda_{k+1}^{3/2}$, then subtracting from both sides of (3.3) and using that $\|P_k A^{3/4}w\|_2^2 \leqslant \lambda_k^{1/2}\|P_k A^{1/2}w\|_2^2$ we have

$$\begin{aligned}\frac{d}{dt}\left\|A^{1/2}Q_k w\right\|_2^2 + \frac{\mu}{2}\lambda_{k+1}^{3/2}\left\|Q_k A^{3/4}u_n\right\|_2^2 + \nu\lambda_{k+1}\left\|Q_k A^{1/2}w\right\|_2^2 \\ \leqslant \frac{8}{\mu}K_1 U_L^2\lambda_k^{1/2}\left\|P_k A^{1/2}w\right\|_2^2 + \frac{4}{\mu}\left\|Q_n A^{-1/2}P(u\cdot\nabla)u\right\|_2^2 + \frac{4}{\nu}\|F_n\|_2^2.\end{aligned} \tag{3.4}$$

We will need to use (3.4) below; meanwhile for the purpose at hand we neglect the term $(\mu/2)\lambda_{k+1}^{3/2}\|Q_k A^{3/4}u_n\|_2^2$ on the left-hand side of (3.4) and integrate both sides to obtain

$$\begin{aligned}&\left\|A^{1/2}Q_k w(t)\right\|_2^2 \\ &\quad\leqslant \left\|A^{1/2}Q_k w(0)\right\|_2^2 e^{-\nu\lambda_{k+1}t} \\ &\qquad + \int_0^t \left(\frac{8}{\mu}K_1 U_L^2\left\|P_k A^{1/2}w\right\|_2^2\right. \\ &\qquad\qquad \left. + \frac{4}{\mu}\left\|Q_n A^{-1/2}P(u\cdot\nabla)u\right\|_2^2 + \frac{4}{\nu}\|F_n\|_2^2\right) e^{-\nu\lambda_{k+1}(t-s)}\,ds.\end{aligned} \tag{3.5}$$

Clearly $\|A^{1/2}Q_k w(0)\|_2^2 = \|A^{1/2}Q_k(u-u_n)(0)\|_2^2 \to 0$ as $n \to \infty$ and $\|F_n(s)\|_2 \to 0$ as $n \to \infty$ for each s. Similarly to (2.17), from (2.2) and (2.24) it follows that there is a constant M_3 such that $\|A^{-1/2}P(u\cdot\nabla)u\|_2^2 \leqslant \|T_\mathrm{b}\|_2^2\|u\otimes u\|_2^2 \leqslant \|T_\mathrm{b}\|_2^2\|u\|_4^4 \leqslant \|T_\mathrm{b}\|_2^2 M_3\|A^{1/2}u\|_4^4 \leqslant \|T_\mathrm{b}\|_2^2 M_3 U_1^4$; since $\|A^{-1/2}P(u\cdot\nabla)u\|_2^2$ is therefore well-defined and bounded we have $\|Q_n A^{-1/2}P(u(s)\cdot\nabla)u(s)\|_2 \to 0$ as $n \to \infty$ for each s. Thus the convergence of the right-hand side of (3.5) to zero as $n \to \infty$ follows by the dominated convergence theorem if we can show that $\|P_k A^{1/2}w(s)\|_2^2 \to 0$ as $n \to \infty$ for each s.

For this we first obtain an estimate for the entire difference w. Taking the inner product of both sides of (3.1) with Aw we obtain similarly to (3.3),

$$\begin{aligned}\frac{d}{dt}\left\|A^{1/2}w\right\|_2^2+\mu\lambda_{k+1}^{3/2}\left\|Q_mA^{3/4}w\right\|_2^2\\ \leqslant\frac{8}{\mu}K_1U_L^2\big(\left\|P_kA^{3/4}w\right\|_2^2+\left\|Q_kA^{3/4}w\right\|_2^2\big)\\ +\frac{4}{\mu}\left\|Q_nA^{-1/2}P(u\cdot\nabla)u\right\|_2^2+\frac{4}{\nu}\left\|F_n\right\|_2^2\end{aligned}\tag{3.6}$$

where we have again used the decomposition $\|A^{3/4}w\|_2^2 = \|P_kA^{3/4}w\|_2^2+\|Q_kA^{3/4}w\|_2^2$ and discarded the term $\nu\|Aw\|_2^2$ on the left-hand side. Again with k large enough so that $(8/\mu)K_1U_L^2\leqslant(\mu/2)\lambda_{k+1}^{3/2}$, we subtract from both sides of (3.6) and obtain similarly to (3.5),

$$\begin{aligned}&\left\|A^{1/2}w(t)\right\|_2^2\\ &\leqslant\left\|A^{1/2}w(0)\right\|_2^2e^{-\nu\lambda_{k+1}t}\\ &\quad+\int_0^t\left(\frac{8}{\mu}K_1U_L^2\lambda_k^{1/2}\left\|P_kA^{1/2}w\right\|_2^2\right.\\ &\quad\left.+\frac{4}{\mu}\left\|Q_nA^{-1/2}P(u\cdot\nabla)u\right\|_2^2+\frac{4}{\nu}\left\|F_n\right\|_2^2\right)e^{-\nu\lambda_{k+1}(t-s)}ds.\end{aligned}\tag{3.7}$$

Using the fact that $\|P_kA^{1/2}w(t)\|_2^2\leqslant\|A^{1/2}w(t)\|_2^2$ and neglecting the exponential terms, where not needed, we obtain from (3.7) for all t in an interval $[0, T]$,

$$\begin{aligned}&\left\|P_kA^{1/2}w(t)\right\|_2^2\\ &\leqslant\left\|A^{1/2}w_0\right\|_2^2\\ &\quad+\int_0^T\left(\frac{4}{\mu}\left\|Q_nA^{-1/2}P(u\cdot\nabla)u\right\|_2^2+\frac{4}{\nu}\left\|F_n\right\|_2^2\right)e^{-\nu\lambda_{k+1}(t-s)}ds\\ &\quad+\int_0^t\frac{8}{\mu}K_1U_L^2\lambda_k^{1/2}\left\|P_kA^{1/2}w(s)\right\|_2^2ds.\end{aligned}\tag{3.8}$$

Applying Gronwall's inequality to (3.8) we obtain for

$$U_{Q,F}(s)=\frac{4}{\mu}\left\|Q_nA^{-1/2}P(u\cdot\nabla)u\right\|_2^2+\frac{4}{\nu}\left\|F_n\right\|_2^2,$$

the inequality

$$\begin{aligned}&\left\|P_kA^{1/2}w(t)\right\|_2^2\\ &\leqslant\left[\|A^{1/2}w_0\|_2^2+\int_0^TU_{Q,F}(s)e^{-\nu\lambda_{k+1}(t-s)}ds\right]e^{8K_1U_L^2\lambda_k^{1/2}t/\mu}\end{aligned}\tag{3.9}$$

and by the same observations as those following (3.5), $\|P_k A^{1/2}(u(t)-u_n(t))\|_2^2 \to 0$ uniformly on $[0, T]$ as $n \to \infty$, and hence $\|A^{1/2}(u(t)-u_n(t))\|_2^2 \to 0$ uniformly on $[0, T]$ as $n \to \infty$. Thus Theorem 1.1 follows.

In the case of decaying turbulence, in the transition from (3.2) to (3.3) we retain the term $\|u_n\|_2^2 + \|u\|_2^2$ rather than using (2.12) and from Poincaré's inequality use that $\|u_n\|_2^2 + \|u\|_2^2 \leqslant \lambda_1^{-1}(\|Au_n^{1/2}\|_2^2 + \|A^{1/2}u\|_2^2)$. We then proceed as in the developments resulting in (3.9) but now the exponential term is replaced by $\exp \frac{8}{\mu} K_1 U_L^2 \lambda_k^{1/2} \lambda_1^{-1} \int_0^t (\|Au_n^{1/2}\|_2^2 + \|A^{1/2}u\|_2^2)\, ds$. Meanwhile, from the remarks following (3.5) and from (2.24) we have

$$\left\| Q_n A^{-1/2} P(u\cdot\nabla)u \right\|_2^2 \leqslant \|T_{\mathrm{b}}\|_2^2 M_3 \|A^{1/2}u\|_2^2 \|A^{1/2}u\|_2^2 \leqslant \|T_{\mathrm{b}}\|_2^2 M_3 \|A^{1/2}u\|_2^2 U_1^2$$

and so by (2.7) both $\|Q_n A^{-1/2} P(u\cdot\nabla)u\|_2^2$ and $\|F_n\|_2^2$ are integrable on $[0, +\infty)$. Hence from (2.7) and the above remarks we have for all $t \in [0, +\infty)$,

$$\begin{aligned} &\left\| P_k A^{1/2} w(t) \right\|_2^2 \\ &\quad \leqslant \left[\left\| A^{1/2} w_0 \right\|_2^2 + \int_0^\infty C_n(u)\, ds \right] e^{\frac{2}{\nu}\left(\frac{8}{\mu} K_1 U_L^2 \lambda_k^{1/2} \lambda_1^{-1}\right)\left(\|u_0\|_2^2 + \frac{1}{\nu\lambda_1}\int_0^\infty \|f\|_2^2 ds\right)} \end{aligned} \tag{3.10}$$

which is (1.8), where again $C_n(u) = \frac{4}{\mu}\|Q_n A^{-1/2} P(u\cdot\nabla)u\|_2^2 + \frac{4}{\nu}\|F_n\|_2^2$. Again by the dominated convergence theorem, $\|A^{1/2} P_k w(t)\|_2^2 \to 0$ as $n \to \infty$ uniformly on $[0, +\infty)$. Since $\|P_k A^{1/2} w(t)\|_2^2$ is shown to be bounded in (3.10) as well, and since the other terms in the integrand of (3.5) are bounded, the presence of the term $e^{-\nu\lambda_{k+1}(t-s)} ds$ in the integrand shows that $\|A^{1/2} Q_k w(t)\|_2^2$ is uniformly bounded and that $\|A^{1/2} Q_k w(t)\|_2^2 \to 0$ as $n \to \infty$ uniformly on $[0, +\infty)$. Thus $\|A^{1/2} w(t)\|_2^2 \to 0$ as $n \to \infty$ uniformly on $[0, +\infty)$, which proves Theorem 1.2.

With these new techniques now established we sketch the proof of Theorem 1.5. Let u and v be two solutions of (1.2) and let $w = u - v$. Then proceeding similarly as in (3.3) we obtain

$$\begin{aligned} &\frac{d}{dt}\left\| A^{1/2} Q_k w \right\|_2^2 + \mu\lambda_{k+1}^{3/2} \left\| Q_k A^{3/4} w \right\|_2^2 + \nu \|Q_k A w\|_2^2 \\ &\quad \leqslant \frac{6}{\mu} K_1 U_L^2 \left\| A^{3/4} P_k w \right\|_2^2 + \frac{6}{\mu} K_1 U_L^2 \left\| A^{3/4} Q_k w \right\|_2^2 + \frac{4}{\nu}\|f-g\|_2^2. \end{aligned} \tag{3.11}$$

Choosing $k \geqslant m$ large enough so that $(6/\mu) K_1 U_L^2 \leqslant (\mu/2)\lambda_{k+1}^{3/2}$, we subtract from both sides of (3.11), discard the terms $(\mu/2)\lambda_{k+1}^{3/2}\|A^{3/4} Q_k w\|_2^2$ and $\nu\|Q_k A w\|_2^2$ on the left-hand side, use that $\|A^{3/4} P_k w\|_2^2 = \|P_k A^{3/4} w\|_2^2 \leqslant \lambda_k^{3/2}\|P_k w\|_2^2$ and integrate both sides from t_0 to t for some $t_0 \geqslant 0$ to obtain similarly to (3.5),

$$
\begin{aligned}
&\left\|A^{1/2}Q_k(u-v)(t)\right\|_2^2 \\
&\quad\leqslant \left\|A^{1/2}Q_k(u-v)(t_0)\right\|_2^2 e^{-t\nu\lambda_{k+1}} \\
&\qquad + \int_{t_0}^{t}\left(\frac{6}{\mu}K_1U_L^2\lambda_k^{3/2}\|P_k(u-v)(s)\|_2^2 + \frac{4}{\nu}\|f(s)-g(s)\|_2^2\right)e^{-\nu\lambda_{k+1}(t-s)}\,ds
\end{aligned}
\tag{3.12}
$$

which is (1.11). With Theorem 1.5 thus established we conclude the discussion in this section.

4 Asymptotic convergence and stability in the case of decaying turbulence

We integrate (2.8) from $t_0 \geqslant 0$ to $t \geqslant t_0$ to obtain

$$
\|u(t)\|_2^2 \leqslant \|u(t_0)\|_2^2\, e^{-\nu\lambda_1(t-t_0)} + \frac{1}{\nu\lambda_1}\int_{t_0}^{t}\|f(s)\|_2^2\, e^{-\nu\lambda_1(t-s)}\,ds. \tag{4.1}
$$

We recall the following result [6, Theorem 3]:

Theorem 4.1 *Let* (1.2) *be such that f satisfies* (1.7)*, then*

$$
\|A^{\beta/2}u(t)\|_2 \to 0 \quad as \quad t \to \infty \tag{4.2}
$$

for all $\beta \geqslant 0$ *.*

We first prove Theorem 1.4; adaptations of the techniques established will be used to prove Theorem 1.3. Again we assume without loss of generality that $A_\varphi = Q_m A^2$. For solutions u and v of (1.2) we let $w = u - v$, subtract the two versions of (1.2), take the inner product of both sides with $AQ_k w$ for $k \geqslant m$, and obtain similarly to the development of (3.3),

$$
\begin{aligned}
\frac{d}{dt}\left\|A^{1/2}Q_k w\right\|_2^2 + \mu\lambda_{k+1}^{3/2}\left\|Q_k A^{3/4}w\right\|_2^2 + \nu\|Q_k A w\|_2^2 \\
\leqslant \frac{2}{\mu}K_1\left[\|u\|_2^2 + \|v\|_2^2\right]\|A^{3/4}w\|_2^2 \\
\leqslant \frac{2}{\mu\lambda_1^{1/2}}K_1\left[\|u\|_2^2 + \|v\|_2^2\right]\|Aw\|_2^2
\end{aligned}
\tag{4.3}
$$

where we have again used Poincaré's inequality. Taking the inner product of both sides of (1.2) with $AP_k w$, the term $\mu\|Q_k A^{3/2}w\|_2^2$ is now replaced by $\mu\|(P_k - P_m)A^{3/2}w\|_2^2$ which is best discarded. We note that

$$
\begin{aligned}
(P(w\cdot\nabla)u,\, AP_k u) &= \left(A^{-1/2}P(w\cdot\nabla)u,\, A^{3/2}P_k u\right) \\
&\leqslant \|A^{-1/2}P(w\cdot\nabla)u\|_2\,\|A^{3/2}P_k u\|_2
\end{aligned}
$$

but $\|A^{3/2}P_k u\|_2 \leqslant \lambda_k^{1/2}\|AP_k u\|_2$ so by Young's inequality

$$\left|\left(A^{-1/2}P(w\cdot\nabla)u,\, A^{3/2}P_m u\right)\right| \leqslant \frac{\nu}{4}\,\|AP_m u\|_2^2 + \frac{\lambda_k}{\nu}\,\left\|A^{-1/2}P(w\cdot\nabla)u\right\|_2^2 \quad (4.4)$$

with a similar estimate for the term $(P(u\cdot\nabla)w,\, AP_k u)$. We combine these estimates with (2.17), subtract the term $\frac{\nu}{2}\|AP_k u\|_2^2$, and multiply by 2 to obtain

$$\begin{aligned}\frac{d}{dt}\,\|A^{1/2}P_k w\|_2^2 + \nu\|P_k A w\|_2^2 &= \frac{4\lambda_k}{\nu}\,K_1\big[\|u\|_2^2 + \|v\|_2^2\big]\|A^{3/4}w\|_2^2 \\ &\leqslant \frac{4\lambda_k}{\nu\lambda_1^{1/2}}\,K_1\big[\|u\|_2^2 + \|v\|_2^2\big]\|Aw\|_2^2.\end{aligned} \quad (4.5)$$

Adding (4.5) to (4.3) and omitting the term $\mu\lambda_{k+1}^{3/2}\|Q_k A^{3/4}w\|_2^2$ we obtain

$$\frac{d}{dt}\,\|A^{1/2}w\|_2^2 + \nu\|Aw\|_2^2 \leqslant \left[\frac{2}{\mu\lambda_1^{1/2}} + \frac{4\lambda_m}{\nu\lambda_1^{1/2}}\right]K_1\big[\|u\|_2^2 + \|v\|_2^2\big]\|Aw\|_2^2. \quad (4.6)$$

Using Theorem 4.1 with $\beta = 0$ we assume that t_1 is large enough so that for all $t \geqslant t_1$,

$$\left[\frac{2}{\mu\lambda_1^{1/2}} + \frac{4\lambda_m}{\nu\lambda_1^{1/2}}\right]K_1\big[\|u(t)\|_2^2 + \|v(t)\|_2^2\big] \leqslant \frac{\nu}{2}, \quad (4.7)$$

then combining with (4.6) and again using Poincaré's inequality we have

$$\frac{d}{dt}\,\|A^{1/2}w(t)\|_2^2 + \frac{\nu}{2}\,\lambda_1\|A^{1/2}w(t)\|_2^2 \leqslant 0 \quad (4.8)$$

for all $t \geqslant t_1$, from which it is now straightforward to integrate from t_1 to t to obtain

$$\|A^{1/2}w(t)\|_2^2 \leqslant \|A^{1/2}w(t_1)\|_2^2\, e^{-(\nu/2)\lambda_1(t-t_1)}. \quad (4.9)$$

We note that the same estimate holds for $w = u_n - v_n$ where u_n and v_n are two solutions of (1.3); the time t_1 can be chosen independently of n since $0 \leqslant \int_0^t \|f_n(s)\|_2^2 e^{-\nu\lambda_1(t-s)}\,ds \leqslant \int_0^t \|f(s)\|_2^2\, e^{-\nu\lambda_1(t-s)}\,ds$. Thus (1.10) is established.

For the second statement of Theorem 1.4 we subtract the u- and v-versions of (1.2), take the inner product of both sides with Aw, and obtain similarly to (3.6),

$$\begin{aligned}&\frac{d}{dt}\,\|A^{1/2}w\|_2^2 + \nu\lambda_{k+1}\|Q_k A^{1/2}w\|_2^2 \\ &\qquad\leqslant \frac{4}{\mu}\,K_1 U_L^2\big(\|P_k A^{3/4}w\|_2^2 + \|Q_k A^{3/4}w\|_2^2\big)\end{aligned} \quad (4.10)$$

where we have used again $\|A^{3/4}w\|_2^2 = \|P_k A^{3/4}w\|_2^2 + \|Q_k A^{3/4}w\|_2^2$ and discarded the term $\mu\lambda_{k+1}^{3/2}\|P_k A^{3/4}w\|_2^2$ on the left-hand side. With k large enough so that

$(4/\mu)K_1U_L^2 \leqslant (\mu/2)\lambda_{k+1}^{3/2}$, we subtract from both sides of (4.10) and obtain similarly to (3.7),

$$\begin{aligned} &\|A^{1/2}w(t)\|_2^2 \\ &\quad\leqslant \|A^{1/2}w(0)\|_2^2 e^{-\nu\lambda_{k+1}t} \\ &\qquad\qquad + \int_0^t \frac{4}{\mu} K_1U_L^2\lambda_k^{1/2}\|P_kA^{1/2}w\|_2^2 e^{-\nu\lambda_{k+1}(t-s)}\,ds. \end{aligned} \tag{4.11}$$

Using the fact that $\|P_kA^{1/2}w(t)\|_2^2 \leqslant \|A^{1/2}w(t)\|_2^2$ on the right-hand side of (4.11) and neglecting the exponential factors we obtain for all $t \geqslant 0$,

$$\|A^{1/2}w(t)\|_2^2 \leqslant \|A^{1/2}w(0)\|_2^2 + \int_0^t \frac{4}{\mu} K_1U_L^2\lambda_k^{1/2}\|A^{1/2}w(s)\|_2^2\,ds. \tag{4.12}$$

Applying Gronwall's inequality to (4.12) we obtain

$$\|A^{1/2}w(t)\|_2^2 \leqslant \|A^{1/2}w(0)\|_2^2 e^{4K_1U_L^2\lambda_k^{1/2}t_1/\mu} \tag{4.13}$$

for all $t \in [0, t_1]$. Since $w(0) = u_0 - v_0$, we obtain the second statement of Theorem 1.4 from (4.13); note that similar arguments apply to solutions of (1.3).

To prove Theorem 1.3, for solutions u of (1.2) and u_n of (1.3) we let $w = u - u_n$ and $F_n = f - f_n$ as before and obtain similarly to the development of (4.6) that for $k \geqslant n$,

$$\begin{aligned} &\frac{d}{dt}\|A^{1/2}w\|_2^2 + \nu\|Aw\|_2^2 \\ &\quad\leqslant \frac{4}{\mu}\|Q_nA^{-1/2}P(u\cdot\nabla)u\|_2^2 + \frac{4}{\nu}\|F_n\|_2^2 \\ &\qquad\qquad + 4\left[\frac{1}{\mu\lambda_1^{1/2}} + \frac{\lambda_m}{\nu\lambda_1^{1/2}}\right]K_1\left[\|u\|_2^2 + \|u_n\|_2^2\right]\|Aw\|_2^2. \end{aligned} \tag{4.14}$$

Using Theorem 4.1 with $\beta = 0$ we assume that $t \geqslant t_1$ where t_1 is large enough so that

$$4\left[\frac{1}{\mu\lambda_1^{1/2}} + \frac{\lambda_k}{\nu\lambda_1^{1/2}}\right]K_1\left[\|u(t)\|_2^2 + \|u_n(t)\|_2^2\right] \leqslant \frac{\nu}{2} \tag{4.15}$$

for all $t \geqslant t_1$, then combining with (4.14) and using Poincaré's inequality as before we have

$$\begin{aligned} &\frac{d}{dt}\|A^{1/2}w(t)\|_2^2 + \frac{\nu}{2}\lambda_1\|A^{1/2}w(t)\|_2^2 \\ &\qquad\qquad \leqslant \frac{4}{\mu}\|Q_nA^{-1/2}P(u\cdot\nabla)u\|_2^2 + \frac{4}{\nu}\|F_n\|_2^2 \end{aligned} \tag{4.16}$$

for all $t \geqslant t_1$. Note that by the remarks following (4.9) the same time t_1 can be chosen so that (4.15) and hence (4.16) holds independently of n and simultaneously for u and u_n. For $t_1 \leqslant t_0 \leqslant t$ we then integrate (4.16) from t_0 to t to obtain

$$\begin{aligned} &\|A^{1/2}w(t)\|_2^2 \\ &\quad\leqslant \|A^{1/2}w(t_0)\|_2^2 e^{-(\nu/2)\lambda_1(t-t_0)} \\ &\qquad + \int_{t_0}^{t} \left(\frac{4}{\mu}\|Q_n A^{-1/2}P(u\cdot\nabla)u\|_2^2 + \frac{4}{\nu}\|F_n\|_2^2\right) e^{-(\nu/2)\lambda_1(t-s)}\,ds \end{aligned} \tag{4.17}$$

for all $t \geqslant t_0$, which is (1.9).

To obtain the estimate discussed in the remarks following the statement of Theorem 1.3, we have from (2.17) and the remarks following that $\|Q_n A^{-1/2}P(u\cdot\nabla)u\|_2^2 \leqslant K_1\|u\|_2^2\,\|A^{3/4}u\|_2^2$, but using (2.2) together with the standard interpolation inequality $\|u\|_{\beta,2}^2 \leqslant \|u\|_{\theta\beta,2}^{1/\theta}\,\|u\|_2^{1-1/\theta}$ with $\theta = 4/3$, there exists a constant $K_3^{'}$ such that $\|A^{3/4}u\|_2^2 \leqslant K_3^{'}\|u\|_2^{1/2}\|Au\|_2^{3/2}$ which after applying (2.31) we use in (4.17) with $K_3 \equiv K_1K_3^{'}$ to obtain

$$\begin{aligned} &\|A^{1/2}w(t)\|_2^2 \\ &\quad\leqslant \|A^{1/2}w(t_0)\|_2^2 e^{-(\nu/2)\lambda_1(t-t_0)} \\ &\qquad + \int_{t_0}^{t} \left(\frac{4}{\mu}K_3U_{2,L}^3\|u\|_2^{5/2} + \frac{4}{\nu}\|F_n\|_2^2\right) e^{-(\nu/2)\lambda_1(t-s)}\,ds \end{aligned} \tag{4.18}$$

for all $t \geqslant t_0$. In particular, we have by changing variables that $\int_{t_0}^{t} e^{-(\nu/2)\lambda_1(t-s)}\,ds = \int_0^{t-t_0} e^{-(\nu/2)\lambda_1 s}\,ds \leqslant \int_0^{\infty} e^{-(\nu/2)\lambda_1 s}\,ds = 2/(\nu\lambda_1)$ so that from (4.18) we have

$$\begin{aligned} &\|A^{1/2}w(t)\|_2^2 \\ &\quad\leqslant \|A^{1/2}w(t_0)\|_2^2 e^{-(\nu/2)\lambda_1(t-t_0)} + \frac{8}{\mu\nu\lambda_1}K_3U_{2,L}^3\sup_{t\geqslant t_0}\|u(t)\|_2^{5/2} \\ &\qquad + \int_{t_0}^{t} \frac{4}{\nu}\|F_n\|_2^2 e^{-(\nu/2)\lambda_1(t-s)}\,ds \end{aligned} \tag{4.19}$$

for all $t \geqslant t_0$. With this estimate established we conclude our discussion in this section.

5 Attractor estimates for the SHNSE

With $H = PL^2(\Omega)$ as above we write (1.2) as $du(t)/dt = F(u(t))$, $t > 0$, $u(0) = u_0$ with solution $S(t)\colon u_0 \in H \to H$; the linearized problem is $dU(t)/dt = F'[S(t)u_0]\cdot U(t)$, $U(0) = \xi \in H$. For u_0 fixed in H, let $\xi_1, \ldots, \xi_M$ be M elements of H and let $U_1, \ldots, U_M$ be the corresponding solutions of the linearized problem. Let $q_M = q_M(t, u_0; \xi_1, \ldots, \xi_M)$ be the projection $q_M H = \operatorname{span}\{U_1, \ldots, U_M\}$, and let $\varphi_1(t), \dots, \varphi_M(t)$ be an orthonormal basis for $q_M(t)H$. We need to find M so that uniformly in space and asymptotically in time

$$\begin{aligned} 0 \geqslant \operatorname{Tr} F'(S(t)u_0) \circ q_M(t) &= \sum_{j=1}^{\infty} \big(\operatorname{Tr} F'(u(t)) \circ q_M(t)\varphi_j(t), \varphi_j(t)\big) \\ &= \sum_{j=1}^{M} (F'(u(t))\varphi_j(t), \varphi_j(t)). \end{aligned} \tag{5.1}$$

Now, since $\varphi_j \in H$, we have $(P(\varphi_j \cdot \nabla)u, \varphi_j) = ((\varphi_j \cdot \nabla)u, P\varphi_j) = ((\varphi_j \cdot \nabla)u, \varphi_j)$; we recall also that A commutes with Q_m and that $Q_m^2 = Q_m$. Thus

$$\begin{aligned} (F'(u)\cdot\varphi_j, \varphi_j) =& -\nu(A\varphi_j, \varphi_j) - \mu(A_\varphi\varphi_j, \varphi_j) - (P(\varphi_j \cdot \nabla)u, \varphi_j) \\ \geqslant& -\nu(A\varphi_j, \varphi_j) - \mu(Q_m A^\alpha \varphi_j, \varphi_j) - (P(\varphi_j \cdot \nabla)u, \varphi_j) \\ =& -\nu\big(A^{1/2}\varphi_j, A^{1/2}\varphi_j\big) - \mu\big(Q_m A^{\alpha/2}\varphi_j, Q_m A^{\alpha/2}\varphi_j\big) - ((\varphi_j \cdot \nabla)u, \varphi_j). \end{aligned} \tag{5.2}$$

We have as in [48,49],

$$\left| \sum_{j=1}^{M} ((\varphi_j \cdot \nabla)u, \varphi_j) \right| \leqslant |Du(x)|\rho(x) \tag{5.3}$$

where

$$|Du(x)| = \left\{ \sum_{j,k=1}^{3} |D_i u_k(x)|^2 \right\}^{1/2} \tag{5.4}$$

and

$$\rho(x) = \sum_{i=1}^{3} \sum_{j=1}^{M} (\varphi_{ji}(x))^2. \tag{5.5}$$

Combining (5.2)–(5.5) with (5.1) we have

$$\begin{aligned} &\big|\operatorname{Tr} F'(S(t)u_0) \circ q_M(t)\big| \\ &\quad \leqslant -\nu \sum_{j=1}^{M} \big\|A^{1/2}\varphi_j(t)\big\|_2^2 - \mu \sum_{j=1}^{M} \big\|Q_m A^{\alpha/2}\varphi_j(t)\big\|_2^2 + \int_\Omega |Du|\rho\, dx. \end{aligned} \tag{5.6}$$

We want to apply the generalized Lieb–Thirring inequality developed in [48,49] (see also [50]) in much the same way as we did in [2]; in order to do so we will need to show that $a(u, v) \equiv (A^{\alpha/2}u, A^{\alpha/2}v)$ defines a quadratic form satisfying the conditions prescribed in, e.g., [50]. The most direct path to do this turns out to be through the relation $A^2 u = P(-\Delta)^2 u$ which we establish in the next section. This quadratic form is restricted in, e.g., [50] to a closed subspace V of $H^s(\Omega)$ (in our case $s = 2$) for which H should be the closure of V in $L^2(\Omega)$. This is of course meant to include standard Hilbert spaces used in NSE theory and includes H as defined above. Thus it is reasonable to specify that for smooth enough $u, v \in H$ and for natural-number-valued choices of $\alpha \geqslant 2$, by iterating the identity $A^2 u = P(-\Delta)^2 u$, we have that $a(u, v) \equiv (A^{\alpha/2}u, A^{\alpha/2}v) = (A^{\alpha}u, v) = (P(-\Delta)^{\alpha}u, v)$. But $(P(-\Delta)^{\alpha}u, v) = ((-\Delta)^{\alpha}u, Pv) = ((-\Delta)^{\alpha}u, v) = (B^{\alpha/2}u, B^{\alpha/2}v)$ with $B = -\Delta$ as above. This directly fits into the examples of suitable quadratic forms identified in [50], and in particular $a(\varphi_j, \varphi_j) = \|B^{\alpha/2}\varphi_j(t)\|_2^2 = \|A^{\alpha/2}\varphi_j(t)\|_2^2$. Thus the generalized Lieb–Thirring inequality of order $m = \alpha$ [50, Theorem A.4.1, Remark A.4.2] can be applied to show that for $q = 1 + 3/(2\alpha)$ there exists a constant κ_2 such that

$$\int_{\Omega} \rho(x)^{(2\alpha+3)/3}\, dx \leqslant \kappa_2 \sum_{j=1}^{M} \left\| A^{\alpha/2}\varphi_j(t) \right\|_2^2. \tag{5.7}$$

The constant κ_2 depends only on q, α, and the shape (but not the size) of Ω. With these developments, with (5.6), (5.7), and Young's inequality we can proceed as in the remarks following [2, Theorem 9] to obtain

$$\begin{aligned} &\left|\mathrm{Tr}\, F'(S(t)u_0) \circ q_M(t)\right| \\ &\qquad \leqslant -\nu \sum_{j=1}^{M} \left\| A^{1/2}\varphi_j(t) \right\|_2^2 - \mu \sum_{j=1}^{M} \left\| Q_m A^{\alpha/2}\varphi_j(t) \right\|_2^2 \\ &\qquad\qquad + \epsilon_p \kappa_1 \sum_{j=1}^{M} \left\| A^{\alpha/2}\varphi_j(t) \right\|_2^2 + c_p \|Du\|_q^q. \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} p &= \frac{q}{q-1} = \frac{2\alpha+3}{3}, \\ \epsilon_p &= \frac{\mu}{2\lambda_m^{\alpha-1}\kappa_1}, \\ c_p &= \frac{p-1}{p^q \epsilon_p^{1/(p-1)}} \\ &= \frac{2\alpha}{2\alpha+3}\left(\frac{3}{2\alpha+3}\right)^{3/(2\alpha)} \left(\frac{2\lambda_m^{\alpha-1}\kappa_1}{\mu}\right)^{3/(2\alpha)} \equiv c_\alpha \left(\frac{2\lambda_m^{\alpha-1}\kappa_1}{\mu}\right)^{3/(2\alpha)}. \end{aligned}$$

This is the direct analogue of [2, (3.13)] but now established for solutions of (1.2).

We first establish (1.14); we subtract and add $\mu \sum_{j=1}^{M} \|P_m A^{\alpha/2}\varphi_j(t)\|_2^2$ to both sides of (5.6) and use the estimate $\|P_m A^{\alpha/2}\varphi_j(t)\|_2^2 \leqslant \lambda_m^{\alpha-1} \|A^{1/2} P_m \varphi_j(t)\|_2^2$ to obtain

$$\begin{aligned}&\left|\mathrm{Tr}\, F'(S(t)u_0)\circ q_M(t)\right|\\&\leqslant -\nu\sum_{j=1}^{M}\left\|A^{1/2}\varphi_j(t)\right\|_2^2-\mu\sum_{j=1}^{M}\left\|A^{\alpha/2}\varphi_j(t)\right\|_2^2\\&\qquad+\mu\lambda_m^{\alpha-1}\sum_{j=1}^{M}\left\|A^{1/2}\varphi_j(t)\right\|_2^2+\epsilon_p\int_\Omega\rho(x)^{(2\alpha+3)/3}dx+c_p\|Du\|_q^q\end{aligned}\tag{5.9}$$

where we have used Young's inequality with $\epsilon_p=\mu/(2\kappa_1)$ and $c_p=c_\alpha(2\kappa_2/\mu)^{3/(2\alpha)}$ where $p=q/(q-1)$ and $q=1+3/(2\alpha)$ as before. Applying (5.7) to (5.9) we obtain

$$\begin{aligned}&\left|\mathrm{Tr}\, F'(S(t)u_0)\circ q_M(t)\right|\\&\leqslant -\left(\nu-\mu\lambda_m^{\alpha-1}\right)\sum_{j=1}^{M}\left\|A^{1/2}\varphi_j(t)\right\|_2^2\\&\quad-\mu\sum_{j=1}^{M}\left\|A^{\alpha/2}\varphi_j(t)\right\|_2^2+\frac{\mu}{2}\sum_{j=1}^{M}\left\|A^{\alpha/2}\varphi_j(t)\right\|_2^2+c_p\|Du\|_q^q.\end{aligned}\tag{5.10}$$

We now invoke the size restriction $\mu\lambda_m^{\alpha-1}\leqslant\nu$, omit the resulting non-positive term $(\nu-\mu\lambda_m^{\alpha-1})\sum_{j=1}^{M}\|A^{1/2}\varphi_j(t)\|_2^2$, and combine the other terms involving the $A^{\alpha/2}\varphi_j$ to obtain from (5.10),

$$\left|\mathrm{Tr}\, F'(S(t)u_0)\circ q_M(t)\right|\leqslant -\frac{\mu}{2}\sum_{j=1}^{M}\left\|A^{\alpha/2}\varphi_j(t)\right\|_2^2+c_p\|Du\|_q^q.\tag{5.11}$$

Now by [50, Lemma VI.2.1], $\sum_{j=1}^{M}\|A^{\alpha/2}\varphi_j(t)\|_2^2\geqslant\lambda_1^\alpha+\cdots+\lambda_M^\alpha\geqslant(c\lambda_1)^\alpha(3/(2\alpha+3))M^{(2\alpha+3)/3}$ where again we use $\lambda_j\sim c\lambda_1 j^{2/3}$. Using this and applying Hölder's inequality to the term $\|Du\|_q^q$ as was done in [2] we obtain from (5.11),

$$\begin{aligned}&\left|\mathrm{Tr}\, F'(S(t)u_0)\circ q_M(t)\right|\\&\leqslant -\frac{\mu}{2}(c\lambda_1)^\alpha\frac{3}{2\alpha+3}M^{(2\alpha+3)/3}+c_p|\Omega|^{(2\alpha-3)/(4\alpha)}\left\|A^{1/2}u\right\|_2^q.\end{aligned}\tag{5.12}$$

Using Hölder's inequality again on $\frac{1}{T}\int_0^T\|A^{1/2}u\|_2^q\,ds$ as was done in [2] and recalling the definition of c' we then see from (5.12) that to have

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T\mathrm{Tr}\, F'((S(t))u_0)\circ q_m(t)\,dt\leqslant 0$$

uniformly in space we need that

$$\frac{3\mu}{4\alpha+6}(c\lambda_1)^\alpha M^{(2\alpha+3)/3} \geqslant c_p|\Omega|^{(2\alpha-3)/(4\alpha)}\left[\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|A^{1/2}u\|_2^2\, ds\right]^{(2\alpha+3)/(4\alpha)}. \tag{5.13}$$

With $\epsilon = \lambda_1^{3/2}\nu \limsup_{T\to\infty}\frac{1}{T}\int_0^T \|A^{1/2}u(s)\|_2^2\, ds$ and setting $l_\epsilon = (\nu^3/\epsilon)^{1/4}$ and $l_0 = 1/\lambda_1^{1/2}$, we obtain the condition

$$\frac{3\mu}{4\alpha+6}(c\lambda_1)^\alpha M^{(2\alpha+3)/3} \geqslant c_p|\Omega|^{(2\alpha-3)/(4\alpha)}\nu^{(2\alpha+3)/(2\alpha)}\lambda_1^{(2\alpha+3)/(8\alpha)}\left[\frac{l_0}{l_\epsilon}\right]^{(2\alpha+3)/\alpha}. \tag{5.14}$$

Using the definition of c_p following (5.9), using as noted, e.g., in [22,50] that $\lambda_1 \sim c''|\Omega|^{-2/3}$, and collecting terms in (5.14) we have the condition

$$\lambda_1^\alpha M^{(2\alpha+3)/3} \geqslant \frac{4\alpha+6}{3}(c'')^{(2\alpha+3)/(8\alpha)}c_\alpha \cdot (2\kappa_1)^{3/(2\alpha)}\left(\frac{\nu}{\mu}\right)^{(2\alpha+3)/(2\alpha)}|\Omega|^{(\alpha-3)/(3\alpha)}\left[\frac{l_0}{l_\epsilon}\right]^{(2\alpha+3)/\alpha}. \tag{5.15}$$

Again using the relationship between λ_1 and $|\Omega|$ we have for a constant $d = d(c, c'')$ (which will also absorb various powers of 2),

$$M^{(2\alpha+3)/3} \geqslant \frac{4\alpha+6}{3}dc_\alpha\kappa_1^{3/(2\alpha)}\left(\frac{\nu}{\mu}\right)^{(2\alpha+3)/(2\alpha)}|\Omega|^{(2\alpha+3)(\alpha-1)/(3\alpha)}\left[\frac{l_0}{l_\epsilon}\right]^{(2\alpha+3)/\alpha} \tag{5.16}$$

so that, solving for M in (5.16) and setting $K_\alpha = [((4\alpha+6)/3)dc_\alpha\kappa_1^{3/(2\alpha)}]^{3/(2\alpha+3)}$ results in the condition

$$M \geqslant K_\alpha\left(\frac{\nu}{\mu}\right)^{3/(2\alpha)}|\Omega|^{(\alpha-1)/\alpha}\left[\frac{l_0}{l_\epsilon}\right]^{3/\alpha} \tag{5.17}$$

from which (1.14) follows by the opening remarks of this section.

For the proof of (1.13) we assume that $\mu\lambda_m^{\alpha-1} \geqslant \nu$ and use this in the form $\mu \geqslant \nu\lambda_m^{-(\alpha-1)}$ in (5.6) to obtain

$$
\begin{aligned}
&\left|\mathrm{Tr}\, F'(S(t)u_0) \circ q_M(t)\right| \\
&\quad \leqslant -\nu \sum_{j=1}^{M} \left\| A^{1/2}\varphi_j(t) \right\|_2^2 \\
&\qquad - \nu\lambda_m^{-(\alpha-1)} \sum_{j=1}^{M} \left\| Q_m A^{\alpha/2}\varphi_j(t) \right\|_2^2 + \int_\Omega |Du|\rho\, dx.
\end{aligned}
\tag{5.18}
$$

Now we subtract and add $\nu\lambda_m^{-(\alpha-1)} \sum_{j=1}^{M} \|P_m A^{\alpha/2}\varphi_j(t)\|_2^2$ to (5.18) and again use $\|P_m A^{\alpha/2}\varphi_j(t)\|_2^2 \leqslant \lambda_m^{\alpha-1}\|A^{1/2}P_m\varphi_j(t)\|_2^2$ to obtain

$$
\begin{aligned}
&\left|\mathrm{Tr}\, F'(S(t)u_0) \circ q_M(t)\right| \\
&\quad \leqslant -\nu \sum_{j=1}^{M} \left\| A^{1/2}\varphi_j(t) \right\|_2^2 - \nu\lambda_m^{-(\alpha-1)} \sum_{j=1}^{M} \left\| A^{\alpha/2}\varphi_j(t) \right\|_2^2 \\
&\qquad + \nu \sum_{j=1}^{M} \left\| P_m A^{1/2}\varphi_j(t) \right\|_2^2 + \int_\Omega |Du|\rho\, dx.
\end{aligned}
\tag{5.19}
$$

Canceling terms accordingly we have from (5.19)

$$
\left|\mathrm{Tr}\, F'(S(t)u_0) \circ q_M(t)\right| \leqslant -\nu\lambda_m^{-(\alpha-1)} \sum_{j=1}^{M} \left\| A^{\alpha/2}\varphi_j(t) \right\|_2^2 + \int_\Omega |Du|\rho\, dx. \tag{5.20}
$$

Now we use Young's inequality with $\epsilon_p = \nu/(2\lambda_m^{\alpha-1}\kappa_1)$, $c_p = c_\alpha(2\lambda_m^{\alpha-1}\kappa_1/\nu)^{3/(2\alpha)}$ and p, q as before to obtain

$$
\begin{aligned}
&\left|\mathrm{Tr}\, F'(S(t)u_0) \circ q_M(t)\right| \\
&\quad \leqslant -\nu\lambda_m^{-(\alpha-1)} \sum_{j=1}^{M} \left\| A^{\alpha/2}\varphi_j(t) \right\|_2^2 + \epsilon_p \int_\Omega \rho(x)^{(2\alpha+3)/3}\, dx + c_p\|Du\|_q^q.
\end{aligned}
\tag{5.21}
$$

Applying (5.7) to (5.21) we have

$$
\left|\mathrm{Tr}\, F'(S(t)u_0) \circ q_M(t)\right| \leqslant -\frac{\nu\lambda_m^{-(\alpha-1)}}{2} \sum_{j=1}^{M} \left\| A^{\alpha/2}\varphi_j(t) \right\|_2^2 + c_p\|Du\|_q^q. \tag{5.22}
$$

We now proceed as in the calculations following (5.11); the role of μ will be now played by $\nu\lambda_m^{-(\alpha-1)}$ so that similarly to (5.14) we obtain the condition

$$\frac{3\nu\lambda_m^{-(\alpha-1)}}{4\alpha+6}(c\lambda_1)^{\alpha}M^{(2\alpha+3)/3} \geqslant c_p|\Omega|^{(2\alpha-3)/(4\alpha)}\nu^{(2\alpha+3)/(2\alpha)}\lambda_1^{(2\alpha+3)/(8\alpha)}\left[\frac{l_0}{l_\epsilon}\right]^{(2\alpha+3)/\alpha}. \tag{5.23}$$

Now we again use the relationship between λ_1 and $|\Omega|$ to obtain from (5.23) the following expression involving only λ_1:

$$\frac{3\nu\lambda_m^{-(\alpha-1)}}{4\alpha+6}(c\lambda_1)^{\alpha}M^{(2\alpha+3)/3} \geqslant (c'')^{(9-6\alpha)/(8\alpha)}c_p\nu^{(2\alpha+3)/(2\alpha)}\lambda_1^{(3-\alpha)/(2\alpha)}\left[\frac{l_0}{l_\epsilon}\right]^{(2\alpha+3)/\alpha}. \tag{5.24}$$

Let γ_α be the same as K_α but with d replaced by a similar constant d_1, then proceeding as in the calculations following (5.14) with μ replaced by $\nu\lambda_m^{-(\alpha-1)}$ and noting that $\lambda_1^{-\alpha}\lambda_1^{(3-\alpha)/(2\alpha)} = \lambda_1^{(2\alpha+3)(1-\alpha)/(2\alpha)}$ we derive from (5.24) the condition

$$M \geqslant \gamma_\alpha\left(\frac{\lambda_m}{\lambda_1}\right)^{3(\alpha-1)/(2\alpha)}\left[\frac{l_0}{l_\epsilon}\right]^{3/\alpha} \tag{5.25}$$

which is (1.13), and thus the proof of Theorem 1.6 is complete.

For $\alpha = 5/2$ we have the same estimate $(\mu c_1)^{-1}(2+\nu\lambda_1^{1/4}G)^2$ as in [2] on the dimension of $\mathcal{M}$ where c_1 is a generic constant and $G = L/(\nu^2\lambda_1^{3/4})$ is the 3-D Grashoff number (see, e.g. [22,50]); we have as discussed in [2] that $l_0/l_\epsilon \leqslant G^{1/2}$. The exponent $\alpha = 5/2$ is close enough to 2 that we can again use (1.13) and substituting accordingly we obtain the estimate $\gamma_{5/2}(\lambda_m/\lambda_1)^{9/5}G^{3/5}$. The inertial-manifold dimension estimate above is larger than the attractor estimate because of the higher power on G (again because the power on α is close to 2 we have $(2+\nu\lambda_1^{1/4})(\mu c_1)^{-1} \sim \nu/\mu \sim 1$). Meanwhile, by assuming asymptotically that $U_L \sim L/(\nu\lambda_1) = (\nu/\lambda_1^{1/4})G$ the estimate $(6/\mu)K_1U_L^2 \leqslant (\mu/2)\mu\lambda_{k+1}^{3/2}$ from Theorem 1.5 can be written as $2^{4/3}(\nu/\mu)^{4/3}\lambda_1^{-1/3}G^{4/3} \leqslant \lambda_{k+1}$. This becomes $4c_2(\nu/\mu)^2\lambda_1^2G^2 \leqslant k+1$ where c_2 is a generic constant, again using $\lambda_m \sim c\lambda_1 m^{2/3}$ with m replaced by $k+1$. Since $\nu/\mu \sim 1$ we see that the estimate for the number of determining modes is in fact similar to the estimate for the dimension of the inertial manifold, suggesting that the former (which is easier to obtain), can be used to estimate the latter. That both are larger than the attractor estimate is commensurate with the fact that $A \subseteq \mathcal{M}$.

6 Derivation of the SHNSE on bounded domains

The derivation of (1.2) begins in [5] in the hyperviscous NSE case $A_\varphi = B^\alpha$ by applying the Leray projection P to both sides of (1.1). This requires making sense of the operator PB^α, which is done by first noting as in [42] that the Stokes operator $A = -P\Delta$ satisfies $Au = -\Delta u + \nabla p_s(u)$ where $p_s(u)$ solves the boundary-value problem

$$\begin{aligned} \Delta p_s(u) &= 0, && x \in \Omega, \\ \mathbf{n} \cdot \nabla p_s(u) &= \mathbf{n} \cdot \boldsymbol{\Delta} u, && x \in \Gamma. \end{aligned} \tag{6.1}$$

From the decomposition $P(-\Delta u) = -\Delta u + \nabla p_s(u)$ we obtain

$$\begin{aligned} A^2 u = P(-\Delta)P(-\Delta) &= P(-\Delta)(-\Delta u + \nabla p_s(u)) \\ &= P(-\Delta)^2 u + P(-\Delta)(\nabla p_s(u)) \end{aligned} \tag{6.2}$$

and using (6.1) and the commutativity of spatial derivatives inside Ω we have $P(-\Delta)(\nabla p_s(u)) = -P\nabla(\Delta p_s(u)) = 0$ in Ω. Combining with (6.2) we have in Ω,

$$A^2 u = P(-\Delta)^2 u. \tag{6.3}$$

Since A is well-defined assuming zero boundary conditions, $P(-\Delta)^2 u = A^2 u$ is well-defined as a self-adjoint operator assuming the conditions $u = Au = 0$ on Γ. By induction using (6.3) we have for any integer $\alpha \geqslant 2$ that $P(-\Delta)^\alpha u = A^\alpha u$ is well-defined as a self-adjoint operator assuming the conditions $u = Au = \cdots = A^{\alpha-1} u = 0$ on Γ. That imposing these extra conditions preserves the physics of the NSE was shown in [4] in which these conditions were shown to necessarily hold for the NSE in the no-slip case if the forcing data $f = Pg$ is smooth enough, i.e., satisfies $f \in D(A^{\alpha-1})$, which of course includes the unforced case $f = 0$.

We thus obtain a definition of the hyperviscous NSE on general bounded domains Ω which is physically sound by adding the term $\mu(-\Delta)^\alpha u$ to the NSE, applying P to both sides, invoking (6.3), and associating A^α with the boundary conditions $u = Au = \cdots = A^{\alpha-1} u = 0$ on Γ to obtain

$$\begin{aligned} &\frac{d}{dt} u + \mu A^\alpha u + \nu A u + P(u \cdot \nabla) u = f, \\ &u(x, 0) = u_0(x). \end{aligned} \tag{6.4}$$

To derive the SHNSE from (6.4) we let $0 < \lambda_1 < \lambda_2 < \cdots$ represent the eigenvalues of A with corresponding eigenspaces $E_1, E_2, \ldots$ We set $Q_m = I - P_m$ where P_m is the projection onto $E_1 \oplus \cdots \oplus E_m$, and replace A^α in (6.4) with operators A_φ satisfying $A_\varphi \geqslant A_m = Q_m A^\alpha$, which in particular includes operators in the explicit computational form $A_\varphi = \sum_{j=m_0+1}^{m} d_j (\lambda_j)^\alpha P_{E_j} + Q_m A^\alpha$. For such operators A_φ we thus obtain as in [5] the following formulation of the SHNSE for general bounded domains:

$$\frac{d}{dt}u + \mu A_{\varphi}u + \nu Au + P(u \cdot \nabla)u = f, \\ u(x,0) = u_0(x). \tag{6.5}$$

which is (1.2). In addition to the derivation of (1.2), the capability for solutions of (1.2) to approximate NSE solutions was demonstrated in [5] by establishing $H^1(\Omega)$-convergence of solutions of (1.2) to solutions of the NSE as $\mu \to 0$ or as $m \to \infty$ on intervals $[0, T]$ on which there is a common $H^1(\Omega)$-bound. It was also shown in [5] that the Stokes-pressure formulation of the NSE developed in [42] can be extended to a suitably similar formulation for (1.2).

7 Conclusion

In [2], in the periodic case it was shown for a constant K_0 on the order of unity that

$$\dim_H A \leqslant \dim_F A \leqslant K_0 \left(\frac{\nu}{\mu}\right)^{9/(10\alpha)} \left[\frac{\lambda_m}{\lambda_1}\right]^{9(\alpha-1)/(10\alpha)} \left[\frac{l_0}{l_\epsilon}\right]^{(6\alpha+9)/(5\alpha)}. \tag{7.1}$$

A basic sense of how these estimates compare with those in Theorem 1.6 can be obtained by considering the borderline case $\mu\lambda_m^{\alpha-1} = \nu$; we set, e.g., $\alpha = 2$ and replace ν/μ with λ_m in (7.1) to obtain

$$\dim_H A \leqslant \dim_F A \leqslant K_0(\lambda_m)^{9/20}[\lambda_m/\lambda_1]^{9/20}[l_0/l_\epsilon]^{21/10};$$

again using that $1/\lambda_1$ is proportional to $|\Omega|^{2/3}$ this becomes $\dim_H A \leqslant \dim_F A \leqslant K_0|\Omega|^{3/10}[\lambda_m]^{9/10}[l_0/l_\epsilon]^{21/10}$, and rewriting (1.13) in similar fashion we have $\dim_H A \leqslant \dim_F A \leqslant K_0|\Omega|^{1/2}[\lambda_m]^{3/4}[l_0/l_\epsilon]^{3/2}$. The power on $|\Omega|$ is slightly higher in the latter estimate but the power on the growth term λ_m is comparably lower. Meanwhile the power on the key term l_0/l_ϵ is markedly lower, which would seem to be the overriding consideration with the possible exception of extremely large domains. Similar considerations apply to (1.14). Beyond this the estimates (1.13) and (1.14) not only contribute the lower power on l_0/l_ϵ but isolate the dependencies on λ_m/λ_1 and ν/μ and help enable the favorable comparisons with the Landau–Lifshitz estimates that follow (1.15).

In the formulation derived in [43] on bounded domains the NS-α model takes the form

$$u_t + \nu Au + U^{\alpha_1}(u) + (u \cdot \nabla)u + \nabla p = -(I - \alpha_1\Delta)^{-1}\nabla p + F, \\ \nabla \cdot u = 0 \tag{7.2}$$

where A is again the Stokes operator and $U^{\alpha_1}(u) = \alpha_1^2(I - \alpha_1\Delta)^{-1}\nabla \cdot (\nabla u \cdot \nabla^{\mathrm{T}}u + \nabla u \cdot \nabla u + \nabla^{\mathrm{T}}u \cdot \nabla u)$. (We have used α_1 as the small "alpha" parameter in the above, which is consistent with the notation in [20] and distinguishes this parameter with the exponent α in (1.1)–(1.3).) This form of the NS-α model seems to offer the closest comparison with other SGS models via the term $U^{\alpha_1}(u)$. The following attractor estimates for the NS-α model were established on periodic domains in [20] for a generic constant c:

$$\dim_H \mathcal{A} \leqslant \dim_F \mathcal{A} \leqslant c(\alpha^2\lambda_1)^{-3/4}[l_0/l_\epsilon]^3 \sim c\alpha^{-3/2}|\Omega|^{1/2}[l_0/l_\epsilon]^3. \qquad (7.3)$$

Though it is theoretically remarkable that the power on l_0/l_ϵ in (7.3) matches the Landau–Lifschitz power of 3, at the same time the right-hand side of (7.3) grows rapidly without bound as $\alpha_1 \to 0$, heuristically how the NS-α system approaches the NSE. For $\alpha_1 = 1/100$, we have $\alpha_1^{-3/2} = 1000$ so for $|\Omega| \geqslant 1$ the estimate (7.3) is at least 1000 times larger than the Landau–Lifschitz estimates. Though arguments have been made as in [19] to suggest that the NS-α model reduces the number of degrees of freedom in calculations, this is not exhibited directly by (7.3), whereas the estimates in Theorem 1.6 lead to direct evidence of degrees-of-freedom reduction for (1.2) as discussed following (1.15).

In the spectral hyperviscosity model discussed in [26], the counterparts of the coefficients μ and m_0 in (1.2) are chosen to depend on certain negative and positive powers, respectively, of the truncation order N according to a specific version of spectral vanishing viscosity methodology. In particular, $\mu = \epsilon_N = N^{-\beta}$ in [26] with $\beta < 8/7$ for $\alpha = 2$. Setting $N = 500$ would be a fairly safe representative upper bound given the analogous calculations for high Reynolds number using spectral vanishing viscosity in [44], in which N is roughly on the order of 300. We have for the upper bound $\beta = 8/7$ that $\mu = \epsilon_N$ is larger than 1.3×10^{-3} while in high-Reynolds number turbulence generally ν is significantly smaller than 10^{-4}. Thus for $\alpha = 2$ we have a computational example in which the ratio ν/μ is less than unity. On the other hand, if $\alpha = 8$ as used in [8–10,33] we have in [26] that $\beta < 224/19$ so for, e.g., $\beta = 11$ and N even as low as $N = 50$ we have that $\mu = \epsilon_N$ is smaller than 4.9×10^{-18}. For $\nu = 10^{-6}$ or even $\nu = 10^{-9}$, the ratio ν/μ is clearly large enough so that for computationally realistic values of m the conditions of (1.14) are satisfied.

The high-Reynolds-number wind tunnel results in [44] suggest the potential of the SHNSE as a computational tool for studying high-Reynolds-number turbulence given its similarities to spectral-vanishing viscosity as well as its robust qualities discussed above. Indeed, as noted in [32], spectral vanishing viscosity can be viewed as “using hyperviscous dissipation that will affect only the high Fourier modes”.

Locally in time as is well known, subgrid-scale models do not always effectively capture all of the dynamical effects of the unresolved scales, in particular backscatter, in which significant amounts of dynamic energy are contributed from the higher (yet unresolved) frequencies in the inertial range. At the same time the results and discussion in [38,40] show that subgrid-scale dissipation can in the limit of time-averaging represent a reasonably good model of the effects of the unresolved inertial-range scales; this connects in particular to the attractor results above given the central role of the time-averaged quantity defined by (1.12). Given their time-independence, steady-state solutions of (1.2) would seem to be worthy topics for further study.

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