Global Smooth Solutions to Euler Equations for a Perfect Gas

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ABSTRACT. We consider Euler equations for a perfect gas in $\mathbb{R}^d$, where $d \geq 1$. We state that global smooth solutions exist under the hypotheses (H1)-(H3) on the initial data. We choose a small smooth initial density, and a smooth enough initial velocity which forces particles to spread out. We also show a result of global in time uniqueness for these global solutions.

Introduction. We consider Euler equations for a perfect gas:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \rho(\partial_t u + (u \cdot \nabla)u) + \nabla p = 0, \\ \partial_t S + u \cdot \nabla S = 0, \end{cases} \tag{1}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$ and $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ stands for the velocity, $\rho : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ for the density, $p = (\gamma - 1)\rho e$ for the pressure, with e the internal energy of the gas and $S : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ for the entropy. The adiabatic constant of the gas is denoted by $\gamma > 1$ and $d \geq 1$ is the dimension of the space.

We are interested in the existence of global smooth solutions to the Cauchy problem for (1) with (ρ_0, u_0, S_0) as initial data. There exist few results concerning this problem, especially when d is strictly larger than one. The choice of initial data is decisive for this problem and it depends on whether one wants to prove or to disprove global existence. We aim at finding conditions on (ρ_0, u_0, S_0) as weak as possible which ensure the existence of a global smooth solution. In [4], T. Sideris has shown a result of non global existence: the initial density is close to a constant at infinity—the constant should be different from 0—and some global quantities have to be large. For $d = 1$, in the isentropic case, we have a 2×2 system. In this case, some results can be proved using P. D. Lax's works [3]. In the same case with less restrictive conditions, J. Y. Chemin [2] has also proved a result of non global existence: the initial velocity has to be smaller than the initial sound speed in each point—this quantity depends mostly on ρ_0—. In [1],

D. Serre has proved one result of existence in the multi-dimensionnal case, with $\gamma \leq 1 + 2/d$—which is not a restriction in the realistic case—. One restriction to this result is that the initial velocity must be close to a linear field. We give here a global existence result without this particular hypothesis. Note that the case $\rho = 0$ does occur, that is to say that there can be vacuum in some area. This requires some attention as we will notice later.

This article is divided in two main parts. The first one deals with the isentropic case. This is the simplest one and gives the idea of the methods we use. We give a result of global existence and some estimates on the solution. We show also a result of uniqueness for all time, under some regularity assumptions on the solution. We prove also a corollary of the existence result which improves slightly the hypotheses.

The second part generalizes the previous results to a non-isentropic fluid.

The vector of all the spatial derivatives of order k is denoted by D^k, and ∂^k is one of the component of D^k. The differential with respect to x of $u(\cdot, t)$ will be denoted by Du. We denote by $|.|_p$ the norm of $L^p(\mathbb{R}^d)$ where $1 \leq p \leq \infty$, by $\|.\|_0$ the one of $L^2(\mathbb{R}^d)$ and by $\|.\|_m$ the one of $H^m(\mathbb{R}^d)$. We also set $\|.\|_X$ the norm of the space $X = \{z : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d \mid Dz \in L^\infty(\mathbb{R}^d), D^2 z \in H^{m-1}(\mathbb{R}^d)\}$. The transpose of a vector V is denoted by V^T.

1. The isentropic case. We first consider Euler equations for an isentropic perfect gas:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \rho(\partial_t u + (u \cdot \nabla) u) + \nabla p = 0, \end{cases} \tag{2}$$

with $p = (\gamma - 1)\rho^\gamma$ and the initial data:

$$\begin{cases} u(x, 0) = u_0(x), \\ \rho(x, 0) = \rho_0(x). \end{cases} \tag{3}$$

In this part, we prove a result of global existence for this problem. Then we give a corollary of this result. Finally, we solve the problem of uniqueness. We now find some conditions on the initial data which ensure the global existence. We use a result of local existence of smooth solutions and some energy estimates. Nevertheless, we need to introduce an approximate problem to have a guess of the behaviour of the velocity in our problem. Then, we compare this approximate solution and our local solution in order to obtain accurate energy estimates. The approximate problem is the following:

$$\begin{cases} \partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = 0 & \text{on } \mathbb{R}^d \times \mathbb{R}_+, \\ \bar{u}(x, 0) = u_0(x) & \text{on } \mathbb{R}^d. \end{cases} \tag{4}$$

It is obtained by neglecting ρ in (2). The hypotheses we make on the initial data ensure that there exists a global solution to this problem, and that this approximate solution stays close to the solution of (2)-(3).

1.1. Main result. We state the following result:

Theorem 1. *Let $m > 1 + d/2$ and assume that*

(H1) $\rho_0^{(\gamma-1)/2}$ *is small enough in* $H^m(\mathbb{R}^d)$,
(H2) $D^2u_0 \in H^{m-1}(\mathbb{R}^d)$ *and* $Du_0 \in L^\infty(\mathbb{R}^d)$, *i.e.,* $u_0 \in X$,
(H3) *There exists* $\delta > 0$ *such that for all* $x \in \mathbb{R}^d$, $\mathrm{dist}(\mathrm{Sp}(Du_0(x)), \mathbb{R}_-) \geq \delta$,
(H4) ρ_0 *has a compact support,*

and let $\bar{u}$ *be the global solution of* (4). *Then there exists a global smooth solution to* (2)-(3), *i.e.,* (ρ, u) *such that*

$$(\rho^{(\gamma-1)/2}, u - \bar{u}) \in \mathcal{C}^j([0,\infty[; H^{m-j}(\mathbb{R}^d)) \quad \text{for } j \in \{0,1\}.$$

In (H3), dist stands for the distance and Sp for the spectrum. This means that the spectrum of Du_0 is uniformly bounded away from the real negative numbers. To explain our hypotheses, we first make this simple remark: since our aim is to found a smooth global solution, we choose a small initial density and an initial velocity which make particles to spread out. Thus, the choice of u_0 gives us the existence of a global solution to the simplified problem (4) close to (2). The main hypothesis on ρ_0 is its smallness. The exponent $(\gamma-1)/2$ comes from the proof of local existence which is the first step in the proof of Theorem 1. It is introduced by the symmetrisation of the system. (H1) is not equivalent to $\rho_0 \in H^m(\mathbb{R}^d)$ but these two assumptions can be linked according to the value of γ. Remark that we accept that ρ_0 vanishes in some area. We get rid of (H4) in the corollary following the uniqueness result.

Note that, although the hypotheses have no direct physical interest, since most of the physical problems are given in bounded domains, this result emphasizes the importance of dispersion in the search of a global smooth solution. Moreover, one strongly expects that a physical solution asymptotically behaves like the one obtained here. Our last motivation in studying the Cauchy problem rather than the initial boundary value problem is that the mathematical theory of both is rather poor so far, so that it is legitimate to begin with the simplest one.

The proof is based on local existence of a smooth solution to (2) and energy estimates. But the classical method does not work here. Thus, we consider a simplified problem which gives us a global solution $\bar{u}$ thanks to our hypothesis (H2)-(H3). Then, we prove that there exists a local solution of our problem for (ρ_0, u_0), satisfying (H1)-(H4). We compare the two problems and we use properties of the simpler one to improve the classical energy estimates. Our proof is split in three steps. First, we introduce $\bar{u}$ the global smooth solution to an approximate problem concerning only the velocity. We use a local existence theorem—cf Chemin [2]—and a local uniqueness property of our problem to obtain a local solution with (ρ_0, u_0) as initial data. Then we find some estimates on $\bar{u}$ to precise its behaviour. To conclude, we obtain accurate energy estimates on a certain spatial norm of the difference between the local solution and the approximate one. Thus we obtain the following estimates on our global solution:

Theorem 2. *Under the hypotheses* (H1)-(H4), *the solution of Theorem 1 satisfies:*

- $\|D^k U(t)\|_0 \leq K(1+t)^{-(k+r)}$, *for all* $1 \leq k \leq m$, *for all* $t \geq 0$,
- $|U|_\infty(t) \leq K(1+t)^{1-a}$, *for all* $t \geq 0$,
- $|DU|_\infty(t) \leq K(1+t)^{-a}$, *for all* $t \geq 0$,

where $U = (\pi, u - \bar{u})^T$, K *depends on* δ, $\|u_0\|_X$, $\|\rho_0^{(\gamma-1)/2}\|_m$,

$$r = \begin{cases} 1 - \dfrac{d}{2} & \text{if } \gamma \geq \gamma_c = 1 + \dfrac{2}{d}, \\ \dfrac{\gamma - 1}{2} d - \dfrac{d}{2} & \text{if } 1 < \gamma < \gamma_c, \end{cases}$$

and $a = 1 + r + d/2 > 1$.

Note that, according to the values of k and d, $-(k+r)$ can be positive.

1.1.1. Local existence and approximate problem. We need to suppose that the density is small to expect a global smooth solution. By neglecting ρ and $\nabla(\rho^{\gamma-1})$, we obtain the approximate problem:

$$\begin{cases} \partial_t \bar{u} + (\bar{u} \cdot \nabla)\bar{u} = 0 & \text{on } \mathbb{R}^d \times \mathbb{R}_+, \\ \bar{u}(x, 0) = u_0(x) & \text{on } \mathbb{R}^d. \end{cases}$$

In fact, we have neglected $\rho^{(\gamma-1)/2}$ in H^m, therefore in C^1. Thanks to (H2)-(H3), there is a global solution $\bar{u}$ in $\mathcal{C}^j([0, \infty[; H^{m-j}(\mathbb{R}^d))$ for $j \in \{0, 1\}$, defined by:

$$\bar{u}(X(x_0, t), t) = u_0(x_0), \text{ with } X(x_0, t) = x_0 + t u_0(x_0).$$

Note that this problem does not take in account any forces. That is why the choice of u_0 is decisive in the global existence of the solution $\bar{u}$. We remark that $D\bar{u} \in L^\infty(\mathbb{R}^d \times \mathbb{R}_+)$, as we show in the next section.

We want now to construct a local solution to (2)-(3) such that the difference between this solution and $(0, \bar{u})$ is in $\mathcal{C}^0(H^m(\mathbb{R}^d)) \cap \mathcal{C}^1(H^{m-1}(\mathbb{R}^d))$. Note first that, since $u_0 \notin H^m(\mathbb{R}^d)$, we can not use directly a general local existence theorem to obtain a local solution to our problem.

The first step in the proof of local existence consists in the symmetrisation of the system. The symmetrisation must be cautiously chosen because the case $\rho = 0$ can occur. Following T. Makino, S. Ukai, S. Kawashima [6], we take

$$\pi = \sqrt{\frac{(\gamma - 1)}{4\gamma}} \rho^{(\gamma-1)/2}$$

to obtain:

$$\begin{cases} (\partial_t + u \cdot \nabla)\pi + C_1 \pi \operatorname{div}(u) = 0, \\ (\partial_t + (u \cdot \nabla))u + C_1 \pi \nabla \pi = 0, \end{cases} \tag{5}$$

where $C_1 = (\gamma - 1)/2$.

Actually, this system is not equivalent to (2) because of the case $\rho = 0$, but we can pass from (5) to (2) by multiplying by ρ. Thus if we find a global smooth solution to this problem, we obtain one solution for (2) such that $\rho^{(\gamma-1)/2}$ is smooth. But we loose the property of uniqueness of the solution.

We write (5) in the following way:

$$\partial_t V + \sum_{\alpha=1}^{d} A^\alpha(V)\partial_\alpha V = 0, \tag{6}$$

where $V = (\rho, u)^T$, $A^\alpha(V) \in M_{d+1}(\mathbb{R})$ is symmetric:

$$A^\alpha(V) = \begin{pmatrix} u_\alpha & 0 & \dots & C_1\pi & \dots & 0 \\ 0 & u_\alpha & 0 & & & \\ \vdots & 0 & \ddots & & & \vdots \\ C_1\pi & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & 0 \\ 0 & 0 & \dots & \dots & 0 & u_\alpha \end{pmatrix}.$$

We now construct a local solution to this problem with initial data satisfying (H1)-(H4).

Let $R > 0$ such that $\text{supp}\rho_0 \subset B(0, R)$. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi \equiv 1$ on $B(0, R + 2\eta)$, where η is some positive constant. We consider $(\pi_0, u_0\varphi)$ as an initial data for the problem (5) and we use the theorem of local existence of solution for symmetric hyperbolic systems, since $(\pi_0, u_0\varphi) \in H^m(\mathbb{R}^d)$—see in [2]—. Therefore we obtain (π^φ, u^φ) a solution in $\mathcal{C}^j([0, T_{ex}[; H^{m-j}(\mathbb{R}^d))$ for $j \in \{0, 1\}$. Note that $(0, \bar{u})$ is a solution to (5) with $(0, u_0)$ as initial data.

Let

$$\begin{aligned} K &= \{(x,t) \mid 0 \le t \le T,\ x \in B(0, R + \eta + Mt)\},\ \text{with:} \\ M &= \sup_{0 \le t \le T_{ex}-\varepsilon} (C_1|\pi^\varphi|_{L^\infty} + |u^\varphi|_{L^\infty}),\ \text{and} \\ T &= \min(T_{ex} - \varepsilon, \eta/(2M) - \varepsilon) \text{ for } \varepsilon > 0 \text{ given.} \end{aligned}$$

Now we take

$$(\pi, u) = \begin{cases} (\pi^\varphi, u^\varphi) & \text{in } K, \\ (0, \bar{u}) & \text{outside } K. \end{cases}$$

We have to show that (π, u) is actually a solution on $\mathbb{R}^d \times [0, T]$ of (5) with (π_0, u_0) as initial data. In fact, (π, u) is a solution in K and outside K. Thus we have just to show that it is continuous across ∂K. In order to do this, we show that (π^φ, u^φ) and $(0, \bar{u})$ are equal on

$$D = \{(x,t) \mid 0 \le t \le T,\ x \in B(x_0, \eta - Mt) \text{ for } x_0 \in S(0, R + \eta)\}.$$

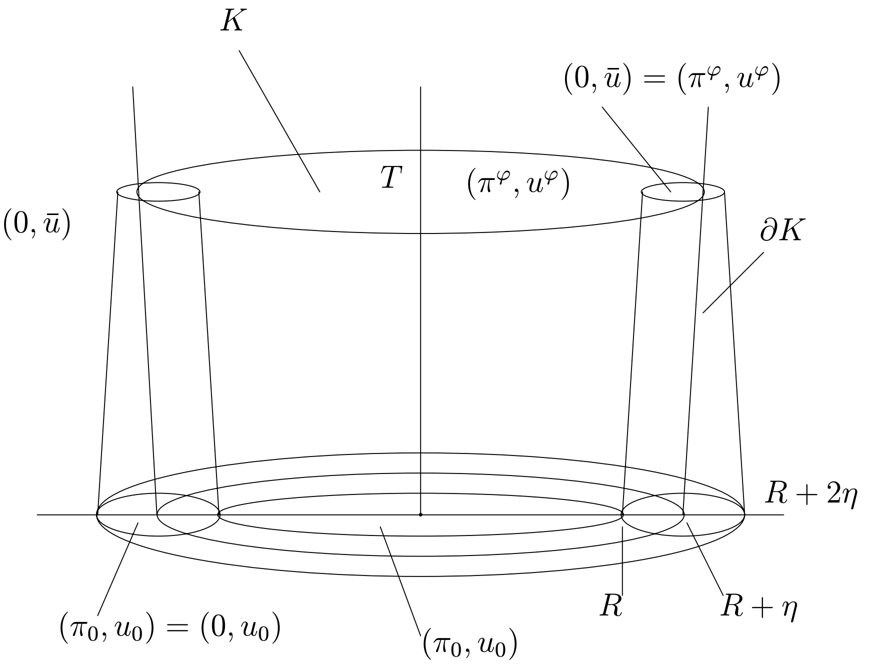

Figure 1: LOCAL EXISTENCE

This is true thanks to a property of local uniqueness of the solutions of the system (5).

Proposition 1. *Let V_0^1, V_0^2 be two initial data for* (5)*. Assume that $V_0^1 \in H^m(\mathbb{R}^d)$. Let $V^1 = (\pi, u)^T$, V^2 be two associated solutions of* (5) *defined for $0 \leq t \leq T_0$, and let $M \geq sup_{(x,t)\in Q}\{(C_1|\pi| + |u|)(x,t)\}$, where $Q = B(x_0, \eta) \times [0, T_0]$. Suppose that $V_0^1 = V_0^2$ on $B_0 = B(x_0, \eta)$ and take*

$$C_T = \{(x,t) \mid 0 \leq t \leq T,\ x \in B_t = B(x_0,\ \eta - Mt)\}$$

for $0 \leq T \leq T_1 = \min(T_0, \eta/M)$. Suppose moreover that $|DV^2|_\infty < \infty$. Then $V^1 = V^2$ on C_{T_1}.

The proof of this proposition is classical. We use the properties of the symmetric system (5).

We apply this proposition to $V_0^1 = (\pi_0, u_0\varphi)^T$, which is in $H^m(\mathbb{R}^d)$, and $V_0^2 = (0, u_0)^T$. Then we have $V^1 = (\pi^\varphi, u^\varphi)^T$ and $V^2 = (0, \bar{u})^T$. We know that $D\bar{u} \in L^\infty(\mathbb{R}^d \times \mathbb{R}_+)$, and we have chosen M to satisfy the condition in the proposition. As $(\pi_0, u_0\varphi) = (0, u_0)$ on each $B_0 = B(x_0, \eta)$ for $x_0 \in S(0, R + \eta)$, we have $(\pi^\varphi, u^\varphi) = (0, \bar{u})$ on D. This implies that (π, u) is smooth across ∂K since D contains ∂K.

Thus we have found (π, u) a local solution to our problem such that

$$(\pi, u-\bar{u}) \in \mathcal{C}^j([0,T[, H^{m-j}) \quad \text{for } j=0,1.$$

1.1.2. Estimates for the approximate solution. We now precise our knowledge of $\bar{u}$. We use hypothesis (H3), which requires some uniformity, to obtain estimates on the spatial norm of the derivatives of $\bar{u}$. We have the following result:

Proposition 2. *Suppose* (H2), (H3). *Let $\bar{u}$ be the global smooth solution of* (4). *Then:*

(i) $D\bar{u}(x,t) = \dfrac{1}{(1+t)} I + \dfrac{1}{(1+t)^2} K(x,t)$, *for all* $x \in \mathbb{R}^d$, *all* $t \in \mathbb{R}_+$,

(ii) $\left\| D^\ell \bar{u}(\cdot,t) \right\|_0 \le K_\ell (1+t)^{d/2-(\ell+1)}$, *for* $2 \le \ell \le m+1$,

(iii) $\left| D^2 \bar{u}(\cdot,t) \right|_\infty \le C(1+t)^{-3}$,

with $I = Id_{\mathbb{R}^d}$ *and* $K : \mathbb{R}^d \times \mathbb{R}_+ \to M_d(\mathbb{R})$, $|K|_{L^\infty(\mathbb{R}^d\times\mathbb{R}_+)} \le M$, *where* M, C, *and* K_ℓ, *for* $2 \le \ell \le m+1$, *are some positive constants which depend on* m, d, δ, $\|u_0\|_X$.

We remark that the decay in t of the derivatives of $\bar{u}$ improves itself with the order of the derivatives. We will show that $V = (\pi, u)^T$ behaves similarly. We show that (ii) is true for $\ell \in \mathbb{N}$, $\ell \ge 2$, then by interpolation we obtain the result for all $\ell \in \mathbb{R}$, $\ell \ge 2$. Note that since $m-1 > d/2$, $D^2 u_0$ and $D^2 \bar{u}$ are in $L^\infty \cap \mathcal{C}^0$.

Proof of the proposition.

(i) Let $V(x,t) = D\bar{u}(x,t)$ and $V_0(x_0) = Du_0(x_0)$. We have

$$V(X(x_0,t),t) = (\mathrm{I} + tV_0(x_0))^{-1} V_0(x_0).$$

We write

$$V(X(x_0,t),t) = \frac{1}{(1+t)}\mathrm{I} + \frac{1}{(1+t)^2} K(x_0,t),$$

where $K(x_0,t) = (1+t)^2(\mathrm{I}+tV_0)^{-1}V_0 - (1+t)\mathrm{I}$. Then K is bounded on every compact subset of $\mathbb{R}^d \times \mathbb{R}_+$. We now show that K stays bounded for t and x_0 large.

Remark 1. *Thanks to* (H3), *we show that:*

- *there exists a constant* $K = K(\delta, |V_0|_\infty)$ *such that* $|V_0^{-1}|_\infty \le K$,
- *there exists a constant* $L = L(\delta, |V_0|_\infty)$ *such that* $|(I+tV_0)^{-1}|_\infty \le L/(1+t)$.

To prove that, consider

$$V_0^{-1} = \frac{1}{\det(V_0)}(\mathrm{adj}V_0)^T,$$

where $\mathrm{adj}V_0$ stands for the matrix of the cofactors of V_0. With this formula, it is easy to see that $|V_0^{-1}|_\infty \leq C\delta^{-d}|V_0|_\infty^{d-1}$.

For the second point, we notice first that if μ is an eigenvalue of $(I + tV_0)$, then $\lambda = (\mu - 1)/t$ is an eigenvalue of V_0, and that $|1 + t|/|1 + \lambda t| \leq C$ for $\lambda \in \mathrm{Sp}V_0$. We write the same formula as in the previous case for $(I + tV_0)^{-1}$ to conclude.

Then for t large enough, one has $\|t^{-1}V_0^{-1}(x_0)\| < 1$ for all x_0, and

$$\begin{aligned} K(x_0,t) &= \frac{(1+t)^2}{t}(\mathrm{I} + t^{-1}V_0^{-1})^{-1} - (1+t)\mathrm{I} \\ &= \frac{(1+t)^2}{t}\left(\mathrm{I} - \frac{V_0^{-1}}{t} + O\left(\frac{1}{t^2}\right)\right) - (1+t)\mathrm{I} \\ &= \frac{(1+t)}{t}\mathrm{I} - \frac{(1+t)^2}{t^2}V_0^{-1} + O\left(\frac{1}{t}\right). \end{aligned}$$

This is bounded independently of (x_0, t). This proves part (i) of Proposition 2.

(ii) We know $\bar{u}$ and its derivatives with respect to x_0 on the curves $X(x_0, t)$. We just have to deduce from that the expression of the derivatives with respect to x. Let $W(x_0, t) = V(X(x_0, t), t)$. By induction, we show that, for $k \geq 1$:

$$D_{x_0}^k W = (\mathrm{I} + tV_0(x_0))^{-1}\Lambda_k(\mathrm{I} + tV_0(x_0))^{-1},$$

where Λ_k is a sum of products of $t(\mathrm{I} + tV_0(x_0))^{-1}$ and (D^jV_0), $j \in \{1, \ldots, k\}$, appearing β_j times with $\sum_j j\beta_j = k$.

Then we use:

$$D_{x_0}^k W(x_0,t) = \sum_{j=1}^{k} D_x^j V(X(x_0,t),t)\Big(\sum_{1\leq k_i \leq k} D_{x_0}^{k_1}X \otimes \ldots \otimes D_{x_0}^{k_j}X\Big)$$

with $\sum_{i=1}^{j} k_i = k$, and

$$\begin{aligned} D_{x_0}X &= \mathrm{I} + tV_0(x_0), \\ D_{x_0}^{\ell}X &= tD^{\ell-1}V_0(x_0), \text{ for } \ell \geq 2. \end{aligned}$$

By induction, we show that for all $j \geq 1$:

IH(j): D_x^jV is a sum of terms which are products in a certain order of: $(\mathrm{I} + tV_0(x_0))^{-1}$, $t\mathrm{I}$, or $(\mathrm{I} + tV_0(x_0))$ and $D^\ell V_0$ appearing β_ℓ times, with

$\sum_\ell \ell\beta_\ell = j$. Moreover, the L^∞-norm of the terms with t is bounded by a constant times $(1+t)^{-(j+2)}$, and we have $\|D_x^j V(X(\cdot,t),t)\|_0 \leq C(1+t)^{-(j+2)}$, with $C = C(\delta, \|u_0\|_X)$.

Suppose IH(k-1), then:

$$D_{x_0}^k V(X(x_0,t),t) = \Big(D_{x_0}^k W(x_0,t) - \sum_{j=1}^{k-1} D_x^j V\Big(\sum_{1\leq k_i\leq k-1} D_{x_0}^{k_1} X \otimes \ldots \otimes D_{x_0}^{k_j} X\Big)\Big) \circ\Big((\mathrm{I}+tV_0(x_0))^{-1}\Big)^{\otimes k}.$$

In the right-hand side term, one has the norm of the following terms to estimate:

a) $(\mathrm{I}+tV_0(x_0))^{-1}\Lambda_k(\mathrm{I}+tV_0(x_0))^{-(k+1)}$.

b) $D_x^j V(\mathrm{I}+tV_0(x_0))^{j-s} \prod_{k_i\neq 1} tD^{k_i-1}V_0(\mathrm{I}+tV_0(x_0))^{-k}$,

with $\sum_{k_i\neq 1}(k_i-1) = k-j$.

This last term correspond to the term where s of the k_j are distinct from 1. Thus, the factor $(I+tV_0)$, which corresponds to $D_{x_0}X$, appears $j-s$ times.

For each of these terms, we apply first the induction hypothesis and we consider the L^∞-norm in space for the terms with t and we use the remark to bound it. We use IH(j) for $j \leq k-1$ to show that b) is a product of $(D^j V_0(\cdot,t))^{\beta_j}$ with $\sum_j j\beta_j = k$ and of tI, $(I+tV_0)^{-1}$, such that the the L^∞-norm of the terms with t is bounded by a constant times $(1+t)^{-(k+2)}$. Then we find an upper bound in L^2-norm for

$$\prod_{1\leq j\leq k} (D^j V_0(\cdot,t))^{\beta_j}$$

with $\sum_j j\beta_j = k$, by using Gagliardo-Nirenberg inequality. To conclude, we obtain the upper bound in IH(k) which depends on δ, $|V_0|_\infty$, and $\|D^k V_0\|_0$ for $1 \leq k \leq m$, that is to say on δ and $\|u_0\|_X$.

Finally, we have to make a change of variables to obtain:

$$\|D_x^j V(\cdot,t)\|_0 \leq (1+t)^{d/2}\|D_x^j V(X(\cdot,t),t)\|_0.$$

This gives (ii) since $D_x^j V = D^2\bar{u}$.

(iii) Since $m-1 > d/2$, we know that $D^2 u_0 \in L^\infty$. Thus $D^2\bar{u} \in L^\infty$, and we have:

$$D_x V(X(x_0,t),t) = -(\mathrm{I}+tV_0(x_0))^{-1} DV_0(x_0)(\mathrm{I}+tV_0(x_0))^{-2}.$$

Using the remark again, we obtain: $|D_x V(X(x_0,t),t)|_\infty = O((1+t)^{-3})$. □

1.1.3. Energy estimates. We have now a guess of the behaviour of the velocity. Therefore, we introduce $\bar{u}$ in the equations in order to estimate $(\pi, w = u - \bar{u})$. Moreover, we will use the dispersive properties of $\bar{u}$ described in Proposition 2 and also the fact that $(\pi, u - \bar{u})(\cdot, t) \in H^m(\mathbb{R}^d)$. The system for (π, w) is:

$$\begin{cases} (\partial_t + w \cdot \nabla)\pi + C_1 \pi \mathrm{div}(w) = -\bar{u} \cdot \nabla \pi - C_1 \pi \mathrm{div}(\bar{u}), \\ (\partial_t + (w \cdot \nabla))w + C_1 \pi \nabla \pi = -(\bar{u} \cdot \nabla) w - (w \cdot \nabla)\bar{u}, \end{cases} \tag{7}$$

We note $\bar{U} = (0, \bar{u})^T$, $U = (\pi, w)^T$, and U is solution of the following system:

$$\partial_t U + \sum_{\alpha=1}^{d} A^\alpha(U) \partial_\alpha U = -B(D\bar{U}, U) - \sum_{\alpha=1}^{d} \bar{u}_\alpha \partial_\alpha U, \tag{8}$$

where

$$B(D\bar{U}, U) = \begin{pmatrix} C_1 \pi \mathrm{div}(\bar{u}) \\ (w \cdot \nabla)\bar{u} \end{pmatrix}.$$

The right-hand side term in (8) provide more precise estimates thanks to Proposition 2. Before we perform the calculus, we have to choose the spatial norm we estimate. We consider the semi-norm which appears naturally in the calculus:

$$Y_k(t) = \left(\int_{\mathbb{R}^d} D^k U(x,t) \cdot D^k U(x,t)\, dx \right)^{1/2}. \tag{9}$$

We expect Y_k for $k = 0, \ldots, m$ to behave more or less like $\|D^k \bar{u}\|_0(t)$, that is to say to decay in time with a rate depending on k. Therefore, instead of using the classical norm in $H^m(\mathbb{R}^d)$, we introduce

$$Z(t) = \sum_{k=0}^{m} (1+t)^{\gamma_k} Y_k(t), \tag{10}$$

where γ_k is chosen such that each term of the sum has the same decay in t. Thus we will obtain an efficient estimate on Z. We take $\gamma_k = k + r - a$, and

$$r = \begin{cases} 1 - \dfrac{d}{2} & \text{if } \gamma \geq \gamma_c = 1 + \dfrac{2}{d}, \\ \dfrac{\gamma - 1}{2} d - \dfrac{d}{2} & \text{if } 1 < \gamma < \gamma_c. \end{cases}$$

We will choose a later. It will ensure that our calculus gives us a good estimate. We emphasize that Z depends on a. Note also that the coefficients γ_k can be negative.

We apply D^k on (8) and we take the inner product with $D^k U$. Then we integrate on $\mathbb{R}^d$ to obtain:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} D^k U(x,t) \cdot D^k U(x,t)\, dx = \int_{\mathbb{R}^d} R_k(U)(x,t)\, dx + \int_{\mathbb{R}^d} S_k(U, \bar{U})(x,t)\, dx, \tag{11}$$

with (dropping x and t):

$$R_k(U) = -D^kU\cdot\Big(D^k\Big(\sum_\alpha A^\alpha(U)\partial_\alpha U\Big)-\sum_\alpha A^\alpha(U)\partial_\alpha D^kU\Big) + \frac{1}{2}\sum_\alpha D^kU\cdot\partial_\alpha A^\alpha(U)D^kU, \tag{12}$$

$$S_k(U,\bar U) = -D^kU\cdot D^k(B(D\bar U,U))+\frac{1}{2}\Big(\sum_\alpha \partial_\alpha\bar u_\alpha\Big)D^kU\cdot D^kU - D^kU\cdot\Big(D^k\Big(\sum_\alpha \bar u_\alpha\partial_\alpha U\Big)-\sum_\alpha \bar u_\alpha\partial_\alpha D^kU\Big). \tag{13}$$

To obtain this form, we used that, at each time, U has a compact support. We used also the symmetric form of the system. We now find an upper bound for the terms in the right-hand side of the inequality. But we first isolate some terms and estimate them more precisely: we will compute exactly the terms where a derivative of order one of $\bar U$ appears. The main reason is that $D\bar u$ has not a so good decay in t, so that we will use the sign of the terms which contain $D\bar u$ to control them in the estimate.We show:

Proposition 3. *There exist $C\in\mathbb{R}_+$ depending only on m, d, and $C'\in\mathbb{R}_+$ depending on m, d, δ, $\|u_0\|_X$, such that:*

$$\left|\int_{\mathbb{R}^d} R_k(U)(x,t)\,dx\right|\le C|DU|_\infty Y_k^2, \tag{14}$$

$$\frac{k+r}{(1+t)}Y_k^2+\int_{\mathbb{R}^d} S_k(U,\bar U)(x,t)\,dx\le C'Y_kZ(1+t)^{-\gamma_k-2}. \tag{15}$$

Before we prove this proposition, we recall some inequalities that we will use in the calculus.

Lemma 1.[*Gagliardo-Nirenberg inequality*] *Let $r>0$, $0\le i\le r$, and $z\in L^\infty\cap H^r$. Then $\partial^iz\in L^{2r/i}$ and*

$$\left|\partial^iz\right|_{2r/i}\le C_{i,r}\left|z\right|_\infty^{1-i/r}\left\|D^rz\right\|_0^{i/r}.$$

For a proof of this lemma, see [7]. We apply this and use Sobolev inequalities to obtain:

Lemma 2. *Let $0<p<d/2$ and $1/q=\frac{1}{2}-p/d$. There exists C such that for all $z\in H^p$, we have $|z|_q\le C\|D^pz\|_0$, with C depending on p, q, d.*

We have also thanks to Sobolev inequalities:

Lemma 3. *Let $p > d/2$ and $z \in H^p(\mathbb{R}^d)$. Then:*

$$\left|z\right|_\infty \le C\left\|z\right\|_0^{1-\theta}\left\|D^p z\right\|_0^\theta, \quad \text{with } \theta = d/2p.$$

We deduce from this:

Lemma 4. *Let $\beta = -\gamma_1 - d/2$. Since $U \in H^m(\mathbb{R}^d)$ and $m > 1 + d/2$, one has:*

(i) $|U|_\infty(t) \le C(1+t)^{\beta+1} Z(t)$.
(ii) $|DU|_\infty(t) \le C(1+t)^{\beta} Z(t)$.
(iii) *If $m > 2 + d/2$, then* $|D^2U|_\infty(t) \le C(1+t)^{\beta-1} Z(t)$.

Proof of Proposition 3.
(i) R_k is a polynomial function in DU, ..., D^kU homogeneous in weight and degree. Its weight is $2k+1$. Remark that R_k is sum of terms like $\partial^k U \partial^\ell U \partial^{k+1-\ell} U$ for $1 \le \ell \le k$. Here, $\partial^k U$ stands for one particular derivative of order k of one component of U, for example for $(\partial^k \pi)/(\partial x_1^{k-1}\partial x_2)$. If $k \ne 0$, 1, we apply Lemma 1 to ∂U, and we obtain:

$$(16) \qquad |\partial^j U|_{p_j} \le C_{j-1,k-1}|DU|_\infty^{1-2/p_j}\|D^k U\|_0^{2/p_j}, \quad \text{for } p_j = 2\frac{k-1}{j-1}.$$

If $\ell \ne k$, $\ell \ne 1$, since $1/p_\ell + 1/p_{k-\ell+1} = \frac{1}{2}$, we have by Hölder's inequality:

$$\begin{aligned}\int |\partial^k U \partial^\ell U \partial^{k+1-\ell} U| &\le \|D^k U\|_0 \|\partial^\ell U \partial^{k+1-\ell} U\|_0 \\ &\le \|D^k U\|_0 |\partial^\ell U|_{p_\ell} |\partial^{k+1-\ell} U|_{p_{k-\ell+1}} \\ &\le C|DU|_\infty \|D^k U\|_0^2.\end{aligned}$$

The constant C depends only on m. We show the same estimate in the other cases and we obtain (14).

(ii) We note first that S_k is a sum of terms which are product of two derivatives in U and one in $\bar{U}$. We split S_k in two terms: S_k^1 where we put all the terms with a derivative of order one of $\bar{U}$, and S_k^2 where we put the terms with a derivative of order at least two of $\bar{U}$. Then we will study precisely S_k^1 and find its sign, and we will find an upper bound for S_k^2.

1. A precise analysis of S_k^1 gives this expression:

$$\int S_k^1 = I_1 + I_2 + I_3,$$

with

$$\begin{aligned} I_1 &= -\int_{\mathbb{R}^d} D^k U \cdot B(D\bar{U}, D^k U), \\ I_2 &= \frac{1}{2}\int_{\mathbb{R}^d} \sum_\alpha \partial_\alpha \bar{u}_\alpha D^k U \cdot D^k U, \\ I_3 &= -\int_{\mathbb{R}^d} \sum_{1\le\beta_1\le\ldots\le\beta_k\le d} \Big(\partial_{\beta_1\ldots\beta_k} U \cdot \sum_{i=1}^k \sum_\alpha \partial_{\beta_i}\bar{u}_\alpha \partial_\alpha \partial_{\beta_1\ldots\beta_{i-1}\beta_{i+1}\ldots\beta_k} U\Big). \end{aligned}$$

Recall that

$$B(D\bar{U}, D^k U) = \begin{pmatrix} C_1 D^k \pi \mathrm{div}(\bar{u}) \\ (D^k w \cdot \nabla)\bar{u} \end{pmatrix}.$$

Using the first part of Proposition 2,

$$D\bar{u}(x,t) = \frac{1}{(1+t)}\mathrm{I} + \frac{1}{(1+t)^2}K(x,t),$$

we may write:

$$\begin{aligned} I_1 &= -\frac{C_1 d}{1+t}\int D^k\pi \cdot D^k\pi - \frac{1}{1+t}\int D^k w \cdot D^k w + R_1, \\ I_2 &= \frac{d}{2}\frac{1}{1+t}{Y_k}^2 + R_2, \\ I_3 &= -\frac{k}{1+t}{Y_k}^2 + R_3. \end{aligned}$$

And the error terms verify:

$$|R_j| \le \frac{\bar{K}}{(1+t)^2}{Y_k}^2 \qquad j = 1,\, 2,\, 3,$$

where $\bar{K}$ is a constant depending on m, δ, and $\|u_0\|_X$.

Now, we have:

$$\int S_k^1 \le \frac{\bar{K}}{(1+t)^2}{Y_k}^2 - \frac{A_k}{(1+t)}\int D^k w \cdot D^k w - \frac{B_k}{(1+t)}\int D^k\pi \cdot D^k\pi,$$

where

$$\begin{cases} A_k = 1 - \dfrac{d}{2} + k, \\ B_k = \dfrac{\gamma-1}{2}d - \dfrac{d}{2} + k. \end{cases}$$

Since $Y_k \le Z(1+t)^{-\gamma_k}$, we have

$$\frac{\bar{K}}{(1+t)^2}{Y_k}^2 \le \bar{K} Y_k Z (1+t)^{-\gamma_k - 2}.$$

We use that

$$Y_k^2 = \int (D^k\pi \cdot D^k\pi + D^k w \cdot D^k w)$$

and we choose

$$r = \min(A_k, B_k) - k.$$

Note that we loose some accuracy in the calculus at this point. We obtain:

$$\frac{k+r}{(1+t)} Y_k^2 + \int_{\mathbb{R}^d} S_k^1(U, \bar{U})(x,t)\,dx \le C' Y_k Z (1+t)^{-\gamma_k - 2}.$$

2. In S_k^2, we have terms of the form: $Q(U) = \partial^k U \partial^\ell \bar{U} \partial^{k+1-\ell} U$ for $1 \le k \le m$ and $2 \le \ell \le k+1$. To estimate these terms, we use the fact that, here, the derivatives of $\bar{U}$ are at least of order two. Hence we know that these terms have a good decay in $(1+t)$ thanks to Proposition 2. When $m > 2 + d/2$, we just apply Lemma 1 to obtain an upper bound where the worst terms are $|D^2\bar{U}|_\infty$ and $|D^2 U|_\infty$. Thanks to part (iii) of Proposition 2, we have a good enough estimate for the first one. When $1 + d/2 < m \le 2 + d/2$, we can not do this since $D^2 U \notin L^\infty$. But we manage to make the worst terms in the upper bound to be $\|D^n \bar{U}\|_0$ and $\|D^n U\|_0$ with some $n \ge 2$, $n \in \mathbb{R}$. Thus we can obtain the same result.

CASE $m > 2 + d/2$

We state first a result we need in the following. We apply Lemma 1 to obtain:

Lemma 5. *Let $z \in H^m$ such that $D^2 z \in L^\infty$. Then we have for all $k \in [4, m]$, for all $i \in [2, k]$, $\partial^i z \in L^p$ with $p = 2(k-3)/(i-2)$ and*

$$|\partial^i z|_p \le C_{i,k} |D^2 z|_\infty^{1-2/p} \|D^{k-1} z\|_0^{2/p}.$$

Now we must study different case according to the values of k and ℓ.

(a) $k > 3$ and $2 \le \ell \le k-1$.

Applying the previous lemma to $z = \bar{U}$ with $i = \ell$, then to $z = U$ with $i = k+1-\ell$, we find that:

$$|\partial^\ell \bar{U}|_p \le C |D^2 \bar{U}|_\infty^{1-2/p} \|D^{k-1}\bar{U}\|_0^{2/p} \quad \text{with } p = 2\frac{k-3}{\ell-2}$$
$$|\partial^{k+1-\ell} U|_q \le C |D^2 U|_\infty^{1-2/q} \|D^{k-1} U\|_0^{2/q} \quad \text{with } q = 2\frac{k-3}{k-\ell-1}.$$

Note that $1/q + 1/p = \frac{1}{2}$. Thus, we obtain:

$$\begin{aligned} \int |Q(U)| &\le \left\|D^k U\right\|_0 \left|\partial^\ell \bar{U}\right|_p \left|\partial^{k+1-\ell} U\right|_q \\ &\le C Y_k \left|D^2 \bar{U}\right|_\infty^{1-2/p} \left\|D^{k-1}\bar{U}\right\|_0^{2/p} \left|D^2 U\right|_\infty^{1-2/q} \left\|D^{k-1} U\right\|_0^{2/q}. \end{aligned}$$

Using that:

$$\begin{aligned} |D^2\bar{U}|_\infty &\le C(1+t)^{-3}, \\ \|D^{k-1}\bar{U}\|_0 &\le C(1+t)^{d/2-k}, \\ |D^2 U|_\infty &\le C' Z(1+t)^{\beta-1}, \\ \|D^{k-1}U\|_0 &\le C' Z(1+t)^{-\gamma_{k-1}}, \end{aligned}$$

where C depends on m, d, $\|u_0\|_X$, δ, and C' depends on m, d, we deduce that:

$$\int |Q(U)| \le C Y_k Z(1+t)^{d_k},$$

with

$$\begin{aligned} d_k &= -3\left(1-\frac{2}{p}\right)+\frac{2}{p}\left(\frac{d}{2}-k\right)+\left(1-\frac{2}{q}\right)(\beta-1)-\gamma_{k-1}\frac{2}{q} \\ &= -3\frac{2}{q}+\left(1-\frac{2}{q}\right)\left(\frac{d}{2}-k\right)+\left(1-\frac{2}{q}\right)\left(-2-(r-a)-\frac{d}{2}\right)-(\gamma_k-1)\frac{2}{q} \\ &= -3\frac{2}{q}+\left(1-\frac{2}{q}\right)(-k-(r-a))-2\left(1-\frac{2}{q}\right)-\gamma_k\frac{2}{q}+\frac{2}{q} \\ &= -\gamma_k-2. \end{aligned}$$

(b) If $\ell = k$ and $k \ge 2$, $\ell = k+1$ and $k \ge 1$, or $k = 3$ and $\ell = 2$, an easy computation gives the same estimate.

CASE $1 + d/2 < m \le 2 + d/2$

Here again, we have to consider several cases.

(a) $2 \le \ell \le k-1$.

Note that $2 \le \ell \le k-1 \le m-1$ occurs only if $d \ge 2$. Hence, we suppose in this case that $d \ge 2$.

Let

$$n = \frac{1}{2}\left(\frac{d}{2}+k+1\right).$$

Remark that $n > 2$ and $n < m$. Thus

$$\frac{1}{2} \le n-\ell < \frac{d}{2} \quad \text{and} \quad \frac{1}{2} \le n-(k+1-\ell) < \frac{d}{2}.$$

We apply Lemma 2 to $z = \partial^\ell \bar{U}$ with $p = n - \ell$, and to $z = \partial^{k+1-\ell} U$ with $s = n-(k+1-\ell)$. We get:

$$\begin{aligned} |\partial^\ell \bar{U}|_q &\le C\|D^n\bar{U}\|_0, && \text{with } 1/q = 1/2-(n-\ell)/d, \\ |\partial^{k+1-\ell}U|_s &\le C\|D^n U\|_0, && \text{with } 1/s = 1/2-(n-(k+1-\ell))/d\,. \end{aligned}$$

We notice that:

$$\frac{1}{q} + \frac{1}{s} = 1 - \frac{(2n-k-1)}{d} = \frac{1}{2}.$$

Finally, we get: $\|\partial^\ell \bar{U} \partial^{k+1-\ell} U\|_0 \leq C |\partial^\ell \bar{U}|_q |\partial^{k+1-\ell} U|_s$, and

$$\begin{aligned} \int Q(U)| &\leq \|D^k U\|_0 |\partial^\ell \bar{U}|_q |\partial^{k+1-\ell} U|_s \leq C Y_k \|D^n \bar{U}\|_0 \|D^n U\|_0 \\ &\leq C Y_k Z (1+t)^{d_k}, \end{aligned}$$

where

$$\begin{aligned} d_k &= \left(\frac{d}{2} - (n+1)\right) - \gamma_n \\ &= -2 - \gamma_k. \end{aligned}$$

We used Proposition 2 with $n \geq 2$.

(b) If $k \geq 2$ and $\ell = k$, or $k \geq 1$ and $\ell = k+1$, we have easily the same estimate.

This finishes the proof of the part (ii) of the estimate since one has in every case:

$$\left| \int S_k^2 \right| \leq C' Z Y_k (1+t)^{-\gamma_k - 2}.$$

with C depending only on m, d, $\|u_0\|_X$, and δ. □

1.1.4. Conclusion. Now we have with (11) and Proposition 3:

$$\frac{1}{2}\frac{d}{dt} {Y_k}^2 + \frac{k+r}{(1+t)} {Y_k}^2 \leq C |DU|_\infty {Y_k}^2 + C' Y_k Z (1+t)^{-\gamma_k - 2}. \tag{17}$$

We simplify by Y_k, multiply by $(1+t)^{\gamma_k}$, and sum over k to obtain:

$$\frac{dZ}{dt}(t) + \frac{a}{(1+t)} Z(t) \leq C |DU|_\infty Z(t) + \frac{C'}{(1+t)^2} Z(t). \tag{18}$$

The constants C, C' are positive, C depends only on γ, m, d, and C' depends on γ, m, d, δ, $\|u_0\|_X$. Since $|DU|_\infty \leq C(1+t)^\beta Z$, we choose $\beta = 0$ to obtain a good estimate, and this leads to

$$a = 1 + r + \frac{d}{2} > 1.$$

Thus:

$$\frac{dZ}{dt}(t) + \frac{a}{(1+t)} Z(t) \leq C (Z(t))^2 + \frac{C'}{(1+t)^2} Z(t). \tag{19}$$

To conclude, we use the following simple result:

Proposition 4. *Since $a > 1$, there exists $\Lambda = \Lambda(a, m, d, \delta, \|u_0\|_X) > 0$ such that the Cauchy problem:*

$$\begin{cases} \dfrac{d\hat{Z}}{dt} + \dfrac{a}{(1+t)}\hat{Z} = C\hat{Z}^2 + \dfrac{C'}{(1+t)^2}\hat{Z}, \\ \hat{Z}(0) = \hat{Z}_0 < \Lambda, \end{cases} \tag{20}$$

has a global solution for $t \geq 0$.

Proof. We claim that the solution of this differential equation is:

$$\hat{Z}(t) = \frac{(1+t)^{-a} \exp\left(C'\left(1 - \dfrac{1}{1+t}\right)\right)}{\left(\dfrac{1}{\hat{Z}_0} - \displaystyle\int_0^t C(1+\tau)^{-a} e^{C'(1-1/(1+\tau))} d\tau\right)}.$$

Thus, $\hat{Z}$ is defined for $t \geq 0$ if and only if:

$$0 < \hat{Z}_0 < \Lambda = \frac{1}{\displaystyle\int_0^\infty C(1+\tau)^{-a} e^{C'} e^{-C' 1/(1+\tau)} d\tau}.$$

This condition can be filled only if the integral converges, that is to say only if $a > 1$. Note that if $a \leq 1$, there is no global solution to the differential equation we consider. In our case, it suffices to choose $\hat{Z}(0)$ small enough to satisfies the condition. Then we have a global solution $\hat{Z}$ which moreover satisfies the estimate:

$$\hat{Z}(t) \leq K(1+t)^{-a} \quad \text{for all } t \geq 0,$$

where K depends only on m, d, δ, $\|u_0\|_X$, $\hat{Z}_0$. □

Finally, if $Z(0)$ is small enough, $Z(t)$ is less than the global solution $\hat{Z}$ given by Proposition 4. The condition on $Z(0)$ is: $Z(0) < \Lambda$. This corresponds to hypothesis (H1) since we have:

$$Z(0) = \sum_{k=0}^{m} \left\|D^k(\rho_0^{(\gamma-1)/2})\right\|_0 = \left\|\rho_0^{(\gamma-1)/2}\right\|_m.$$

We have obtained the estimate:

$$Z(t) \leq K(1+t)^{-a} \quad \text{for all } t \geq 0,$$

where K depends only on m, d, $\|u_0\|_X$, δ, and $\|\rho_0^{(\gamma-1)/2}\|_m$. Then, we deduce that

$$Y_k(t) \leq K(1+t)^{-(k+r)} \quad \text{for all } t \geq 0.$$

These estimates lead to Theorem 2.

We can conclude that the solution is global since the L^2-norm of its derivatives are bounded by functions of t which never blow up. □

1.2. *Global in time uniqueness.* In this part, we state a result of local in space and global in time uniqueness. Then, we prove a corollary of Theorem 1 in which we get rid of hypothesis (H4).

We now show a result of global uniqueness in time. We compare the global solution of Theorem 1 and another solution with enough regularity.

We note $\pi_0 = \rho_0^{(\gamma-1)/2}$.

Proposition 5. *Let* (ρ_0, u_0) *satisfy* (H1)-(H4). *Let* $U = (\pi, u)^T$ *be the global solution of* (5) *given by Theorem 1 and* $\bar{u}$ *be the solution of* (4). *Consider* V *a global solution of* (5) *such that* $DV \in L^\infty(\mathbb{R}^d \times \mathbb{R}_+)$. *Then, for all* $\nu \in]2-a, 1[$, *for all* $R_0 > 0$, *there exists* $T_0 > 0$ *such that, if* $U(\cdot, T_0) = V(\cdot, T_0)$ *on* $B(0, R_0)$, *then* U *and* V *are equal on the domain* $\{(x,t) : |x - x(t)| \leq R(t),$ *for all* $t \geq T_0\}$, *where* $x(t)$ *is the solution of* $x'(t) = u(x(t), t)$, $x(T_0) = 0$, *and* $R(t) = R_0(1+t)^\nu$.

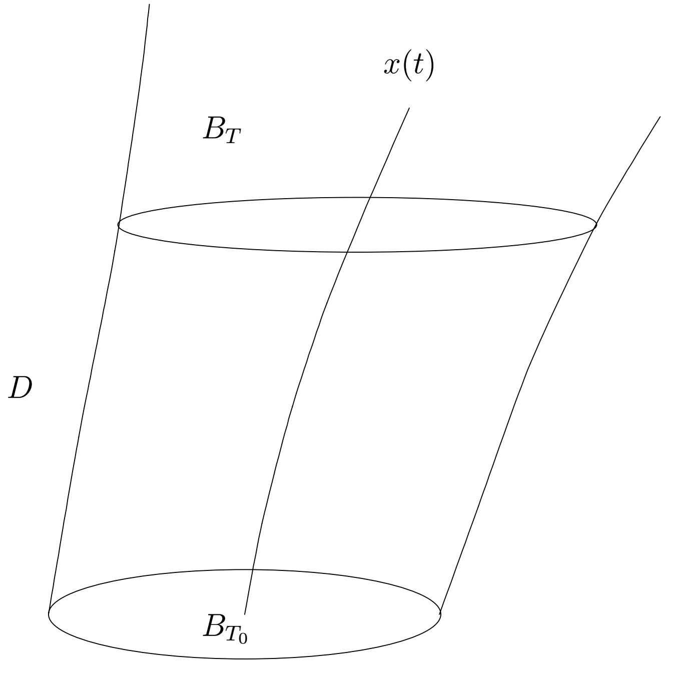

Figure 2: DOMAIN D

We note $B_t = B(x(t), R(t))$. For a T_0 given big enough, we show now that if $U(\cdot, T_0) = V(\cdot, T_0)$ on $B(0, R_0)$, then $U(\cdot, t) = V(\cdot, t)$ on B_t with $R(t) = R_0(1+t)^\nu$, $\nu < 1$, for all $t \geq T_0$. The choice of T_0 depends on ν, R_0, a, $\|u_0\|_X$, δ and $\|\rho_0^{(\gamma-1)/2}\|_m$.

We prove now that there exists T_0, $x(t)$ and $R(t)$ such that $U = (\pi, u)^T$ and $V = (\tilde{\pi}, v)^T$ solutions of (5) satisfy $U = V$ on $D = \{(x,t)/|x - x(t)| \leq R(t)$, for all $t \geq T_0\}$. For that, we consider $T > T_0$ and we evaluate the norm:

$$|U - V|^2_{B_T} = \int_{B_T} (U - V) \cdot (U - V)(x, T)\, dx.$$

We write the equation satisfied by $U - V$ on D:

$$\partial_t(U - V) + \sum_{\alpha=1}^{d} (A^\alpha(U)\partial_\alpha U - A^\alpha(V)\partial_\alpha V) = 0.$$

It is also:

$$\partial_t(U - V) + \sum_{\alpha=1}^{d} A^\alpha(U)(\partial_\alpha U - \partial_\alpha V) = -\sum_{\alpha=1}^{d} (A^\alpha(U) - A^\alpha(V))\partial_\alpha V.$$

Then, we take the inner product with $U - V$ and integrate over

$$D_T = \{(x,t) : |x - x(t)| \leq R(t),\ T_0 \leq t \leq T\} :$$

$$\begin{aligned}\int_{D_T} \frac{1}{2}\partial_t(|U - V|^2) &+ \frac{1}{2}\sum_{\alpha=1}^{d} \partial_\alpha((U - V) \cdot A^\alpha(U)(U - V)) \\ &= \int_{D_T} \frac{1}{2}\sum_{\alpha=1}^{d} (U - V) \cdot \partial_\alpha(A^\alpha(U))(U - V) \\ &\qquad - \int_{D_T} \sum_{\alpha=1}^{d} (U - V) \cdot (A^\alpha(U) - A^\alpha(V))\partial_\alpha V.\end{aligned}$$

By Stokes' formula, we get:

$$\begin{aligned}\frac{1}{2}\int_{\partial D_T} |U - V|^2 n_t &+ \sum_{\alpha=1}^{d} (U - V) \cdot A^\alpha(U)(U - V) n_\alpha \\ &= \frac{1}{2}\int_{D_T} \sum_{\alpha=1}^{d} (U - V) \cdot \partial_\alpha(A^\alpha(U))(U - V) \\ &\qquad - \int_{D_T} \sum_{\alpha=1}^{d} (U - V) \cdot (A^\alpha(U) - A^\alpha(V))\partial_\alpha V,\end{aligned}$$

where $n = (n_t, n_1, \ldots, n_d)$ is the normal vector to D_T.

This gives:

$$\frac{1}{2}\int_{\partial D_T} |U-V|^2 n_t + \sum_{\alpha=1}^{d} (U-V)\cdot A^\alpha(U)(U-V) n_\alpha$$
$$\leq C\int_{T_0}^{T} \big(|DU|_{L^\infty(D)} + |DV|_{L^\infty(D)}\big)\big|U-V\big|_{B_t}^2.$$

To express the first term, we need to precise n and ∂D_T. We have:

$$\partial D_T = (\{T_0\}\times B_{T_0}) \cup (\{T\}\times B_T) \cup \mathcal{M}$$

where

$$\mathcal{M} = \{(t,x) \mid x = \gamma(t) = x(t) + R(t)y, \text{ for } y \in S(0,1) \text{ and } T_0 \leq t \leq T\}.$$

On $\{T_0\}\times B_{T_0}$ we have $n = (-1,0,\dots,0)$, on $\{T\}\times B_T$ we have $n = (1,0,\dots,0)$, and on $\mathcal{M}$,

$$n = \frac{1}{\sqrt{1+\left|y\cdot\frac{\partial\gamma}{\partial t}\right|^2}}\left(-y\cdot\frac{\partial\gamma}{\partial t}, y_1, \dots, y_d\right).$$

We deduce from that the expression of the left-hand side term:

$$\begin{aligned}
&\frac{1}{2}\int_{\partial D_T} |U-V|^2 n_t + \sum_{\alpha=1}^{d}(U-V)\cdot A^\alpha(U)(U-V)n_\alpha \\
=\ & \frac{1}{2}(|U-V|_{B_T}^2 - |U-V|_{B_{T_0}}^2) + \frac{1}{2}\int_{\mathcal{M}} \frac{1}{\sqrt{1+\left|y\cdot\frac{\partial\gamma}{\partial t}\right|^2}}\Bigg(-|U-V|^2 y\cdot\frac{\partial\gamma}{\partial t} \\
&\qquad + \sum_{\alpha=1}^{d}(U-V)\cdot A^\alpha(U)(U-V)y_\alpha\Bigg)\,d\sigma.
\end{aligned}$$

We note

$$\Psi = -|U-V|^2 y\cdot\frac{\partial\gamma}{\partial t} + \sum_{\alpha=1}^{d}(U-V)\cdot A^\alpha(U)(U-V)y_\alpha,$$

and using the expression of $A^\alpha(U)$, we get:

$$\Psi = |U-V|^2\big[(u(x(t)+R(t)y,t) - x'(t))\cdot y - R'(t)\big] + 2C_1\pi(\pi-\tilde{\pi})y\cdot(u-v).$$

We claim that our choice of x and R implies that there exists T_0 such that:

$$\Psi > 0, \text{ for all } t \geq T_0, \text{ for all } y \in S.$$

Indeed, we know that $Du(x,t) = D\bar{u}(x,t) + Dw(x,t)$, where

$$|Dw(\cdot,t)|_\infty \le K(1+t)^{-a},$$

with K a constant depending on $\|u_0\|_X$, δ, and $\|\rho_0^{(\gamma-1)/2}\|_m$. Therefore

$$Du(x,t) = \frac{1}{1+t} I + O((1+t)^{-a}).$$

Using that $x'(t) = u(x(t),t)$, we integrate to obtain:

$$(u(x(t)+R(t)y,t) - x'(t)) \cdot y = \frac{R(t)}{1+t} + O\left(\frac{R(t)}{(1+t)^a}\right),$$

where the notation

$$O\left(\frac{R(t)}{(1+t)^a}\right)$$

means that this function is bounded for all x by $K\big(R(t)/(1+t)^a\big)$, where K is independent of t, x.

We estimate the last term by:

$$\begin{aligned} |2C_1\pi(\pi-\tilde{\pi})y\cdot(u-v)| &\le C_1|\pi|_\infty(|\pi-\tilde{\pi}|^2 + |u-v|^2) \\ &\le \tilde{K}C_1(1+t)^{1-a}|U-V|^2. \end{aligned}$$

Thus we obtain that:

$$\Psi = |U-V|^2\left(\frac{R(t)}{1+t} - R'(t) + O\left(\frac{R(t)}{(1+t)^a}\right) + O\left((1+t)^{1-a}\right)\right).$$

Thus our claim is true if $R(t) = R_0(1+t)^\nu$, for $\nu < 1$, and t big enough. Indeed, we have then

$$\begin{aligned} \frac{R(t)}{1+t} - R'(t) &= R_0(1-\nu)(1+t)^{\nu-1}, \\ K\frac{R(t)}{(1+t)^a} &= KR_0(1+t)^{\nu-a}. \end{aligned}$$

Hence, we choose T_0 such that for $t \ge T_0$, we have:

$$KR(t)(1+t)^{-a} + \tilde{K}C_1(1+t)^{1-a} < \frac{R(t)}{1+t} - R'(t),$$

or

$$\frac{K}{1-\nu}(1+t)^{1-a} + C_1\frac{\tilde{K}}{R_0(1-\nu)}(1+t)^{2-\nu-a} < 1.$$

There exists T_0 such that this is true for $t \ge T_0$ if $\nu \in\,]2-a,\, 1[$.

The size of T_0 depends of $K(\|u_0\|_X, \delta, \|\rho_0^{(\gamma-1)/2}\|_m)$, R_0, ν, and a: the smaller ν is and the bigger a is, the smaller T_0 is.

Now we have for this choice of T_0, x, and R:

$$\frac{1}{2}(|U-V|_{B_T}^2 - |U-V|_{B_{T_0}}^2) \leq C\int_{T_0}^T (|DU|_\infty(t) + |DV|_\infty(t))|U-V|_{B_t}^2 dt.$$

And we deduce that for all $T > T_0$:

$$\frac{1}{2}(|U-V|_{B_T}^2 - |U-V|_{B_0}^2) \leq C\int_{T_0}^T (|DU|_{L^\infty(D)} + |DV|_{L^\infty(D)})|U-V|_{B_t}^2 dt.$$

Now we use the fact that $|DU|_\infty(t) \leq K(1+t)^{-1}$ as we have shown in the first section. And we use also the hypothesis on DV to conclude that:

$$\frac{1}{2}(|U-V|_{B_T}^2 - |U-V|_{B_{T_0}}^2) \leq C\int_{T_0}^T |U-V|_{B_t}^2 dt,$$

where C depends on $\|u_0\|_X$, δ, $\|\rho_0^{(\gamma-1)/2}\|_m$, $|DV|_\infty$. Then we conclude easily by a Gronwall's inequality, since $U(\cdot,T_0) = V(\cdot,T_0)$ on B_{T_0}, that $U(\cdot,T) = V(\cdot,T)$ in B_T for all $T > T_0$. □

1.3. A corollary. Now we prove a corollary of Theorem 1 which improves slightly the hypotheses.

Corollary 1. *Let* (ρ_0, u_0) *be the initial data for Problem* (2). *Suppose* (H1)-(H3). *Then the result of Theorem 1 is still true.*

Proof. We aim at defining a solution for ρ_0 satisfying only (H1). For that we will use the result in the compact support case. We start by introducing a function $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ such that $\psi \equiv 1$ on $B(0,1)$. Then we take: $\rho_0^k(x) = \rho_0(x)\psi(x/k)$ for $k \in \mathbb{N}$. We claim that:

- $\rho_0^k \equiv \rho_0$ on $B(0,k)$ and ρ_0^k has a compact support.
- $\pi_0^k = (\rho_0^k)^{(\gamma-1)/2} \in H^m(\mathbb{R}^d)$ and $\|\pi_0^k\|_m \leq C\|\psi^{(\gamma-1)/2}\|_m \|\rho_0^{(\gamma-1)/2}\|_m$, where C is independent of k.

Thus we can apply Theorem 1 to ρ_0^k noticing that the smallness of $(\rho_0^k)^{(\gamma-1)/2}$ does not depend on k. We obtain $U_k = (\pi_k, u_k)^T$, a solution associated to $U_0^k = (\pi_0^k, u_0)^T$. We have $\pi_0^k \equiv \pi_0^{k+1}$ on $B(0,k)$.

We apply the result of local in space and time uniqueness. The solutions are the same on $\{(x,t) \mid 0 \leq t \leq T_0,\ |x| \leq k - M_k t\}$, provided that the initial data are the same on $B(0,k)$, where

$$M_k \geq \sup_{0\leq t\leq T_0} \left(C_1|\pi^k|_{L^\infty(B(0,k))}(t) + |u^k|_{L^\infty(B(0,k))}(t)\right).$$

Now we have, thanks to our estimates:

$$|\pi^k|_\infty(t) \leq C,$$
$$|u^k|_{L^\infty(B(0,k))}(t) \leq C + |\bar{u}|_{L^\infty(B(0,k))}(t),$$

where C depends on δ, $\|u_0\|_X$, $\|\pi_0\|_m$.

Then we estimate $|\bar{u}|_{L^\infty(B(0,k))}$. We choose $T_0 = 1/(2|Du_0|_\infty)$ to obtain the following lemma:

Lemma 6. *If $(x,t) \in [0,T_0] \times B(0,k)$ and $x = x_0 + tu_0(x_0)$, then $x_0 \in B(0,R_k)$ with $R_k = 2(k + T_0|u_0(0)|)$.*

We deduce from this lemma the estimate:

$$|\bar{u}|_{L^\infty(B(0,k))} \leq |Du_0|_\infty R_k + |u_0(0)|.$$

Then the condition on M_k is:

$$M_k \geq 2C + 2|u_0(0)| + 2|Du_0|_\infty k.$$

Thus, for k big enough, we can take $M_k \leq \tilde{c}k$, where

$$\tilde{c} = \tilde{c}\big(\big|u_0(0)\big|, \big|Du_0\big|_\infty, \delta, \big\|u_0\big\|_X, \big\|\pi_0\big\|_m\big).$$

Now, we consider $T < \min(T_0, 1/(2\tilde{c}))$. Then, we know that $U_k = U_{k+1}$ on $\{(x,t) \mid 0 \leq t \leq T_0,\ |x| \leq k - M_k t\}$. Note that $k - M_k T \geq k - \tilde{c}kT \geq k/2$. Therefore, we have shown that $U_k = U_{k+1}$ on $B(0,k/2) \times [0,T]$.

We can now define

$$U(x,t) = (\pi(x,t), u(x,t))^T$$

by

$$U_k = (\pi_k, u_k)^T \quad \text{on } B\left(0, \frac{k}{2}\right) \times [0,T]$$

for k big enough. We know that $U - (0,\bar{u})^T$ is in $\mathcal{C}^0([0,T]; H^m(B(0,R)))$ for all $R \in \mathbb{R}_+$, by using the property of each U_k. We have to show that $U - (0,\bar{u})^T \in \mathcal{C}^0(0,T; H^m(\mathbb{R}^d))$, and thus U will be a local solution of our problem. Moreover, we will show that U satisfies the same energy estimates as U_k, so that U is a global solution. We set $V = U - (0,\bar{u})^T$. We have:

$$\big\|D^j V\big\|_0^2(t) = \sup_{R>0}\big\|D^j V\big\|_{L^2(B(0,R))}^2(t).$$

Hence for k big enough:

$$\begin{aligned} \big\|D^j V\big\|_{L^2(B(0,R))}^2 &\leq \big\|D^j(U_k - (0,\bar{u})^T)\big\|_0^2(t) \\ &\leq C(1+t)^{-2(a+\gamma_j)}, \end{aligned}$$

since

$$\left\|D^j(U_k-(0,\bar{u})^T)\right\|_0^2(t)\le C(1+t)^{-2(a+\gamma_j)},$$

and C only depends on m, d, δ, $\|u_0\|_X$, $\|\rho_0^{(\gamma-1)/2}\|_m$.

We conclude that $(U-\bar{u})(t)\in H^m(\mathbb{R}^d)$ and that

$$Z(t)=\sum_{j=1}^{m}(1+t)^{\gamma_j}\left\|D^jV\right\|_0$$

satisfies:

$$Z(t)\le C(1+t)^{-a}.$$

In the same way, we verify that

$$U-(0,\bar{u})^T\in\mathcal{C}^j([0,\infty[;H^{m-j}(\mathbb{R}^d))\quad\text{for } j=0,1. \qquad \square$$

2. The non-isentropic case. The results in the general case are mainly the same as in the isentropic one. Nevertheless, some difficulties, especially in the estimates, appear due to the presence of the entropy in the equations.

We express a result of global existence of a smooth solution for (1), and we state some properties of uniqueness which can be proved as in the isentropic case. Recall the system that we consider with (ρ_0,u_0,S_0) as initial data:

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho u)=0,\\ \rho(\partial_t u+(u\cdot\nabla)u)+\nabla p=0,\\ \partial_t S+u\cdot\nabla S=0,\end{cases}$$

and the pressure law is the following: $p=(\gamma-1)\rho e$.

2.1. Global existence. We prove the following result:

Theorem 3. *Let $m>1+d/2$. Suppose that*

(H1) $(\rho_0^{(\gamma-1)/2},S_0)$ *is small enough in* $H^m(\mathbb{R}^d)$,
(H2) $u_0\in X$,
(H3) *there exists $\delta>0$ such that for all $x\in\mathbb{R}^d$,* $\operatorname{dist}(\operatorname{Sp}(Du_0(x)),\mathbb{R}_-)\ge\delta$,
(H4) ρ_0 *and S_0 have a compact support.*

Let $\bar{u}$ be the global solution to (4). *Then there exists a global smooth solution to the Cauchy problem for* (1), *i.e.,* (ρ,u,S) *such that*

$$(\rho^{(\gamma-1)/2},u-\bar{u},S)\in\mathcal{C}^j([0,\infty[;H^{m-j}(\mathbb{R}^d))\text{for } j\in\{0,1\}.$$

In (H3), dist stands for the distance and Sp for the spectrum. Note that we can suppose that $S_0-\bar{S}_0$ is small in $H^m(\mathbb{R}^d)$ where $\bar{S}_0$ is a constant. The scheme of the proof is the same as in the isentropic case. But we have to study more precisely the terms appearing in R_k and S_k, and to adapt our norm since we have a new term with S.

We obtain the following estimates:

Theorem 4. *Under the hypotheses* (H1)-(H4), *the solution of Theorem 3 satisfies:*

- forall $1 \leq k \leq m$, all $t \geq 0$, $\|D^k T\|_0(t) \leq K(1+t)^{-k-b/2+d/2}$,
  forall $1 \leq k \leq m$, all $t \geq 0$, $\|D^k S\|_0(t) \leq K(1+t)^{-k+d/2}$,
- forall $t \geq 0$, $|S|_\infty(t) \leq K$,
  forall $t \geq 0$, $|T|_\infty(t) \leq K(1+t)^{-b/2}$,
- forall $t \geq 0$, $|DT|_\infty(t) \leq K(1+t)^{-1-b/2}$,
  forall $t \geq 0$, $|DS|_\infty(t) \leq K(1+t)^{-1}$.

where T *stands for* π *or* $w = u - \bar{u}$, K *depends on* δ, $\|u_0\|_X$, $\|\rho_0^{(\gamma-1)/2}\|_m$, $\|S_0\|_m$, *and* b *is a real number such that*

$$0 < b < \min\left(1, \frac{\gamma-1}{2} d\right).$$

2.1.1. Local existence. We have first to symmetrize the system in taking into account S. As in [6], we take

$$\pi = \left(\frac{\gamma-1}{2}\frac{p}{\gamma-1}\right)^{(\gamma-1)/2\gamma},$$

and we obtain the following system from (1):

$$\begin{cases} e^{S/\gamma}(\partial_t + u\cdot\nabla)\pi + C_1 e^{S/\gamma}\pi \mathrm{div}(u) = 0\,, \\ (\partial_t + (u\cdot\nabla))u + C_1 e^{S/\gamma}\pi\nabla\pi = 0\,, \\ \partial_t S + u\cdot\nabla S = 0\,. \end{cases} \tag{21}$$

The construction of a local solution is the same as in the isentropic case. Let $R > 0$ be such that supp$\rho_0 \subset B(0,R)$ and supp$S_0 \subset B(0,R)$. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be such that $\varphi \equiv 1$ on $B(0, R+2\eta)$ where $\eta > 0$. We consider the same approximate problem on the velocity (4) and we use the result of Propositions 2. We use also the general theorem of local existence of smooth solution for symmetric hyperbolic systems —cf [2]— to obtain a local solution $(\pi^\varphi, u^\varphi, S^\varphi)$ with $(\pi_0, u_0\varphi, S_0) \in H^m$ as initial data. A result of local uniqueness can be proved as in the isentropic case -see in the next section-. With this result, we obtain (π, u, S) a solution of (21) defined by:

$$\begin{cases} (\pi^\varphi, u^\varphi, S^\varphi) & \text{on } K, \\ (0, \bar{u}, 0) & \text{outside } K, \end{cases}$$

where

$$K \;=\; \{(x,t) \mid 0 \leq t \leq T\ ,\ x \in B(0, R+\eta+Mt)\} \quad \text{with:}$$

$$\begin{aligned} M &= \sup_{0\le t\le T_{ex}-\varepsilon} (C_1 e^{|S_0|_\infty/2\gamma}|\pi^\varphi|_{L^\infty} + |u^\varphi|_{L^\infty}), \quad \text{and} \\ T &= \min\left(T_{ex}-\varepsilon, \frac{\eta}{2M}-\varepsilon\right), \quad \text{for } \varepsilon > 0 \text{ given.} \end{aligned}$$

And by construction $(\pi, u-\bar{u}, S) \in \mathcal{C}^j([0,\infty[; H^{m-j}(\mathbb{R}^d))$ for $j \in \{0,1\}$ and has a compact support.

2.1.2. Estimates in the general case. Thus, we compare V and $\bar{U} = (0,\bar{u},0)^T$ and we note $w = u - \bar{u}$, $U = (\pi, w, S)^T$. Using the fact that the equation on S is a simple transport equation, which is not deeply linked to the rest of the system, we put a weight in t on S to control its role in the estimates. We write the system in the following way:

$$\begin{cases} e^{S/\gamma}(\partial_t + w\cdot\nabla)\pi + C_1 e^{S/\gamma}\pi \mathrm{div}(w) = -e^{S/\gamma}(\bar{u}\cdot\nabla\pi - C_1\pi\mathrm{div}(\bar{u})), \\ (\partial_t + (w\cdot\nabla))w + C_1 e^{S/\gamma}\pi\nabla\pi = -(w\cdot\nabla)\bar{u} - (\bar{u}\cdot\nabla)w, \\ (1+t)^{-b}\partial_t S + w\cdot\nabla((1+t)^{-b}S) = -\bar{u}\cdot\nabla((1+t)^{-b}S), \end{cases}$$

where b is a real number that we will choose later. We write this also:

$$A^0(t,U)\partial_t U + \sum_{\alpha=1}^d A^\alpha(U)\partial_\alpha U = -B(D\bar{U},U) - \sum_{\alpha=1}^d C^\alpha(\bar{U})\partial_\alpha U, \tag{22}$$

where $A^0(t,U) = \mathrm{diag}(e^{S/\gamma}, 1, \ldots, 1, (1+t)^{-b}) \in M_{d+2}(\mathbb{R})$ is symmetric positive definite—we denote it by A^0 in the following—. Each $A^\alpha(V) \in M_{d+2}(\mathbb{R})$ is symmetric:

$$A^\alpha(V) = \begin{pmatrix} e^{S/\gamma}u_\alpha & 0 & \ldots & C_1 e^{S/\gamma}\pi & \ldots & 0 & 0 \\ 0 & u_\alpha & 0 & & & \vdots & \vdots \\ \vdots & 0 & \ddots & & & & \\ C_1 e^{S/\gamma}\pi & \vdots & & \ddots & & \vdots & \\ \vdots & & & & \ddots & 0 & \vdots \\ \vdots & \vdots & & \ldots & 0 & u_\alpha & 0 \\ 0 & 0 & \ldots & & \ldots & 0 & (1+t)^{-b}u_\alpha \end{pmatrix},$$

$C^\alpha(\bar{U}) = \bar{u}_\alpha A^0 \in M_{d+2}(\mathbb{R})$ is symmetric, and

$$B(D\bar{U},U) = \begin{pmatrix} C_1 e^{S/\gamma}\pi\mathrm{div}(\bar{u}) \\ (w\cdot\nabla)\bar{u} \\ 0 \end{pmatrix} \in M_{d+2}(\mathbb{R}).$$

Note that S is bounded in L^∞-norm since it is the solution of a transport equation and S_0 is in $L^\infty(\mathbb{R}^d)$. We choose the semi-norm:

$$Y_k(t) = \left(\int_{\mathbb{R}^d} D^k U(x,t) \cdot A^0(t,U) D^k U(x,t)\, dx \right)^{1/2}.$$

Then, we introduce

$$Z(t) = \sum_{k=0}^{m} (1+t)^{\gamma_k} Y_k(t),$$

where γ_k is chosen such that each term of the sum has the same decay in t. In fact, we choose:

$$\gamma_k = k + r - a, \ r = \frac{b}{2} - \frac{d}{2}, \text{ with } 0 < b \leq \min\left(1, \frac{\gamma - 1}{2} d\right).$$

We determine a later on. As in the isentropic case, the expression $k+r$ appears in the calculus when we study the role of $D\bar{U}$ in the estimates, i.e., when we compute S_k^1.

We apply D^k on (22), we take the inner product with $D^k U$. Then we integrate on $\mathbb{R}^d$ and we use Stokes' formula. We obtain:

$$\frac{1}{2}\frac{d}{dt}(Y_k)^2 = \int_{\mathbb{R}^d} R_k(U)(x,t)\, dx + \int_{\mathbb{R}^d} S_k(U,\bar{U})(x,t)\, dx, \tag{23}$$

with -dropping x and t-:

$$\begin{aligned} R_k(U) = & -\sum_\alpha D^k U \cdot A^0 (D^k((A^0)^{-1} A^\alpha(U) \partial_\alpha U) - (A^0)^{-1} A^\alpha(U) \partial_\alpha D^k U) \\ & + \frac{1}{2} \sum_\alpha D^k U \cdot \partial_\alpha A^\alpha(U) D^k U - \frac{1}{2\gamma} e^{S/\gamma} D^k \pi \cdot (w \cdot \nabla S) D^k \pi, \end{aligned}$$

$$\begin{aligned} S_k(U,\bar{U}) = & -D^k U \cdot A^0 D^k ((A^0)^{-1} B(D\bar{U}, U)) + \frac{1}{2} \sum_\alpha D^k U \cdot \partial_\alpha C^\alpha(\bar{U}) D^k U \\ & - \sum_\alpha D^k U \cdot A^0 (D^k((A^0)^{-1} C^\alpha(\bar{U}) \partial_\alpha U) - (A^0)^{-1} C^\alpha(\bar{U}) \partial_\alpha D^k \bar{U}) \\ & - \frac{1}{2\gamma} e^{S/\gamma} D^k \pi \cdot (\bar{u} \cdot \nabla S) D^k \pi - \frac{b}{2}(1+t)^{-b-1} D^k S \cdot D^k S. \end{aligned}$$

We show the estimates:

Proposition 6. *There exist $C \in \mathbb{R}_+$ depending only on m, d, and $C' \in \mathbb{R}_+$ depending on m, d, δ, $\|u_0\|_X$, such that:*

$$\left| \int_{\mathbb{R}^d} R_k(U)(x,t)\, dx \right| \leq C \sum_{x \in \mathcal{E}_k} Y_k Z^{2+x} (1+t)^{e(x)}, \tag{24}$$

where $\mathcal{E}_k = \{0,\, 1,\, 2,\, \ldots,\, k,\ p/(k-1)\ \textit{for}\ 1 \le p \le k-1\}$ *and* $e(x) = -\gamma_k + \beta + (\beta + 1 + b/2)\, x = -\gamma_k + \beta + ax$.

$$\frac{k+r}{(1+t)} Y_k^2 + \int_{\mathbb{R}^d} S_k(U, \bar{U})(x,t)\, dx \le C' Y_k Z (1+t)^{-\gamma_k - 2}. \tag{25}$$

Recall that

$$\beta = -\gamma_1 - \frac{d}{2} = -1 - \frac{b}{2} + a.$$

The main difference between this proposition and the Proposition 3 stands in the point (i) and comes from the term $e^{S/\gamma}$ in our matrices. We first show that S_k is not modified by this term and then we find an upper bound for the new terms appearing in R_k because of this factor $e^{S/\gamma}$.

Proof.

(i) First we consider S_k and we make a cancellation in its terms. We have indeed:

$$\partial_\alpha C^\alpha(\bar{U}) = \begin{pmatrix} e^{S/\gamma} \left(\frac{1}{\gamma} (\partial_\alpha S) \bar{u}_\alpha + \partial_\alpha \bar{u}_\alpha \right) & 0 & 0 \\ 0 & \left(\partial_\alpha \bar{u}_\alpha \ \mathrm{I} \right) & 0 \\ 0 & 0 & (1+t)^{-b} \partial_\alpha \bar{u}_\alpha \end{pmatrix}.$$

Thus:

$$\begin{aligned} \frac{1}{2} \sum_{\alpha=1}^{d} D^k U \cdot \partial_\alpha C^\alpha(\bar{U}) D^k U \quad &= \quad \frac{1}{2\gamma} e^{S/\gamma} D^k \pi \cdot (\bar{u} \cdot \nabla S) D^k \pi \\ &\quad + \frac{1}{2} \sum_\alpha \partial_\alpha \bar{u}_\alpha (D^k U \cdot A^0 D^k U). \end{aligned}$$

Therefore we have:

$$\begin{aligned} S_k \quad = \quad & -D^k U \cdot A^0 D^k ((A^0)^{-1} B(D\bar{U}, U)) + \frac{1}{2} \mathrm{div} \bar{u} (D^k U \cdot A^0 D^k U) \\ & - \sum_\alpha D^k U \cdot A^0 (D^k ((A^0)^{-1} C^\alpha(\bar{U}) \partial_\alpha U) - (A^0)^{-1} C^\alpha(\bar{U}) \partial_\alpha D^k U) \\ & - \frac{b}{2} (1+t)^{-b-1} D^k S \cdot D^k S. \end{aligned}$$

Now, we note that $(A^0)^{-1}$ and $C^\alpha(\bar{U})$ are two diagonal matrices and that the product $(A^0)^{-1} C^\alpha(\bar{U})$ equals to $\bar{u}_\alpha \mathrm{I}$. Remark also the simple expression of the product $(A^0)^{-1} B(D\bar{U}, U) = \bar{B}(D\bar{U}, U)$ with

$$\bar{B}(D\bar{U}, U) = \begin{pmatrix} C_1 \pi \mathrm{div}(\bar{u}) \\ (w \cdot \nabla) \bar{u} \\ 0 \end{pmatrix} \in M_{d+2}(\mathbb{R}).$$

As a consequence, we claim that the calculus for S_k are the same as in the isentropic case, except for the presence of a term $D^kS\cdot D^kS$. We use here that S is bounded in L^∞-norm by $|S_0|_\infty$ to find an upper bound for $e^{S/\gamma}$. As a consequence, we have:

$$\begin{aligned} S_k &= -D^kU\cdot A^0D^k(\bar{B}U)+\frac{1}{2}\mathrm{div}\bar{u}(D^kU\cdot A^0D^kU) \\ &\quad -\sum_\alpha D^kU\cdot A^0(D^k(u_\alpha\partial_\alpha U)-u_\alpha\partial_\alpha D^k\bar{U}) \\ &\quad -\frac{b}{2}(1+t)^{-b-1}D^kS\cdot D^kS. \end{aligned}$$

We note $S_k = S_k^1+S_k^2$, where S_k^1 contains the terms with a derivative of order one of $\bar{U}$, and S_k^2 contains the terms with a derivative of order at least two of $\bar{U}$. We have:

$$\begin{aligned} \int S_k^1 = &-\int_{\mathbb{R}^d} D^kU\cdot A^0\bar{B}(D\bar{U},D^kU)-\int_{\mathbb{R}^d}\frac{b}{2}(1+t)^{-b-1}D^kS\cdot D^kS \\ &-\int_{\mathbb{R}^d}\sum_{1\le\beta_1\le\ldots\le\beta_k\le d}\Big(\partial_{\beta_1\ldots\beta_k}U\cdot A^0\sum_{i=1}^k\sum_\alpha(\partial_{\beta_i}\bar{u}_\alpha)\partial_\alpha\partial_{\beta_1\ldots\beta_{i-1}\beta_{i+1}\ldots\beta_k}U\Big) \\ &+\frac{1}{2}\int_{\mathbb{R}^d}\mathrm{div}\bar{u}(D^kU\cdot A^0D^kU). \end{aligned}$$

Recall Proposition 2 and replace $D\bar{u}$ in the previous expression:

$$\begin{aligned} \int S_k^1 &= \int_{\mathbb{R}^d}\frac{\frac{d}{2}-k}{1+t}D^kU\cdot A^0D^kU-\int_{\mathbb{R}^d}\frac{C_1d}{1+t}(e^{S/\gamma}D^k\pi\cdot D^k\pi) \\ &\quad -\int_{\mathbb{R}^d}\frac{1}{1+t}D^kw\cdot D^kw-\int_{\mathbb{R}^d}\frac{b}{2(1+t)}(1+t)^{-b}D^kS\cdot D^kS \\ &\quad +\int_{\mathbb{R}^d}O\left(\frac{1}{(1+t)^2}\right)D^kU\cdot A^0D^kU. \end{aligned}$$

Now we use the definition of Y_k to write this:

$$\begin{aligned} \int S_k^1 &= \frac{\frac{d}{2}-k}{1+t}Y_k^2-\frac{1}{1+t}\left(\frac{\gamma-1}{2}d\int_{\mathbb{R}^d}e^{S/\gamma}D^k\pi\cdot D^k\pi\right. \\ &\quad \left.+\int_{\mathbb{R}^d}D^kw\cdot D^kw+\frac{b}{2}\int_{\mathbb{R}^d}(1+t)^{-b}D^kS\cdot D^kS\right) \\ &\quad +O\left(\frac{1}{(1+t)^2}Y_k^2\right). \end{aligned}$$

We have

$$r=\frac{b}{2}-\frac{d}{2}=\min\left(\frac{\gamma-1}{2}d,1,\frac{b}{2}\right)-\frac{d}{2},$$

since we have already chosen b such that

$$\frac{b}{2} \leq \min\left(1, \frac{\gamma - 1}{2} d\right).$$

Then we obtain:

$$\int S_k^1 + \frac{k+r}{1+t} Y_k^2 \leq \frac{C}{(1+t)^2} Y_k^2 \leq \frac{C}{(1+t)^2} Y_k Z (1+t)^{-\gamma_k}.$$

And for S_k^2 there is no difference from the isentropic case. It is a sum of terms like

$$Q(T) = \partial^k T \partial^\ell \bar{U} \partial^{k+1-\ell} T$$

for $1 \leq k \leq m$, $2 \leq \ell \leq k+1$, and $T = \pi$ or $T = w$. Thus we can show:

$$\left| \int S_k^2 \right| \leq \frac{C}{(1+t)^2} Y_k Z (1+t)^{-\gamma_k}.$$

Then we have proved the estimate (25).

(ii) Now we consider R_k. We note that the product $(A^0)^{-1} A^\alpha(U)$ contains some terms with $e^{S/\gamma}$. As a consequence, when we compute

$$D^k((A^0)^{-1} A^\alpha(U) \partial_\alpha U)$$

some new terms arise in addition of those we had in the isentropic case. They are of the form:

$$Q(U) = \partial^k T \left(\prod_{j=1}^{p} (\partial^j S)^{\beta_j} \right) \partial^{\ell - p} T \partial^{k+1-\ell} T,$$

with $1 \leq \ell \leq k$, $1 \leq p \leq \ell$, and $\sum_j j\beta_j = p$, for $T = \pi$ or $T = w$. We have also the usual terms and the terms corresponding to S:

$$\begin{cases} Q'(U) = \partial^k T \partial^\ell T \partial^{k+1-\ell} T & \text{for } 1 \leq \ell \leq k \text{ and } T = \pi \text{ or } T = w, \\ Q'(U) = (1+t)^{-b} \partial^k S \partial^\ell T \partial^{k-\ell+1} S & \text{for } 1 \leq \ell \leq k. \end{cases}$$

For these last terms, we find an upper bound as in the Proposition (i) of 3, that is to say:

$$\int |Q'(U)| \leq C |DU|_\infty Y_k^2 \leq C Y_k Z^2 (1+t)^{\theta_k},$$

with $\theta_k = -\gamma_k + \beta$. This corresponds to the case $x = 0$ in the proposition. Now we have to find an upper bound for the terms like $Q(U)$. We consider four cases.

(a) $1 \leq p \leq \ell - 1$. Note that in this case we have necessarily $k \geq 2$.

Let

$$p_j = 2\frac{k-1}{j}, \quad q = 2\frac{k-1}{\ell-p-1}, \quad s = 2\frac{k-1}{k-\ell}.$$

We have

$$\sum_{j=1}^{p} \frac{\beta_j}{p_j} + \frac{1}{q} + \frac{1}{s} = \frac{1}{2}.$$

Then:

$$\begin{aligned} \int |Q(U)| &\leq \|\partial^k T\|_0 \left\| \prod_{j=1}^{p} (\partial^j S)^{\beta_j} \partial^{\ell-p} T \partial^{k+1-\ell} T \right\|_0, \\ &\leq \|\partial^k T\|_0 \prod_{j=1}^{p} \left|\partial^j S\right|_{p_j}^{\beta_j} \left|\partial^{\ell-p} T\right|_q \left|\partial^{k+1-\ell} T\right|_s. \end{aligned}$$

We use Lemma 1

$$\begin{aligned} \left|\partial^j S\right|_{p_j} &\leq C\left|S\right|_\infty^{1-2/p_j} \left\|D^{k-1} S\right\|_0^{2/p_j}, \\ \left|\partial^{\ell-p} T\right|_q &\leq C\left|DT\right|_\infty^{1-2/q} \left\|D^k T\right\|_0^{2/q}, \\ \left|\partial^{k+1-\ell} T\right|_s &\leq C\left|DT\right|_\infty^{1-2/s} \left\|D^k T\right\|_0^{2/s}. \end{aligned}$$

Then since $T = \pi$ or $T = w$, $\|D^k T\|_0 \leq C Y_k$, and going back to the definition of Y_k, we see that $\|D^{k-1} S\|_0 \leq C(1+t)^{b/2} Y_{k-1}$. The same remarks can be made for the L^∞-norm

$$\begin{aligned} |S|_\infty &\leq C, \\ |DT|_\infty &\leq CZ(1+t)^\beta, \quad \text{for } T = \pi \text{ or } T = w, \\ |DS|_\infty &\leq CZ(1+t)^{\beta+b/2}. \end{aligned}$$

Thus, we obtain:

$$\begin{aligned} \int |Q(U)| &\leq C Y_k \left|S\right|_\infty^{\sum_j (1-2/p_j)\beta_j} \left\|D^{k-1} S\right\|_0^{2\sum_j \beta_j/p_j} \left|DT\right|_\infty^{2-2/q-2/s} \left\|D^k T\right\|_0^{2/q+2/s} \\ &\leq C Y_k Z^{2+p/(k-1)} (1+t)^{\theta_k}, \end{aligned}$$

with

$$\begin{aligned} \theta_k &= \left(-\gamma_{k-1} + \frac{b}{2}\right)\left(2\sum_j \frac{\beta_j}{p_j}\right) + \beta\left(2 - \frac{2}{q} - \frac{2}{s}\right) - \gamma_k\left(\frac{2}{q} + \frac{2}{s}\right) \\ &= \left(-\gamma_k + 1 + \frac{b}{2}\right)\left(2\sum_j \frac{\beta_j}{p_j}\right) + \beta\left(1 + 2\sum_j \frac{\beta_j}{p_j}\right) - \gamma_k\left(\frac{2}{q} + \frac{2}{s}\right) \quad = \end{aligned}$$

$$\begin{aligned} &= -\gamma_k + \beta + \left(1 + \frac{b}{2} + \beta\right)\left(2\sum_j \frac{\beta_j}{p_j}\right) \\ &= -\gamma_k + \beta + \left(1 + \frac{b}{2} + \beta\right)\frac{p}{k-1}. \end{aligned}$$

This gives $x = p/(k-1)$ for $1 \leq p \leq \ell - 1$ in the proposition.

(b) $1 \leq p = \ell$ and $\sum \beta_j = 1$. Note that $1 \leq p = \sum_j j\beta_j$. Thus we have necessarily $\beta_\ell = 1$ and $\beta_j = 0$ for $j \neq \ell$. That is to say $Q(U) = \partial^k T \partial^\ell S T \partial^{k+1-\ell} T$. Let

$$q = 2\frac{k-1}{\ell-1}, \quad s = 2\frac{k-1}{k-\ell}.$$

We have

$$\int |Q(U)| \leq \|\partial^k T\|_0 \, |T|_\infty \, |\partial^\ell S|_q \, |\partial^{k+1-\ell} T|_s.$$

We use that

$$\begin{aligned} |T|_\infty &\leq Z(1+t)^{\beta+1}, \\ |\partial^l S|_q &\leq C|DS|_\infty^{1-2/q} \, \|D^k S\|_0^{2/q}, \\ |\partial^{k+1-\ell} T|_s &\leq C|DT|_\infty^{1-2/s} \, \|D^k T\|_0^{2/s}. \end{aligned}$$

Then we obtain easily

$$\int |Q(U)| \leq C Y_k Z^3 (1+t)^{\theta_k},$$

with $\theta_k = -\gamma_k + \beta + (1 + b/2 + \beta)$. This gives the value $x = 1$ in the proposition.

(c) $1 \leq p = \ell$ and $\sum \beta_j = k$. This case occurs only when $\beta_1 = k$ and $\beta_j = 0$ for $j \neq 1$. That is to say $Q(U) = \partial^k T (\partial S)^k T \partial T$. We have:

$$\int |Q(U)| \leq C Y_k |DS|_\infty^k \, \|T\|_0 \, |DT|_\infty,$$

and this leads easily to

$$\int |Q(U)| \leq C Y_k Z^{2+k} (1+t)^{\theta_k},$$

with $\theta_k = -\gamma_k + \beta + (1 + b/2 + \beta)\, k$. This gives $x = k$ in the proposition.

(d) $p = \ell$ and $1 < \sum \beta_j < k$. We set

$$p_j = 2\frac{k-1}{j-1}, \; s = 2\frac{k-1}{k-\ell}, \; \frac{1}{q} = \frac{1}{2} - \left(\frac{1}{s} + \sum \frac{\beta_j}{p_j}\right).$$

Thus we have

$$2 < q = \frac{2(k-1)}{\sum \beta_j - 1}$$

and

$$\begin{aligned}\int Q(U) &\leq \|\partial^k T\|_0 \left\| \left(\prod_{j=1}^{\ell} (\partial^j S)^{\beta_j} \right) T \partial^{k+1-\ell} T \right\|_0 \\ &\leq \|\partial^k T\|_0 \prod_{j=1}^{\ell} |\partial^j S|_{p_j}^{\beta_j} |T|_q |\partial^{k+1-\ell} T|_s.\end{aligned}$$

We use Lemma 1:

$$\begin{aligned}|\partial^j S|_{p_j} &\leq C|DS|_\infty^{1-2/p_j} \|D^k S\|_0^{2/p_j}, \\ |\partial^{k+1-\ell} T|_s &\leq C|DT|_\infty^{1-2/s} \|D^k T\|_0^{2/s}.\end{aligned}$$

And we use the Sobolev's imbeddings theorem to find an upper bound to $|T|_q$. Indeed, for $n = d/2$ and for all $q \geq 2$, $H^n(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$. We deduce from that the following estimate:

$$|T|_q \leq CZ(1+t)^{-\gamma_0 - d(1/2-1/q)}.$$

Thus we obtain

$$\int |Q(U)| \leq CY_k Z^{2+\sum \beta_j} (1+t)^{\theta_k},$$

with $\theta_k = -\gamma_k + \beta + (1 + b/2 + \beta) \sum \beta_j$. This gives $x = 2, \cdots, k-1$ in the proposition. □

2.1.3. Conclusion. Now we conclude using (23) and Proposition 6. We have proved that:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}{Y_k}^2 + \frac{k+r}{(1+t)}{Y_k}^2 &\leq \frac{C}{(1+t)^2} Y_k Z(1+t)^{-\gamma_k} \\ &\quad + C \sum_{x \in \mathcal{E}_k} Y_k Z^{2+x} (1+t)^{e(x)},\end{aligned} \tag{26}$$

with $\mathcal{E}_k = \{0,\ p/(k-1) \text{ for } 1 \leq p \leq k-1,\ 1,\ 2,\ \ldots,\ k\}$ and $e(x) = -\gamma_k + \beta + (\beta + 1 + b/2)x$.

Recall that $\gamma_k = k + r - a$ and $r = b/2 - d/2$. We simplify by Y_k, multiply by $(1+t)^{\gamma_k}$, and make the sum over k to obtain:

$$\frac{dZ}{dt}(t) + \frac{a}{(1+t)} Z(t) \leq C \left(\frac{1}{(1+t)^2} Z(t) + \sum_k \sum_{x \in \mathcal{E}_k} Z(t)^{2+x} (1+t)^{\beta + ax} \right).$$

The constant C is positive and depends only on γ, m, d, δ, $\|u_0\|_X$. We used that

$$\beta = -\gamma_1 - \frac{d}{2} = -1 - r + a - \frac{d}{2} = -1 - \frac{b}{2} + a.$$

Now we claim that

$$\begin{aligned} \frac{dZ}{dt}(t) + \frac{a}{(1+t)} Z(t) \quad &\leq \quad C\left(\frac{1}{(1+t)^2} Z(t) + Z(t)^2 (1+t)^{\beta} \right. \\ &\qquad \left. + Z(t)^{2+m}(1+t)^{\beta+am}\right), \end{aligned}$$

since we keep the smaller and the biggest $e(x)$ in the sum. We choose $a = 1+b/2$, that is to say $\beta = 0$ to obtain

$$\begin{aligned} \frac{dZ}{dt}(t) + \frac{a}{(1+t)} Z(t) \quad &\leq \quad C\left(\frac{1}{(1+t)^2} Z(t) + Z(t)^2 \right. \\ &\qquad \left. + Z(t)^{2+m}(1+t)^{am}\right). \end{aligned} \tag{27}$$

We set $\zeta(t) = (1+t)^a \exp(C/(1+t))Z(t)$ to get

$$\frac{d\zeta}{dt}(t) \leq C \frac{\zeta^2}{(1+t)^a}(1+\zeta(t)^m). \tag{28}$$

Using that $\zeta^2 + \zeta^{2+m} \leq 2\zeta(1+\zeta^{1+m})$, we modify (28)

$$\frac{1}{\zeta(1+\zeta^{1+m})}\frac{d\zeta}{dt}(t) \leq \frac{\tilde{C}}{(1+t)^a}. \tag{29}$$

Now we can integrate (29). Consider

$$f(x) = \frac{1}{m+1} \ln\left(\frac{x^{m+1}}{(1+x^{m+1})}\right) \quad \text{and} \quad \psi(t) = f(\zeta(t)) + \frac{\tilde{C}}{a-1}(1+t)^{1-a}.$$

We derivate ψ to obtain

$$\psi'(t) = \frac{1}{\zeta(1+\zeta^{1+m})}\zeta'(t) - \frac{\tilde{C}}{(1+t)^a}.$$

Since (29) holds, we conclude that:

$$\psi(t) \leq \psi(0) \quad \text{for all } t \geq 0.$$

And this leads to

$$f(\zeta(t)) \leq f(\zeta(0)) + \frac{\tilde{C}}{a-1} \quad \text{for all } t \geq 0.$$

To finish, we remark that f is strictly increasing and that $f(x) \to -\infty$ when $x \to 0$. Thus if $\zeta(0) \leq \varepsilon_0$, then

$$f(\zeta(t)) \leq f(\varepsilon_0) + \frac{\tilde{C}}{a-1} \quad \text{for all } t \geq 0.$$

This implies that $\zeta(t) \leq M_0 = M(\varepsilon_0, a, \tilde{C})$ for all $t \geq 0$. As a consequence, we have:

$$Z(t) \leq M_0(1+t)^{-a} \exp\left(-\frac{C}{1+t}\right) \quad \text{for all } t \geq 0.$$

We conclude that if we have $Z(0) \leq \varepsilon_0 \exp(-C)$, then

$$Z(t) \leq M_0(1+t)^{-a} \quad \text{for all } t \geq 0.$$

That corresponds to hypothesis (H1), since $Z(0) \sim \|\pi_0\|_m + \|S_0\|_m$.

2.2. Uniqueness results and corollary. In the isentropic case, we prove two results of uniqueness: the first one is local in time and space, the second one is local in space and global in time. In the proofs, we used the symmetric form of the system (21):

$$A^0(V)\partial_t V + \sum_{\alpha=1}^{d} A^\alpha(V)\partial_\alpha V = 0. \tag{30}$$

The proofs work in the same way in the general case, one has just to deal with $A^0(V)$ instead of A^0.

Proposition 7. *Let V_0^1, V_0^2 be two initial data for* (21)*. Assume that $V_0^1 \in H^m(\mathbb{R}^d)$. Let $V^1 = (\pi, u, S)^T$, V^2 be two associated solutions of* (21) *defined for $0 \leq t \leq T_0$ and let*

$$M \geq \sup_{0 \leq t \leq T_0} \left(C_1 e^{|S_0|_\infty/2\gamma}|\pi|_\infty + |u|_\infty\right)(t).$$

Suppose that $V_0^1 = V_0^2$ on $B_0 = B(x_0, \eta)$. We take $C_T = \{(x,t) \mid 0 \leq t \leq T$, $x \in B_t = B(x_0, \eta - Mt)\}$ for $0 \leq T \leq T_1 = \min(T_0, \eta/M)$. Then $V^1 = V^2$ on C_{T_1}.

Proposition 8. *Let (ρ_0, u_0, S_0) satisfy* (H1)-(H4)*. Let $U = (\pi, u, S)^T$ be the global solution of* (21) *given by Theorem (3) and $\bar{u}$ be the solution of* (4)*. Consider V a global solution of* (21) *such that $DV \in L^\infty(\mathbb{R}^d \times \mathbb{R}_+)$.*

Then for all $\nu \in \,]2-a, 1[$, for all $R_0 > 0$, there exists $T_0 > 0$ such that, if $U(\cdot, T_0) = V(\cdot, T_0)$ on $B(0, R_0)$, then U and V are equal on the domain $\{(x,t) : |x - x(t)| \leq R(t)$, for all $t \geq T_0\}$, where $x(t)$ is the solution of $x'(t) = u(x(t), t)$, $x(T_0) = 0$, and $R(t) = R_0(1+t)^\nu$.

Using the local uniqueness result, we can show as in the isentropic case that the hypothesis (H4) can be forgotten:

Corollary 2. *Let (ρ_0, u_0, S_0) be the initial data for Problem* (1)*. Suppose* (H1)-(H3)*. Then the result of Theorem 3 is still true.*

REFERENCES

[1] D. SERRE, *Solutions classiques globales des équations d'Euler pour un fluide parfait compressible,* Annales de l'Institut Fourier **47** (1952), 139–153.

[2] J. Y. CHEMIN, *Dynamique des gaz à masse totale finie,* Asymptotic Analysis **3** (1990), 215–220.

[3] P. D. LAX, *Development of singularities of solutions of nonlinear hyperbolic partial differential equations,* J. Math. Phys. **5** (1964), 611–513.

[4] T. SIDERIS, *Formation of singularities in three-dimensional compressible fluids,* Commun. Math. Phys. **101** (1985), 475–485.

[5] LIU FAGUI, *Global smooth resolvability for one dimensional gas dynamics systems,* communication personnelle.

[6] T. MAKINO, S. UKAI & S. KAWASHIMA, *Sur la solution à support compact de l'équation d'Euler compressible,* Japan J. Appl. Math. **3** (1986), 249–257.

[7] A. FRIEDMAN, *Partial Differential Equations,* Holt, Rinehart, Winston, 1969.

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Received: April 10th, 1998; revised: June 22nd, 1998.