Energy conservative and -stable schemes for the two-layer shallow water equations.

Ulrik Skre Fjordholm
ETH Zurich, Rämistrasse 101, 8092 Zurich
E-mail: ulrikf@sam.math.ethz.ch

Abstract

By extending the work in [4] we design an energy stable finite difference scheme for the two-layer shallow water equations.

1 Introduction

The two-layer shallow water equations model the flow of two fluids of different densities, superimposed on each other, under the influence of gravity. The main assumption in the derivation is that the horizontal length scales are much bigger than the vertical scales, and that one can therefore neglect variations in depth of density and velocity.Furthermore, it is assumed that no mixing occurs between the fluids. In one spatial dimension the equations have the form

$$\begin{aligned} (h_1)_t + (h_1u_1)_x = 0, \qquad (h_2)_t + (h_2u_2)_x = 0, \\ (h_1u_1)_t + \left(\frac{1}{2}gh_1^2 + h_1u_1^2\right) = -gh_1\left(b + rh_2\right)_x, \\ (h_2u_2)_t + \left(\frac{1}{2}gh_2^2 + h_2u_2^2\right) = -gh_2\left(b + h_1\right)_x, \end{aligned} \tag{1.1}$$

where subscripts 1 and 2 denote the lower and upper layers, respectively, $b = b(x)$ is the bottom topography, h is the layer height, u is layer velocity, $r := \frac{\rho_2}{\rho_1} < 1$ and ρ is the layer density, with $\rho_1 > \rho_2$. In this equation, mass (h_1 and h_2) and total momentum ($\rho_1h_1u_1 + \rho_2h_2u_2$) are the conserved variables. (1.1) is a *balance law*, which in general has the form

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u}_x) = \mathbf{s}(\mathbf{u}, \mathbf{u}_x, x).$$

The system (1.1) is equipped with an entropy pair, the *energy* of the solution:

$$\eta = \frac{1}{2}\left(\rho_1\left(h_1u_1^2 + gh_1^2\right) + \rho_2\left(h_2u_2^2 + gh_2^2\right)\right) + g\rho_1h_1b + g\rho_2h_2(h_1+b)$$
$$q = \sum_{i=1}^{2}\rho_i\left(\frac{1}{2}h_iu_i^3 + gh_iu_i(h_i+b)\right) + g\rho_2h_1h_2(u_1+u_2). \tag{1.2}$$

The relevant entropy condition for (1.1) is therefore

$$\eta(\mathbf{u})_t + q(\mathbf{u})_x \leq 0. \tag{1.3}$$

The two-layer shallow water equations entails several difficult problems:

- The right-hand side products (e.g. $h_1\partial_xh_2$) are undefined at discontinuities, as we get the product of a distribution (h_1) with a measure (∂_xh_2), a *nonconservative product*. The theoretical framework for nonconservative products was set by DalMaso, LeFloch and Murat in [3], and numerical methods (so-called path-conservative schemes) have been developed to fit into this framework. However, as has been recently reported [1], such schemes may not converge to the correct weak solution.
- As (1.1) is a 4-by-4 system, a direct calculation of its eigenvalues can be hard. However, a first-order approximation in $u_2 - u_1$ was found in [2] to be

$$\lambda_{\text{int}}^{\pm} = U_c \pm \sqrt{g'\frac{h_1h_2}{h_1+h_2}\left(1 - \frac{(u_2-u_1)^2}{g'(h_1+h_2)}\right)},$$
$$\lambda_{\text{ext}}^{\pm} = U_m \pm \sqrt{g(h_1+h_2)}, \tag{1.4}$$

  where $U_c := \frac{h_1u_2+h_2u_1}{h_1+h_2}$, $U_m := \frac{h_1u_1+h_2u_2}{h_1+h_2}$, $g' := g(1-r)$. From this we see that the system is only hyperbolic in the regime

$$\frac{(u_2-u_1)^2}{g'(h_1+h_2)} \leq 1.$$

  This loss of hyperbolicity can be linked with Kevin-Helmholtz instabilities – violent mixing that occurs when the relative difference in velocities between the two layers becomes too large.
  From the above approximation of eigenvalues we also get a bound on the wave speeds of the system (1.1):

$$|\lambda| \leq |U_m| + \sqrt{g(h_1+h_2)}.$$

  This will be useful when determining CFL conditions and adding diffusion in numerical computations.

- It is easily seen that (1.1) has the steady state

$$u_1, u_2 \equiv 0, \qquad b + h_1 \equiv \text{const}, \qquad h_2 \equiv \text{const}. \tag{1.5}$$

  This is the so-called *lake at rest* steady state. A major challenge in numerical schemes for shallow water models is the preservation of the lake at rest. Its importance is evident in the modelling of lakes and oceans, where the flows of interest are small perturbations of the lake at rest.

1.1 Numerical methods

We consider finite difference schemes to solve (1.1). Our spatial domain is partitioned into a uniform grid $\{x_j\}_j$ with $x_{j+1} - x_j \equiv \Delta x$, and we solve for the point values $u_j(t) \approx u(x_j, t)$. The general form of a finite difference scheme for (1.1) is then

$$\frac{d}{dt}\mathbf{u}_j + \frac{1}{\Delta x}\left(\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}\right) = \mathbf{S}_j, \tag{1.6}$$

where we suppress the t-dependence of u_j for notational convenience. Temporal integration is performed with the explicit Euler method for first-order schemes and strong stability preserving Runge Kutta method for second-order schemes.

Our main goal in this paper is to design a finite difference scheme that satisfies a discrete entropy inequality

$$\frac{d}{dt}\eta(\mathbf{u}_j) + \frac{1}{\Delta x}\left(Q_{j+1/2} - Q_{j-1/2}\right) \leq 0. \tag{1.7}$$

for some numerical entropy flux $Q_{j+1/2} = Q(\mathbf{u}_j, \mathbf{u}_{j+1})$ that is consistent with the entropy flux q. Such a scheme will be called *energy stable* after the work of Tadmor [6]. To this end, we first construct an *entropy conservative* scheme – one which satisfies

$$\frac{d}{dt}\eta(\mathbf{u}_j) + \frac{1}{\Delta x}\left(\widehat{Q}_{j+1/2} - \widehat{Q}_{j-1/2}\right) = 0, \tag{1.8}$$

and then add a numerical diffusion operator to obtain entropy stability. Define the *entropy variables* as $\mathbf{v}(\mathbf{u}) := \nabla\eta(\mathbf{u})$ and the *entropy potential* as $\psi(\mathbf{u}) := \mathbf{v}(\mathbf{u})^\top \mathbf{f}(\mathbf{u}) - q(\mathbf{u})$. For the two-layer shallow water equations, the entropy variables are

$$V = \begin{bmatrix} \rho_1\left(g(h_1 + rh_2 + b) - \frac{u_1^2}{2}\right) \\ \rho_2\left(g(h_1 + h_2 + b) - \frac{u_2^2}{2}\right) \\ \rho_1 u_1 \\ \rho_2 u_2 \end{bmatrix}. \tag{1.9}$$

It was shown in [6] that a scheme

$$\frac{d}{dt}\mathbf{u}_j + \frac{1}{\Delta x}\left(\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}\right) = 0 \tag{1.10}$$

is entropy conservative if its numerical flux $\mathbf{F}_{j+1/2}$ satisfies

$$[\![\mathbf{v}]\!]_{j+1/2}^{\top}\mathbf{F}_{j+1/2} = [\![\psi]\!]_{j+1/2},$$

where $[\![x]\!]_{j+1/2} := x_{j+1} - x_j$. Moreover, it was shown that if a numerical flux function $\mathbf{F}$ can be written as $\mathbf{F}_{j+1/2} = \mathbf{F}^*_{j+1/2} - D_{j+1/2}[\![\mathbf{v}]\!]_{j+1/2}$ for an entropy conservative flux $\mathbf{F}^*$ and a positive definite diffusion matrix $D_{j+1/2}$, then the resulting scheme is entropy *stable*. Using these results, the authors designed energy conservative and energy stable schemes for the shallow water equations in [4]. (The entropy used is the total energy of the solution, hence the word "energy".) A consistent discretization of the bottom topography source term in [5] led to energy conservative and energy stable, *well-balanced* methods for the shallow water equations with variable bottom topography.

2 The EEC scheme

It turns out that the generalization from the setting of [5] to the two-layer shallow water equations is more or less trivial. Without going into details we give the spatial discretization and refer to [4, 5] for further information:

$$\begin{aligned}
\mathbf{F}^*_{j+1/2} &= \begin{bmatrix} \overline{h_1}\,\overline{u_1} \\ \overline{h_2}\,\overline{u_2} \\ \frac{1}{2}g\overline{h_1^2} + \overline{h_1}\,\overline{u_1}^2 \\ \frac{1}{2}g\overline{h_2^2} + \overline{h_2}\,\overline{u_2}^2 \end{bmatrix}_{j+1/2}, \\
\mathbf{S}_j &= -\frac{g}{2\Delta x}\begin{bmatrix} 0 \\ 0 \\ \overline{(h_1)}_{j+1/2}[\![b + rh_2]\!]_{j+1/2} + \overline{(h_1)}_{j-1/2}[\![b + rh_2]\!]_{j-1/2} \\ \overline{(h_2)}_{j+1/2}[\![b + h_1]\!]_{j+1/2} + \overline{(h_2)}_{j-1/2}[\![b + h_1]\!]_{j-1/2} \end{bmatrix},
\end{aligned} \tag{2.1}$$

where we use the notation $\overline{x}_{j+1/2} := \frac{x_j + x_{j+1}}{2}$. We denote the scheme (1.6), (2.1) the *EEC* (Explicit Energy Conservative) scheme.

Theorem 2.1. *The EEC scheme is consistent, second-order accurate and energy conservative – solutions $\{\mathbf{u}_j\}_j$ satisfy the discrete entropy* *equality* (1.8) *with*

$$\widehat{Q}_{j+1/2} = \sum_{i=1}^{2} \rho_i \Bigg(\frac{1}{2} \overline{(h_i)}_{j+1/2} \overline{(u_i)}_{j+1/2} (u_i)_j (u_i)_{j+1} + g \overline{(h_i)}_{j+1/2} \left(\frac{(u_i)_j (h_i + b)_{j+1} + (u_i)_{j+1} (h_i + b)_j}{2} \right) \Bigg) + \rho_2 g \Bigg(\overline{(h_1)}_{j+1/2} \left(\frac{(u_1)_j (h_2)_{j+1} + (u_1)_{j+1} (h_2)_j}{2} \right) + \overline{(h_2)}_{j+1/2} \left(\frac{(u_2)_j (h_1)_{j+1} + (u_2)_{j+1} (h_1)_j}{2} \right) \Bigg)$$

(compare to (1.2)*). Furthermore, it is well-balanced – given initial data satisfying* (1.5)*, the computed solution is constant in time.*

Proof. Consistency and order of accuracy is straight-forward to check. For entropy conservation, we take the inner product of (1.6) with $\mathbf{v}(\mathbf{u}_j)$ (see (1.9)) and obtain (1.8). Last, if the solution at some point of time satisfies the lake at rest conditions (1.5), then the two first components of the flux and source in (1.6) drop out immediately, and the third reduces to

$$\begin{aligned}
&\overline{(h_1^2)}_{j+1/2} - \overline{(h_1^2)}_{j-1/2} + \overline{(h_1)}_{j+1/2} [\![b + rh_2]\!]_{j+1/2} + \overline{(h_1)}_{j-1/2} [\![b + rh_2]\!]_{j-1/2} \\
&= \frac{[\![h_1^2]\!]_{j+1/2} + [\![h_1^2]\!]_{j-1/2}}{2} + \overline{(h_1)}_{j+1/2} [\![b + rh_2]\!]_{j+1/2} + \overline{(h_1)}_{j-1/2} [\![b + rh_2]\!]_{j-1/2} \\
&= \overline{(h_1)}_{j+1/2} [\![b + h_1 + rh_2]\!]_{j+1/2} + \overline{(h_1)}_{j-1/2} [\![b + h_1 + rh_2]\!]_{j-1/2} \\
&= 0
\end{aligned}$$

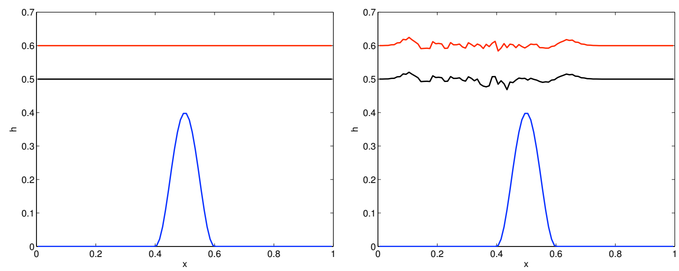
by (1.5). An analogous argument holds for the fourth component. Thus, we end up with $\frac{d}{dt}\mathbf{u}_j = 0$, whence $\mathbf{u}_j(t) \equiv$ constant for all j. □

2.1 Numerical experiments

We test the EEC scheme on a problem taken from [2]. The bottom topography is set to be

$$b(x) = \begin{cases} \left(\cos\left(10\pi(x - 0.5)\right) + 1\right)/5 & \text{if} \quad |x - 0.5| \leq 0.1 \\ 0 & \text{otherwise.} \end{cases}$$

The initial data is the lake at rest data $u_1, u_2 \equiv 0$, $h_1 + b(x) \equiv 0.5$, $h_2 \equiv 0.1$. We discretize the domain $[0, 1]$ into 100 grid cells and compute up to $t = 1$. We set $g = 1$, $\rho_1 = 1$ and $\rho_2 = 0.9$ in this and the remaining



(a) Lake at rest (b) Lake at rest with perturbation

Figure 2.1: Lake at rest computed with the EEC scheme

experiments. As can be seen in Figure 2.1(a), there is no unphysical generation of spurious waves, and hence the initial lake at rest data is preserved.

Figure 2.1(b) shows the same experiment at $t = 0.4$, but with a small initial perturbation of $+0.05$ in h_2 in the range $x \in [0.38, 0.42]$. The two resulting shock waves are resolved, but in-between there are unphysical oscillations that are due to a lack of numerical diffusion. To fix this we will add numerical diffusion to obtain an *energy stable* scheme.

3 The ERus scheme

To obtain energy decay near discontinuities we add a Rusanov-type numerical diffusion operator of the form $c_{j+1/2}[\![\mathbf{u}]\!]_{j+1/2}$ to the numerical flux, where $c_{j+1/2}$ is some approximation of the largest eigenvalues of the system. Assume for the moment that the bottom topography is flat, $b \equiv 0$. As the energy η is strictly convex, the mapping $\mathbf{u} \mapsto \mathbf{v}(\mathbf{u})$ is injective, and so the change-of-variables matrix $\partial_{\mathbf{v}}\mathbf{u}(\mathbf{v})$ is well-defined and positive definite. By the mean value theorem there is a state $\mathbf{v}_{j+1/2}$ such that

$$[\![\mathbf{u}]\!]_{j+1/2} = \partial_{\mathbf{v}}\mathbf{u}(\mathbf{v}_{j+1/2})[\![\mathbf{v}]\!]_{j+1/2}. \tag{3.1}$$

This is the form of our numerical diffusion operator, and we define the *ERus* (Energy stable Rusanov) flux to be

$$\mathbf{F}_{j+1/2} = \mathbf{F}^*_{j+1/2} - \frac{1}{2}c_{j+1/2}\partial_{\mathbf{v}}\mathbf{u}(\mathbf{v}_{j+1/2})[\![\mathbf{v}]\!]_{j+1/2}, \tag{3.2}$$

where $\mathbf{F}^*$ is the EEC flux (2.1). In the more general case $b \neq 0$ we still use the above expression, although the identity (3.1) no longer holds since $\mathbf{v}$, unlike $\mathbf{u}$, depends explicitly on b. We set the diffusion coefficient $c_{j+1/2}$ to be

$$c_{j+1/2} = \max\left(c_j, c_{j+1}\right),$$

where $c_j := |(U_m)_j| + \sqrt{g((h_1)_j + (h_2)_j)}$ (see (1.4)).

Theorem 3.1. *The ERus scheme is consistent, first-order accurate and energy stable – solutions* $\{\mathbf{u}_j\}_j$ *satisfy* (1.7) *with* $Q_{j+1/2} = \widehat{Q}_{j+1/2} - \frac{1}{2}\overline{\mathbf{v}}_{j+1/2}^\top D_{j+1/2}[\![\mathbf{v}]\!]_{j+1/2}$ *and* $D_{j+1/2} := \frac{1}{2}c_{j+1/2}\partial_{\mathbf{v}}\mathbf{u}(\mathbf{v}_{j+1/2})$. *Furthermore, it is well-balanced – given initial data satisfying* (1.5), *the computed solution is constant in time.*

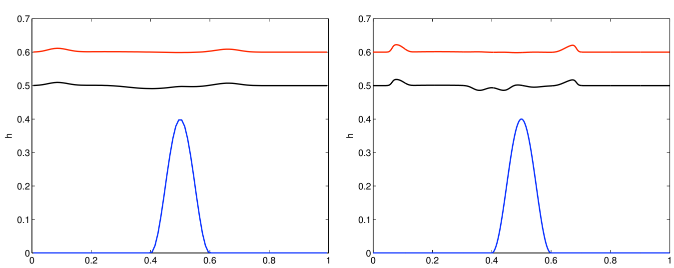
Proof. First-order accuracy comes from the $\mathcal{O}(\Delta x)$ term $[\![\mathbf{v}]\!]_{j+1/2}$. The proof of the discrete entropy inequality follows [6, Theorem 5.2]: Multiply the ERus scheme (1.6) by $\mathbf{v}_j^\top$, use the entropy conservativity of the EEC flux $\mathbf{F}^*$ and rearrange the diffusion terms to obtain (**??**).

For well-balancedness, we note that when the solution satisfies (1.5), the entropy variables (1.9) are constant in space. Hence, the ERus flux reduces to the EEC flux, which we have already shown is well-balanced. □

3.1 Numerical experiments

We repeat the numerical experiments from the previous section with the ERus scheme. The lake at rest is preserved exactly; even after long-time simulations the relative error stays within 10^{-16}. Figure 3.1(a) shows the perturbed lake at rest experiment. Clearly, the unphysical oscillations of the EEC scheme are gone.

Figure 3.2 shows computed h_1 and u_1 on a problem from [2]. The results compare well to those of [2].



(a) 100 grid points. (b) 1000 grid points.

Figure 3.1: Perturbed lake at rest computed with the ERus scheme.

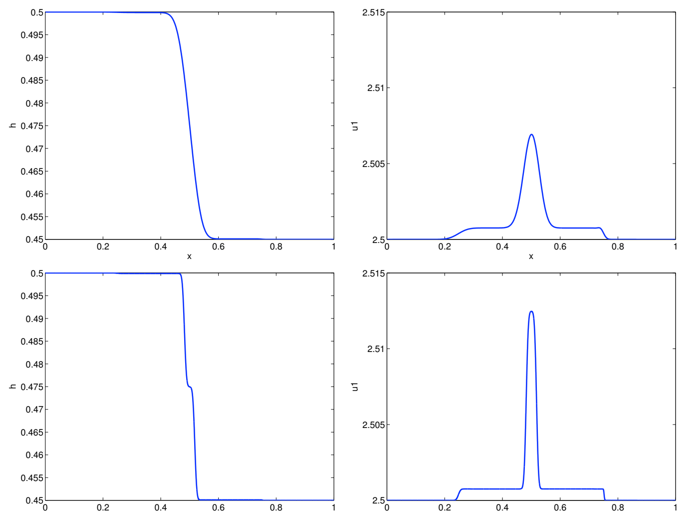


Figure 3.2: Problem from [2]. h_1 to the left, u_1 to the right. Top row with 500 grid points, bottom row with 10000.

References

[1] R. Abgrall and S. Karni. A comment on the computation of non-conservative products. *Journal of Computational Physics*, 229:2759–2763, April 2010.

[2] Rémi Abgrall and Smadar Karni. Two-layer shallow water system: A relaxation approach. *SIAM Journal on Scientific Computing*, 31(3):1603–1627, 2009.

[3] G. DalMaso, P. G. LeFloch, and F. Murat. Definition and weak stability of nonconservative products. *Journal de Mathematiques Pures et Appliquees*, 74(6):483–548, 1995.

[4] U. S. Fjordholm, S. Mishra, and E. Tadmor. Energy preserving and energy stable schemes for the shallow water equations. In *"Foundations of Computational Mathematics", Proc. FoCM held in Hong Kong 2008*, volume 363 of *London Math. Soc. Lecture Notes Series*, pages 93–139, 2009.

[5] U. S. Fjordholm, S. Mishra, and E. Tadmor. Well-balanced, energy stable schemes for the shallow water equations with varying topography. Submitted, 2010.

[6] E. Tadmor. The numerical viscosity of entropy stable schemes for systems of conservation laws. I. *Mathematics of Computation*, 49:91–103, 1987.