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CONVERGENCE RESULTS FOR SOME CONSERVATION LAWS WITH A REFLUX BOUNDARY CONDITION AND A RELAXATION TERM ARISING IN CHEMICAL ENGINEERING*

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Abstract. This paper deals with a system of $2N$ semilinear transport equations with a boundary condition of imposed flux. The right-hand side models some kinetic exchange between two phases. It is thus a stiff term involving a small parameter which will tend to 0. Using compensated compactness, one proves, under some assumptions on the flux, that the solution to this system converges to a solution to a system of N quasilinear equations, a solution which satisfies a set of entropy inequalities. Thus the reflux boundary condition for the quasi-linear system is given a meaning.

Key words. hyperbolic systems, boundary conditions, relaxation, entropy, compensated compactness, chromatography, distillation

AMS subject classifications. 35L65, 35L67, 35Q20

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1. Introduction.

We are interested in the following system of $2N$ equations, $N \geq 1$,

$$\left\{ \begin{aligned} &\partial_t \mathbf{c}^1_\varepsilon + \partial_x u \mathbf{c}^1_\varepsilon = \tfrac{1}{\varepsilon}\big(\mathbf{c}^2_\varepsilon - \mathbf{h}(\mathbf{c}^1_\varepsilon)\big), \\ &\partial_t \mathbf{c}^2_\varepsilon + \partial_x v \mathbf{c}^2_\varepsilon = -\tfrac{1}{\varepsilon}\big(\mathbf{c}^2_\varepsilon - \mathbf{h}(\mathbf{c}^1_\varepsilon)\big), \end{aligned} \right. \tag{1.1}$$

which is a simplified model of diphasic propagation arising in chemical engineering. In this kind of problem, two phases labelled 1 and 2 are in motion with respective velocities $u > 0$ and $v \leq 0$, which are assumed here to be constant. The case $v = 0$ corresponds to a model of chromatography (a mobile phase and a stationary one), and the case $v < 0$ corresponds to distillation (two phases moving countercurrent).

In equations (1.1), $\mathbf{c}^1_\varepsilon$ and $\mathbf{c}^2_\varepsilon$ are related to the concentrations in phase 1 and 2, respectively, and therefore should be nonnegative. The right-hand side rules the matter exchanges between the two phases. Without motion, the two phases would reach a state of thermodynamical equilibrium: the concentration in phase 2 is therefore related to the concentration in phase 1 by the function $\mathbf{h}$, which enjoys several properties coming from the thermodynamics.

In the case we are considering, the equilibrium cannot be reached because of the motion. The time needed to reach the equilibrium is not negligible with respect to the characteristic times induced by the velocities u and v. This phenomenon is known as a finite exchange kinetic: the actual concentration $\mathbf{c}^2_\varepsilon$ in phase 2 differs from $\mathbf{h}(\mathbf{c}^1_\varepsilon)$. The right-hand side of the equations quantifies the attraction of the system to the equilibrium state: it is a pulling-back force, and the constant parameter $1/\varepsilon$ is the "velocity" of exchange between the two phases.

A natural question arises here: how do the solutions of (1.1) behave when ε tends to 0, that is, when the exchange kinetic becomes instantaneous (the process is then quasi-static)? The limit system is obtained in a natural way by summing the $2N$

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equations in (1.1) and by putting $\mathbf{c}_\varepsilon^1 = \mathbf{c}$, $\mathbf{c}_\varepsilon^2 = \mathbf{h}(\mathbf{c})$, which means indeed that the concentration in phase 2 is actually the equilibrium concentration. We are led to the following nonlinear hyperbolic system, which expresses the conservation of matter:

$$\partial_t \left(\mathbf{c} + \mathbf{h}(\mathbf{c})\right) + \partial_x \left(u\mathbf{c} + v\mathbf{h}(\mathbf{c})\right) = 0. \tag{1.2}$$

The aim of this paper is to analyze the behavior of the solutions of (1.1) when ε tends to 0, when it is provided with boundary conditions

(1.3)

$$\mathbf{c}_\varepsilon^1(0,t) = \mathbf{a}(t) \in L^\infty(]0,+\infty[)^N, \quad u\mathbf{c}_\varepsilon^1(1,t) + v\mathbf{c}_\varepsilon^2(1,t) = \mathbf{b}(t) \in L^\infty(]0,+\infty[)^N,$$

together with Cauchy data in $L^\infty(]0,1[)^N$. To avoid any initial layer, we shall assume that the initial data are at equilibrium, that is, $\mathbf{c}_\varepsilon^1(\cdot,0) = \mathbf{c}^0 \in L^\infty(]0,1[)^N$ and $\mathbf{c}_\varepsilon^2(\cdot,0) = \mathbf{h}(\mathbf{c}^0)$. From the point of view of distillation, the boundary conditions are natural: the first one is a Dirichlet-like "injection" at one end of a column and acts only on the incoming variable ($u > 0$); the second one looks like a Neumann condition on the other end and imposes $v < 0$ (it is a simplified model of the "reflux" in a distillation column).

Concerning the standard Cauchy problem in the scalar case, i.e., $c(0,x) = c^0(x)$, $x \in \mathbb{R}$, $c^0 \in L^\infty$, the analysis is straightforward, and the solution of (1.1) tends to the entropy solution of (1.2), thus providing an alternative to the artificial viscosity method. Such results were obtained, for instance, by Tveito and Winther in [29], where the rate of convergence is estimated, and by Natalini [22]. Let us mention also the work by Katsoulakis and Tzavaras [19], where they give contraction properties for the solution of the system with relaxation. For systems of conservation laws, we refer to Chen, Levermore, and Liu [7], where a convergence result is proved for a 2×2 genuinely nonlinear system. This point of view can be successfully used for numerical purpose, see Jin and Xin [16] for a general setting for systems and Aregba-Driollet and Natalini [2] for convergence results in the scalar case.

On the other hand, the problem with boundary conditions is not as well behaved when ε tends to 0: it is well known that the setting of a Dirichlet boundary condition for a nonlinear hyperbolic scalar equation is difficult. Bardos, Leroux, and Nédélec [3] gave such a setting in the Kružkov sense, using the artificial viscosity method in the context of BV functions. We shall not recover this formulation here, since the Dirichlet data act only on incoming variables. For systems, the first existence result was given by Benabdallah and Serre [4] for systems of two equations. We refer also to works by Dubois and LeFloch [8], where the Dirichlet boundary condition appears as a Riemann problem on a half-plane, Gisclon [10], and Gisclon and Serre [11]. We mention also Goodman's work [12], where global existence is proved for strictly hyperbolic systems of conservation laws with initial and boundary data of small BV norm. The solutions also have small total variation and therefore have strong traces on the boundary. On the other hand, in [18] Kan, Santos, and Xin consider a general system of conservation laws and compare various notions of boundary conditions (vanishing viscosity, half-space Riemann problem). Their solution is built by a Godunov method. In the same spirit, we also mention the paper by Joseph and LeFloch [17], who also compare different approximations and the resulting boundary layers.

The reflux boundary condition at $x = 1$ seems to have been very little studied. For the scalar Burgers equation with the boundary condition $u^2(.,t) = 0$, Gisclon proved in [9] that the solution satisfies $u(.,t) \leq 0$ on the boundary (which coincides with the solution in the sense of [3]).

Finally, let us mention one work which is concerned with both relaxation and boundary conditions. Wang and Xin [30] consider a 2×2 system with relaxation. The boundary condition is chosen so that uniform BV estimates hold, and they prove convergence to a scalar conservation law satisfying a boundary-entropy condition, for which uniqueness holds.

We are going to prove that, under suitable conditions on the flux $u\mathbf{c}+v\mathbf{h}(\mathbf{c})$ with respect to $\mathbf{b}$, there exists a subsequence of solutions of (1.1) which converges to a weak solution of (1.2). This solution is characterized by a set of entropy inequalities. Since we have no BV estimates for the solution with $\varepsilon > 0$, we are led to work with bounded measurable functions, and use the compensated compactness method. This can be done in two cases: first for scalar equations with any smooth function $\mathbf{h}$ and then for a system of N equations, for a specific $\mathbf{h}$, the so-called Langmuir isotherm. Notice that the Langmuir system is not hyperbolic on the whole physical domain of interest. However, we use a specific set of entropies, namely the so-called kinetic entropies, which were introduced in [14], that allows us to achieve compactness.

Finally, we prove that the weak solutions are indeed solutions in the sense of distributions and that they satisfy in a strong sense the initial condition as well as the reflux boundary condition at $x = 1$. The incoming boundary condition seems to be lost in the limiting system. This is not very surprising, since we fall from a $2N$ equations system to N equations. Some boundary layer phenomena probably occur at $x = 0$, which we do not investigate here. This may indicate that the system of conservation laws with the reflux boundary condition is well-posed, but the precise study of this is left for future research.

The paper is organized as follows. In section 2 we state a few results and notations which hold for both the scalar equation and the system. Section 3 and 4 are devoted to the proof of a priori estimates and compactness, respectively, for the scalar equation and the system. Section 5 deals with boundary conditions.

2. Preliminary results.

We state here a few results and remarks that are common to both the scalar equation and the Langmuir model. Namely, we prove that equation (1.1) is well-posed for $\varepsilon > 0$, and we also define a particular set of entropies, which appears to be natural from the structure of the equations. In the following, we shall set $\Omega \stackrel{\text{def}}{=}]0,1[\times]0,T[$.

2.1. Existence for $\varepsilon > 0$.

THEOREM 2.1. *For a given $T > 0$, assume that $\mathbf{a}$ and $\mathbf{b}$ are in $L^\infty(]0,T[)^N$, $\mathbf{c}^0 \in L^\infty \cap L^1(]0,1[)^N$, and that the function $\mathbf{h}$ is of class C^1. Then there exists a unique solution to* (1.1), *which lies in* $L^\infty(]0,T[;L^1(]0,1[))$.

Proof. We first rewrite (1.1) in an equivalent integral form by using Duhamel's principle; then we prove a contraction estimate to apply a fixed point theorem. This is rather tedious, because of the initial and boundary conditions. The set $[0,1]\times[0,+\infty[$ is indeed divided into four zones, namely, $Z_1 = \{(x,t) \mid x \geq ut, x \leq 1+vt\}$, $Z_2 = \{(x,t) \mid x \geq ut, x \geq 1+vt\}$, $Z_3 = \{(x,t) \mid x \leq ut, x \leq 1+vt\}$, $Z_4 = \{(x,t) \mid x \leq ut, x \geq 1+vt\}$, depending upon whether the characteristics encounter $\{t=0\}$, $\{x=0\}$, or $\{x=1\}$.

We shall fully write the contraction estimate for t large enough so that $(x,t) \in Z_4$ for every $x \in [0,1]$. We omit in this proof the dependence in ε. Taking into account the reflux boundary condition on $x = 1$, Duhamel's principle writes, for almost every

$(x,t) \in Z_4$,

(2.1)

$$\begin{cases} \mathbf{c}^1(x,t) = \mathbf{a}\left(t - \frac{x}{u}\right) + \frac{1}{\varepsilon}\int_{t-\frac{x}{u}}^{t} \left[\mathbf{c}^2(x+u(s-t),s) - \mathbf{h}(\mathbf{c}^1(x+u(s-t),s))\right] ds, \\ \mathbf{c}^2(x,t) = \frac{1}{v}\mathbf{b}\left(t + \frac{1-x}{v}\right) - \frac{u}{v}\mathbf{a}\left(t + \frac{1-x}{v} - \frac{1}{u}\right) \\ -\frac{u}{v}\frac{1}{\varepsilon}\int_{t+\frac{1-x}{v}-\frac{1}{u}}^{t+\frac{1-x}{v}} \left[\mathbf{c}^2(1+u(s-t-\frac{1-x}{v}),s) - \mathbf{h}(\mathbf{c}^1(1+u(s-t-\frac{1-x}{v}),s))\right] ds \\ \qquad -\frac{1}{\varepsilon}\int_{t+\frac{1-x}{v}}^{t} \left[\mathbf{c}^2(x+v(s-t),s) - \mathbf{h}(\mathbf{c}^1(x+v(s-t),s))\right] ds. \end{cases}$$

Denote by $\mathcal{T}$ the application from $X = L^\infty(]0,T[;L^1_x)^{2N}$ into itself which associates the right-hand side of the equations in (2.1) with a pair $C = (\mathbf{c}^1, \mathbf{c}^2) \in X$. For two elements C and $\hat{C}$ in X, with the same initial and boundary data, the terms involving $\mathbf{a}$ and $\mathbf{b}$ in (2.1) disappear when computing $\mathcal{T}(C) - \mathcal{T}(\hat{C})$, so, for a given (x,t), we have

$$|\mathcal{T}(C)(x,t) - \mathcal{T}(\hat{C})(x,t)| \leq \frac{1}{\varepsilon}\max(|T_1(x,t)|, |T_2(x,t)|),$$

where T_i follows from the difference of the integral terms and $|\cdot|$ is a norm on $\mathbb{R}^{2N}$. One has easily

$$\begin{aligned} |T_1(x,t)| \leq & \int_{t-\frac{x}{u}}^{t} \left|\mathbf{c}^2(x+u(s-t),s) - \hat{\mathbf{c}}^2(x+u(s-t),s)\right| ds \\ & + \int_{t-\frac{x}{u}}^{t} \left|\mathbf{h}(\mathbf{c}^1(x+u(s-t),s)) - \mathbf{h}(\hat{\mathbf{c}}^1(x+u(s-t),s))\right| ds \\ \leq & \int_{t-\frac{x}{u}}^{t} \left|\mathbf{c}^2(x+u(s-t),s) - \hat{\mathbf{c}}^2(x+u(s-t),s)\right| ds \\ & + K\int_{t-\frac{x}{u}}^{t} \left|\mathbf{c}^1(x+u(s-t),s) - \hat{\mathbf{c}}^1(x+u(s-t),s)\right| ds \end{aligned}$$

if K is the Lipschitz constant of $\mathbf{h}$. We can estimate $\|T_1(.,t)\|_{L^1_x}$ by Fubini's theorem, which gives

$$\begin{aligned} \|T_1(.,t)\|_{L^1_x} & \leq \int_{t-1/u}^{t} \|\mathbf{c}^2(.,s) - \hat{\mathbf{c}}^2(.,s)\|_{L^1_x} ds + K\int_{t-1/u}^{t} \|\mathbf{c}^1(.,s) - \hat{\mathbf{c}}^1(.,s)\|_{L^1_x} ds \\ & \leq t\left(\max_{s\in[0,t]} \|\mathbf{c}^2(.,s) - \hat{\mathbf{c}}^2(.,s)\|_{L^1_x} + K\max_{s\in[0,t]} \|\mathbf{c}^1(.,s) - \hat{\mathbf{c}}^1(.,s)\|_{L^1_x}\right). \end{aligned}$$

A similar formula can be obtained for T_2, involving the quantity $u/|v|$.

Now, if (x,t) changes zone with x, we proceed in the same way in each zone and separate the integral for the L^1_x norm. We do not write these straightforward computations, which lead to the existence of a constant $M > 0$, which depends on K and $u/|v|$, such that

$$\|\mathcal{T}(C) - \mathcal{T}(\hat{C})\|_{L^\infty(]0,t[;L^1_x)^{2N}} \leq \frac{tM}{\varepsilon}\|C - \hat{C}\|_{L^\infty(]0,t[;L^1_x)^{2N}}.$$

Now, choose T_0 such that $T_0 M/\varepsilon < 1$, and apply the fixed point theorem on $L^\infty(]0,T_0[;L^1_x)^{2N}$. This gives existence and uniqueness of the solution on $[0,T_0]$.

Since the contraction estimate does not depend on the initial data, we can perform again the same argument on $[T_0, 2T_0]$, and so on, to finally reach any prescribed $T > 0$. Thus the theorem is proved. ◻

2.2. Diphasic entropies. We introduce here a set of entropies which is quite natural in view of the structure of the equations. They are actually a discrete version (with two velocities only) of the kinetic entropies introduced by Perthame and Tadmor in [23].

DEFINITION 2.1. *We shall say that a function* $\eta : \mathbb{R}^N \longrightarrow \mathbb{R}$ *is a "diphasic entropy" for* (1.2) *if there exist two convex functions* $\eta_1, \eta_2 : \mathbb{R}^N \longrightarrow \mathbb{R}$ *satisfying*

$$\nabla_{\mathbf{c}}\eta_1(\mathbf{c}) = \nabla_{\mathbf{c}}\eta_2\big(\mathbf{h}(\mathbf{c})\big) \qquad \forall \mathbf{c} \in \mathbb{R}^N, \tag{2.2}$$

such that $\eta(\mathbf{c}) = \eta_1(\mathbf{c}) + \eta_2\big(\mathbf{h}(\mathbf{c})\big)$.

Remark 2.1. The function $\mathbf{h}$ itself is, in general, defined by such a pair of functions, which are given, for instance, by statistical thermodynamics models (see [15] and the quoted references therein for examples and more information). Actually, the pair $(\mathbf{c}, \mathbf{h}(\mathbf{c}))$ is a stable state of equilibrium for the diphasic system. Thus it achieves the infimum of $\eta_1(\mathbf{c}_1) + \eta_2(\mathbf{c}_2)$ under the constraint that the total amount of matter $\mathbf{c}_1 + \mathbf{c}_2$ is constant. The relation (2.2) is nothing but the characterization of the minimum and is a generalized version of the well-known "chemical potential equalities" in thermodynamics. Consequently, $\mathbf{h}'(\mathbf{c})$ is positive in the scalar case and is diagonable with positive eigenvalues for a system. This leads also to the existence of a natural "physical" entropy for such systems.

Our main concern in the following is to obtain a priori estimates on the solution $(\mathbf{c}_\varepsilon^1, \mathbf{c}_\varepsilon^2)$ to (1.1) which are uniform in ε. The classical method here is to prove that the entropy production associated with (1.1) is nonpositive for several well-chosen entropies. Consider any pair (η_1, η_2) satisfying (2.2); multiply the two equations in (1.1), respectively, by $\nabla_{\mathbf{c}}\eta_1(\mathbf{c}_\varepsilon^1)$ and $\nabla_{\mathbf{c}}\eta_2(\mathbf{c}_\varepsilon^2)$; sum; then use (2.2). We formally obtain the following law for the entropy production:

$$\begin{aligned}\partial_t\left(\eta_1(\mathbf{c}_\varepsilon^1) + \eta_2(\mathbf{c}_\varepsilon^2)\right) + \partial_x\left(u\eta_1(\mathbf{c}_\varepsilon^1) + v\eta_2(\mathbf{c}_\varepsilon^2)\right)\\ = \frac{1}{\varepsilon}\left[\left(\nabla_{\mathbf{c}}\eta_2\big(h(\mathbf{c}_\varepsilon^1)\big) - \nabla_{\mathbf{c}}\eta_2(\mathbf{c}_\varepsilon^2)\right)\cdot\left(\mathbf{c}_\varepsilon^2 - \mathbf{h}(\mathbf{c}_\varepsilon^1)\right)\right].\end{aligned} \tag{2.3}$$

It remains to notice that the right-hand side is always nonpositive, since η_2 is convex. Integrating on $[0, 1]$ therefore gives, at least formally,

$$\frac{d}{dt}\int_0^1 \left[\eta_1(\mathbf{c}_\varepsilon^1(x,t)) + \eta_2(\mathbf{c}_\varepsilon^2(x,t))\right]dx \le -\left[u\eta_1(\mathbf{c}_\varepsilon^1(.,t)) + v\eta_2(\mathbf{c}_\varepsilon^2(.,t))\right]\Big|_0^1, \tag{2.4}$$

and all the technical work is now to estimate the boundary terms. To give a precise meaning to this differential inequality, we have to rewrite it in a weak form by multiplying by a test function $\varphi \ge 0$ and integrating by parts.

Provided we have enough entropies, (2.4) will give a priori estimates as well as compactness of a subsequence of solutions to (1.1). We can exhibit such entropies in the scalar case on the one hand and for the system of chromatography with the Langmuir isotherm on the other hand. In both cases, the local solution of Theorem 2.1 is therefore global for fixed ε.

2.3. Subcharacteristic condition. Before proceeding to estimates, we would like to relate system (1.1) with the usual form of systems with relaxation. This is done easily in the particular case $v = -u$ by setting $\mathbf{U}^\varepsilon = \mathbf{c}_\varepsilon^1 + \mathbf{c}_\varepsilon^2 \in \mathbb{R}^N$ and $\mathbf{V}^\varepsilon = u\mathbf{c}_\varepsilon^1 - u\mathbf{c}_\varepsilon^2 \in \mathbb{R}^N$. System (1.1) is therefore rewritten as

$$\partial_t \mathbf{U}^\varepsilon + \partial_x \mathbf{V}^\varepsilon = 0, \qquad \partial_t \mathbf{V}^\varepsilon + u^2 \partial_x \mathbf{U}^\varepsilon = \frac{2}{\varepsilon}\left(\mathbf{V}^\varepsilon - u(\mathbf{c}_\varepsilon^1 - \mathbf{h}(\mathbf{c}_\varepsilon^1))\right). \tag{2.5}$$

Now, we notice that, since by Remark 2.1 $\mathbf{h}'(\mathbf{c})$ has positive eigenvalues, the function $\mathbf{c} + \mathbf{h}(\mathbf{c})$ is one-to-one. Let us denote $\mathbf{U} = \mathbf{c} + \mathbf{h}(\mathbf{c})$, its inverse by $\mathbf{c} = \mathbf{g}(\mathbf{U})$, and $\mathbf{F}(\mathbf{U}) \stackrel{\text{def}}{=} u[\mathbf{g}(\mathbf{U}) - \mathbf{h}(\mathbf{g}(\mathbf{U}))]$. The usual form of this kind of system should involve $\mathbf{F}(\mathbf{U})$ instead of $u(\mathbf{c}_\varepsilon^1 - \mathbf{h}(\mathbf{c}_\varepsilon^1))$ in (2.5). This discrepancy appears because the right-hand side of system (1.1) is not symmetric with respect to $\mathbf{c}^1$ and $\mathbf{c}^2$. Another possible writing would make use of the "Maxwellians" $\mathbf{M}_1(\mathbf{U}) = \mathbf{g}(\mathbf{U})$ and $\mathbf{M}_2(\mathbf{U}) = \mathbf{h}(\mathbf{g}(\mathbf{U}))$. The convergence results would not be affected by this change.

In [21], Liu introduced a necessary condition on $\mathbf{F}'(\mathbf{U})$ to ensure the convergence of a subsequence of solutions of (1.1) to a solution of (1.2). This condition is known as the subcharacteristic condition, and we would like to point out that it is satisfied here because the function $\mathbf{h}$ is, in some sense, monotone (see Remark 2.1). Indeed, we have

$$\mathbf{F}'(\mathbf{U}) = [I_N + \mathbf{h}'(\mathbf{g}(\mathbf{U}))]\,\mathbf{g}'(\mathbf{U}),$$

where I_N stands for the identity matrix in $\mathbb{R}^N$. However, $\mathbf{g}'(\mathbf{U}) = (I_N + \mathbf{h}'(\mathbf{g}(\mathbf{U})))^{-1}$, so $\mathbf{F}'(\mathbf{U})$ is diagonable, and its eigenvalues are given for general values of u and v by

$$\lambda_i(\mathbf{U}) = \frac{u + v\mu_i(\mathbf{U})}{1 + \mu_i(\mathbf{U})},$$

where $\mu_i > 0$ are the eigenvalues of $\mathbf{h}'$, $1 \le i \le N$. It is readily seen that $v < \lambda_i(\mathbf{U}) < u$, which is the specific version of Liu's condition in this context.

3. Scalar equation. This section is devoted to the proof of the strong convergence of a subsequence of solutions to (1.1) in the scalar case. The function h is therefore a scalar function, which satisfies

$$h(0) = 0 \qquad \text{and} \qquad h'(c) > 0 \quad \forall c. \tag{3.1}$$

Also, for any given convex η_2, we can define η_1 by $\eta_1(c) = \int_0^c \eta_2'\big(h(\sigma)\big)\,d\sigma$. We have, obviously, $\eta_1'(c) = \eta_2'\big(h(c)\big)$ and $\eta_1''(c) = \eta_2''\big(h(c)\big)h'(c) > 0$. Two particular cases are interesting. These are nonsmooth entropies, but a classical regularization argument, omitted in the following, allows us to deal with them.

- "Kružkov-like" entropies. For $k \in \mathbb{R}$, we set

$$\varphi_1^k(c^1) = |c^1 - k|, \qquad \varphi_2^k(c^2) = |c^2 - h(k)|.$$

It is easily checked that φ_1^k and φ_2^k satisfy (2.2), since h is increasing.

- L^∞ entropies. For $k \in \mathbb{R}$, we define

$$\psi_1^k(c^1) = (c^1 - k)^+, \qquad \psi_2^k(c^2) = \big(c^2 - h(k)\big)^+.$$

With these last entropies, the entropy estimates on c_ε^i become L^∞ estimates.

We begin in a classical way with some entropy and a priori estimates and first notice that, to prove entropy estimates for a given pair (η_1, η_2) giving a diphasic entropy, we need the condition

$$f(c) \stackrel{\text{def}}{=} uc + vh(c) \leq \min_{t>0} b(t) \qquad \text{for } c \geq M. \tag{3.2}$$

This is not a very satisfactory condition to impose, since it is not satisfied by such an usual isotherm as the Langmuir one,

$$h(c) = Kc/(1+c), \qquad K > 0. \tag{3.3}$$

Condition (3.2) actually implies some restrictions on the initial and boundary data, which lead to uniform L^∞ estimates for the solution to (1.1), for a broader class of fluxes.

THEOREM 3.1. *Assume $a \geq 0$, $b \leq 0$, $c^0 \geq 0$, and*

$$c^\star \stackrel{\text{def}}{=} \sup\{c \geq 0; \exists c' \leq c, f(c') \leq \min b(t)\} \geq \max[\|a\|_\infty, \|c^0\|_\infty]. \tag{3.4}$$

Then there exists a constant C depending only on c_1^0, a, and b such that $0 \leq c_\varepsilon^i(t,.) \leq C$, $i = 1, 2$.

Remark 3.1. Of course the result is meaningful only if $c^\star > 0$. This occurs only if $f(c)$ becomes nonpositive for some c. For instance, consider again the case of the Langmuir isotherm (3.3). It is easily seen that for $b = 0$, $c^\star > 0$ only if $u/|v| < K$. More generally, f achieves its minimum for $c_{\min} = \sqrt{K|v|/u} - 1$, which is positive if $u/|v| < K$, and $c^\star > 0$ if we have $f(c_{\min}) \leq b$, that is, $(\sqrt{K|v|} - \sqrt{u})^2 \geq -b$.

Remark 3.2. The choice $k = \max(\|a\|, \|c^0\|)$ is possible only if $f(a(t)) - b(t) \geq 0$ and $f(c^0(x)) - b(x) \geq 0$ for all $t > 0$ and a.e. $x \in]0,1[$. Otherwise, the L^∞ norm of the solution may not be bounded by the initial and Dirichlet boundary data.

From the above L^∞ estimate, we can easily obtain a weak convergence result by considering the weak form of the first equation in (1.1). Since c_ε^1 is L^∞ bounded uniformly in ε, and ε tends to 0, then $c_\varepsilon^2 - h(c_\varepsilon^1)$ tends to 0 in the sense of distributions on Ω when ε tends to 0. But we actually have the following stronger result.

LEMMA 3.2. *Under the assumptions of Theorem* 3.1, *ensuring the L^∞ bounds on the solution, $c_\varepsilon^2 - h(c_\varepsilon^1)$ tends to 0 in $L^2_{\text{loc}}(\Omega)$.*

From this result we can deduce, using the compensated compactness method, the following main result of this section.

THEOREM 3.3. *Consider $a, b \in L^\infty(]0,T[)$, and $c^0 \in L^1 \cap L^\infty(]0,1[)$. Under the assumptions of Theorem* 3.1, *that is,*

$$a \geq 0, \quad b \leq 0, \quad c^0 \geq 0, \quad c^\star \geq \max[\|a\|_\infty, \|c^0\|_\infty],$$

there then exists a subsequence of solutions to (1.1), *still denoted by c_ε^1, which converges a.e. and strongly in $]0,1[\times]0,T[$ to $c \in L^\infty(]0,T[; L^1(]0,1[))$. Moreover, c satisfies, for any $\varphi \in \mathcal{D}(\overline{\Omega})$, $\varphi \geq 0$, $k \in \mathbb{R}$,*

$$\begin{aligned} -\int_0^T \int_0^1 \Big[\big(|c-k| + |h(c) - h(k)|\big)\partial_t \varphi + \big(u|c-k| + v|h(c)-h(k)|\big)\partial_x \varphi \Big] dxdt \\ \leq \int_0^T u|a(t) - k|\varphi(0,t)dt + \int_0^T |b(t) - f(k)|\varphi(1,t)dt \\ - \int_0^1 \big(|c^0(x) - k| + |h(c^0(x)) - h(k)|\big)\, \varphi(x,0)dx. \end{aligned} \tag{3.5}$$

Section 3.1 contains the entropy estimates and the proof of Theorem 3.1, while section 3.2 is devoted to the convergence results. Finally, we give a few remarks on viscous regularization is section 3.3.

3.1. A priori estimates. First we briefly show how condition (3.2) gives general entropy estimates. Consider a pair (η_1, η_2) which satisfies (2.2), and assume for simplicity that η_2 is bounded from below by 0. We start from equation (2.4) and estimate the boundary terms.

At $x = 0$, we have $c_\varepsilon^1(0, \cdot) = a(\cdot)$. We have, since $v < 0$, $\big[u\eta_1(c_\varepsilon^1) + v\eta_2(c_\varepsilon^2)\big]\big|_0 \leq u\eta_1(a) \leq C$ if $a \in L^\infty$. Next at $x = 1$, we rewrite the boundary condition in the form

$$c_\varepsilon^2 = \frac{u}{|v|} c_\varepsilon^1 - \frac{b}{|v|}.$$

We want to make $-[u\eta_1(c_\varepsilon^1) + v\eta_2(\frac{u}{|v|}c_\varepsilon^1 - \frac{b}{|v|})] \leq K$, K being a constant. A sufficient condition to ensure this is

$$\zeta(c) \stackrel{\text{def}}{=} -\left[u\eta_1(c) + v\eta_2\left(\frac{u}{|v|}c - \frac{b}{|v|}\right)\right] \leq K,$$

or $\zeta'(c) \leq 0$, for c large. Differentiating ζ and using (2.2) shows that this occurs if $\eta_1'(c) = \eta_2'(h(c)) \leq \eta_2'(\frac{u}{|v|}c - \frac{b}{|v|})$. Now, the fact that η_2' is nondecreasing and condition (3.2) lead to

$$\frac{d}{dt}\int_0^1 \big[\eta_1(c_\varepsilon^1(x,t)) + \eta_2(c_\varepsilon^2(x,t))\big]dx \leq C(a, b, \eta_1, \eta_2, M, K),$$

where $K = \sup_{0 \leq c \leq M} \zeta(c)$. By integration, this leads to

$$\int_0^1 \big[\eta_1(c_\varepsilon^1(x,t)) + \eta_2(c_\varepsilon^2(x,t))\big]dx$$
$$\leq \int_0^1 \big[\eta_1(c^0(x)) + \eta_2(h(c^0)(x))\big]dx + C(a, b, \eta, M, K)t.$$

We point out again the fact that condition (3.2) is not to be used as it stands, since it depends on the flux. We prefer to put restrictions on the initial and boundary data, as in Theorem 3.1, which we are going to prove now. Actually, we perform the same computations as above, with two particular choices for η_i.

Proof of Theorem 3.1. (i) We first take $\eta_j(c^j) = [c^j]^-$, which happens to be a diphasic entropy since h is increasing. With this choice, the right-hand side of (2.4) is clearly bounded by $ua(t)^- - [uc_\varepsilon^1(1,t) + vc_\varepsilon^2(1,t)]^- = ua(t)^- + [-b(t)]^-$ (by using the boundary condition at $x = 1$). This becomes nonnegative provided $a \geq 0$ and $b \leq 0$. Integrating in time now gives

$$\int_0^1 \big[\eta_1(c_\varepsilon^1(x,t)) + \eta_2(c_\varepsilon^2(x,t))\big]dx \leq \int_0^1 \big[(c_1^0(x))^- + (c_2^0(x))^-\big]dx \leq 0$$

if the initial data are nonnegative. Thus (1.1) preserves the positivity.

(ii) We now choose $\eta_i = \psi_i^k$, for an adequate k, which will give the upper bound. Indeed, the ψ_i^k are bounded from below (by 0!), and for $k \geq \|a\|_\infty$, the term on $x = 0$ becomes nonpositive. Now, for $x = 1$, we have with our choice for η_i,

$$\zeta(c) = -u(c-k)^+ - v\left[\frac{u}{v}c + \frac{b}{v} - h(k)\right]^+ \leq [f(k) - b]^+$$

by a triangle inequality. To make the right-hand side nonpositive, we must find k such that $f(k) \le b$. This implies $k \le c^\star$ and is compatible with the constraint at $x = 0$ only if $\|a\|_\infty$ is less than $c^\star$.

Finally, by integration, provided k satisfies $c^\star \ge k \ge \|a\|_\infty$, we have

$$\int_0^1 \left[(c_\varepsilon^1(x,t) - k)^+ + \big(c_\varepsilon^2(x,t) - h(k)\big)^+\right] dx \\ \le C(b,k)t + \int_0^1 \left[(c^0(x) - k)^+ + \big(h(c^0(x)) - h(k)\big)^+\right] dx.$$

Now, provided $c^\star \ge \max[\|a\|_\infty, \|c^0\|_\infty]$, we can choose k such that the right-hand side is nonpositive. ◻

3.2. Strong convergence. We turn to the proof of the convergence results.

Proof of Lemma 3.1. We begin from (2.3) with the diphasic entropy given by $\eta_2(c_2) = (1/2)c_2^2$, $\eta_1(c_1) = \int_0^{c_1} h(\sigma)\, d\sigma$, then multiply by φ with compact support in $]0,1[\times]0,T[$, and integrate by parts:

$$-\int_0^T \int_0^1 \big([\eta_1(c_\varepsilon^1) + \eta_2(c_\varepsilon^2)]\partial_t\varphi + [u\eta_1(c_\varepsilon^1) + v\eta_2(c_\varepsilon^2)]\partial_x\varphi\big)\, dx\, dt \\ = -\frac{1}{\varepsilon}\int_0^T \int_0^1 \big(c_\varepsilon^2 - h(c_\varepsilon^1)\big)^2 \varphi dx\, dt.$$

Since c_ε^1 and c_ε^2 are L^∞-bounded uniformly in ε, multiplying this relation by ε gives the result. ◻

We now wish to prove a strong convergence property on c_ε^i by using Murat–Tartar's compensated compactness argument [27].

Proof of Theorem 3.2. *Step* 1. First we prove that, up to a subsequence, c_ε^1 converges strongly. Since c_ε^1 is L^∞ bounded uniformly in ε, and the functions h and η_i are smooth, the sequences c_ε^1, $h(c_\varepsilon^1)$, and $\eta_i(c_\varepsilon^i)$ converge in $L^\infty - w*$, respectively, to $\bar{c}$, $\bar{h}$, and $\bar{\eta}_i$, $i = 1, 2$. Now consider the following two quantities:

$$S^\varepsilon \stackrel{\text{def}}{=} \partial_t\left(c_\varepsilon^1 + h(c_\varepsilon^1)\right) + \partial_x\left(uc_\varepsilon^1 + vh(c_\varepsilon^1)\right),$$
$$T^\varepsilon \stackrel{\text{def}}{=} \partial_t\left(\eta_1(c_\varepsilon^1) + \eta_2\big(h(c_\varepsilon^1)\big)\right) + \partial_x\left(u\eta_1(c_\varepsilon^1) + v\eta_2\big(h(c_\varepsilon^1)\big)\right).$$

We want to apply the classical div-curl lemma, which asserts that the quantity

$$(c_\varepsilon^1 + h(c_\varepsilon^1))[u\eta_1(c_\varepsilon^1) + v\eta_2\big(h(c_\varepsilon^1)\big)] - [\eta_1(c_\varepsilon^1) + \eta_2\big(h(c_\varepsilon^1)\big)](uc_\varepsilon^1 + vh(c_\varepsilon^1))$$

passes to the L^∞ weak-$*$ limit (see [27]), provided S^ε and T^ε are compact in $H^{-1}_{\text{loc}}(\Omega)$. But, for any pair (η_1, η_2) of diphasic entropies (in particular for the trivial entropies $(c_\varepsilon^1, c_\varepsilon^2)$ which give back S^ε), $T^\varepsilon = \mu^\varepsilon + g^\varepsilon$, where

$$\mu^\varepsilon \stackrel{\text{def}}{=} \partial_t\left(\eta_1(c_\varepsilon^1) + \eta_2(c_\varepsilon^2)\right) + \partial_x\left(u\eta_1(c_\varepsilon^1) + v\eta_2(c_\varepsilon^2)\right),$$
$$g^\varepsilon \stackrel{\text{def}}{=} \partial_t\left(\eta_2(h(c_\varepsilon^1)) - \eta_2(c_\varepsilon^2)\right) + \partial_x v\left(\eta_2(h(c_\varepsilon^1)) - \eta_2(c_\varepsilon^2)\right).$$

Now, T^ε is bounded in $W^{-1,\infty}$ since c_ε^1 is bounded in L^∞, and μ^ε is a nonpositive measure (it is actually 0 for the trivial entropies). By Lemma 3.1, we have that

$c_\varepsilon^2 - h(c_\varepsilon^1)$ tends to 0 in $L^2_{\text{loc}}(\Omega)$; hence $\eta_2(c_\varepsilon^2) - \eta_2(h(c_\varepsilon^1))$ also tends to 0 in $L^2_{\text{loc}}(\Omega)$. Since the operators ∂_t and ∂_x are continuous from $L^2_{\text{loc}}(\Omega)$ to $H^{-1}_{\text{loc}}(\Omega)$, g^ε tends to 0, and hence is compact, in $H^{-1}_{\text{loc}}(\Omega)$. Thus, by Murat's lemma, T^ε is compact in $H^{-1}_{\text{loc}}(\Omega)$.

With the obvious notation denoting the weak-$*$ limit with an overline, we obtain, after trivial simplifications,

$$\overline{h(c_1)\eta_1(c_1)} - \overline{h}\;\overline{\eta_1} = \overline{c_1\eta_2\big(h(c_1)\big)} - \bar{c}_1\overline{\eta_2}. \tag{3.6}$$

We now proceed classically by introducing the Young measure $\nu = \nu_{x,t}$ associated with the sequence c_ε^1: for every function α,

$$\alpha(c_\varepsilon^1) \rightharpoonup \overline{\alpha} = \int_{\mathbb{R}} \alpha(\xi)\, d\nu(\xi) = \langle \alpha(\xi), \nu\rangle \qquad \text{in } L^\infty - w * .$$

Equation (3.6) therefore becomes

$$\Big\langle (\xi - \bar{c})\eta_2\big(h(\xi)\big) - \Big(h(\xi) - \overline{h(c)}\Big)\,\eta_1(\xi), \nu \Big\rangle = 0.$$

If we now introduce the aforementioned Kružkov-like entropies $\eta_1(c_1) = |\xi - c_1|$, $\eta_2(c_2) = |h(\xi) - c_2|$, the preceding equality becomes

$$\Big\langle (\xi - \bar{c})|h(\xi) - h(\bar{c})| - \big(h(\xi) - \overline{h(c)}\big)|\xi - \bar{c}|, \nu \Big\rangle = 0.$$

But the fact that h is increasing implies easily that $(\xi-\bar{c})|h(\xi)-h(\bar{c})| = |\xi-\bar{c}|\big(h(\xi)-h(\bar{c})\big)$, so we finally obtain

$$\big(\overline{h(c)} - h(\bar{c})\big)\,\langle |\xi - \bar{c}|, \nu\rangle = 0.$$

The conclusion now follows exactly in the same way as in [27]: ν is a Dirac mass, except where h is affine.

Proof of Theorem 3.2. *Step* 2. First notice that any solution $(c_\varepsilon^1, c_\varepsilon^2)$ to (1.1) with the boundary conditions (1.3) satisfies

$$\begin{aligned} &-\int_0^T\!\!\int_0^1 \Big[\big(|c_\varepsilon^1 - k| + |c_\varepsilon^2 - h(k)|\big)\partial_t\varphi + \big(u|c_\varepsilon^1 - k| + v|c_\varepsilon^2 - h(k)|\big)\partial_x\varphi\Big] dxdt \\ &\qquad \le \int_0^T u|a(t) - k|\varphi(0,t)dt + \int_0^T |b(t) - f(k)|\varphi(1,t)dt \\ &\qquad\quad - \int_0^1 \big(|c^0(x) - k| + |h(c^0(x)) - h(k)|\big)\,\varphi(x,0)dx. \end{aligned} \tag{3.7}$$

Indeed, rewrite (2.3) with $\eta_1(c^1) = |c^1 - k|$ and $\eta_2(c^2) = |c^2 - h(k)|$, multiply by $\varphi(x,t) \ge 0$, and integrate by parts with respect to x and t. We obtain, using the boundary condition on $x = 0$ and the fact that $v < 0$,

$$-\int_0^T \int_0^1 \Big[\big(|c_\varepsilon^1 - k| + |c_\varepsilon^2 - h(k)|\big)\partial_t\varphi + \big(u|c_\varepsilon^1 - k| + v|c_\varepsilon^2 - h(k)|\big)\partial_x\varphi\Big] dxdt$$
$$\le \int_0^T u|a(t) - k|\varphi(0,t)dt$$
$$- \int_0^T \big(u|c_\varepsilon^1(1,t) - k| + v|c_\varepsilon^2(1,t) - h(k)|\big)\, \varphi(1,t)dt$$
$$- \int_0^1 \big(|c^0(x) - k| + |h(c^0)(x) - h(k)|\big)\, \varphi(x,0)dx.$$

For $x = 1$, we use the boundary condition to get

$$\int_0^T (u|c_\varepsilon^1(1,t) - k| + v|c_\varepsilon^2(1,t) - h(k)|)\varphi(1,t)dt$$
$$= \int_0^T \left(u|c_\varepsilon^1(1,t) - k| + v\frac{1}{|v|}|b(t) - uc_\varepsilon^1(1,t) - vh(k)|\right) \varphi(1,t)dt.$$

Again since $v < 0$, $v/|v| = -1$, we add and subtract uk in the second term of the right-hand side, and we use the triangle inequality to conclude. Finally, the first step of this proof allows us to pass to the limit in the left-hand side of (3.7). ∎

3.3. Remarks on viscous regularization. We consider here another possible perturbation of the hyperbolic equation, by means of a viscous regularization. We go back to the classical form of conservation law,

$$\partial_t w + \partial_x f(w) = \varepsilon\partial_{xx} w, \qquad x < 1, \tag{3.8}$$

provided with a perturbed Neumann condition on $x = 1$:

$$-\varepsilon\partial_x w(t,1) + f\big(w(t,1)\big) = b(t).$$

This is exactly the context considered by Gisclon in [9] for the Burgers equation.

We drop the Dirichlet condition on $x = 0$: it has been fully considered by Bardos, Leroux, and Nédélec in [3] and cannot be treated without a priori BV estimates, since the entropy condition on the boundary involves the trace of the solution. Notice that our boundary condition differs from the one in [3], since we do not impose the equilibrium at the boundary.

We are going to formally recover the L^∞ estimate from this perturbation, under the same assumptions as in Theorem 3.1, that is, $b \le 0$ and condition (3.4). After that, classical compactness arguments can be performed in order to obtain strong convergence of the sequence w^ε to a weak solution.

Indeed, multiply (3.8) by $\eta'(w)$, where (η, q) is any pair entropy-flux; then integrate in x. We obtain

$$\frac{d}{dt}\int_{-\infty}^1 \eta(w(x,t))\,dx + q(w(1,t)) = (\varepsilon\eta'(w(1,t))\partial_x w(1,t)) \tag{3.9}$$
$$-\varepsilon\int_{-\infty}^1 \eta''(w(x,t))(\partial_x w(x,t))^2\,dx$$

Now, the term involving η'' is nonnegative since η is convex, and we wish to control the quantity $q(w) - \varepsilon\eta'(w)\partial_x w$ on the boundary. Using the boundary condition, we have $q(w) - \varepsilon\eta'(w)\partial_x w = q(w) + \eta'(w)\,(b - f(w))$. But, assuming $f(0) = 0$, we can write

$$q(w) = \int_0^w \eta'(v) f'(v)\, dv = -\int_0^w \eta''(v) f(v)\, dv + \eta'(w) f(w),$$

so that

$$\frac{d}{dt}\int_{-\infty}^1 \eta(w(x,t))\, dx \le \int_0^{w(1,t)} \eta''(v)\,[f(v) - b]\; dv. \tag{3.10}$$

Now, for a general η, if we assume that

$$f(w) \le \min_{t>0} b(t) \qquad \text{if} \quad |w| \ge M, \tag{3.11}$$

for some $M > 0$ then, since $\eta'' \ge 0$, the right-hand side in (3.10) is bounded by

$$\inf_{w \le M} \int_0^w \eta''(v)\,(f(v) - b)\; dv \overset{\text{def}}{=} C.$$

This proves an entropy estimate for any entropy η, provided (3.11) is satisfied. Notice that, in the particular case $f(w) = ug(w) + vh\big(g(w)\big)$, condition (3.11) is exactly (3.2). Notice also that such a flux condition on a Burgers-like equation does not satisfy the assumption, since the function $w \mapsto w^2$ is not bounded.

To recover the L^∞ estimates, we first consider $\eta(w) = w^-$. Then we have $\eta''(w) = -\delta_0(w)$, so that (3.10) becomes

$$\frac{d}{dt}\int_{-\infty}^1 w(x,t)^-\, dx \le b(t) \le 0.$$

Hence $w(x,t) \ge 0$ if $w(x,0) \ge 0$. For the upper bound, we choose $\eta(w) = (w-k)^+$, for a given $k \in \mathbb{R}$, which gives $\eta''(w) = \delta_k(w)$. Thus

$$\frac{d}{dt}\int_{-\infty}^1 [w(x,t) - k]^+\, dx \le \begin{cases} 0 & \text{if } k \notin [0, w(1,t)], \\ f(k) - b(t) & \text{if } k \in [0, w(1,t)]. \end{cases}$$

If one can choose k such that $f(k) - b(t) \le 0$, then we are done. This can be done precisely if condition (3.4) is satisfied.

4. The Langmuir model. We now consider an $N \times N$ system which appears in chemical engineering both in chromatography and distillation. The unknowns are N functions $c_i(x,t)$ solutions of

$$\partial_t\big(c_i + h_i(\mathbf{c})\big) + \partial_x\big(uc_i + vh_i(\mathbf{c})\big) = 0, \quad t \ge 0,\ x \in]0,1[,\ 1 \le i \le N, \tag{4.1}$$
$$c_i(x,0) = c_i^\circ(x) \ge 0,$$

where the vector-valued function $\mathbf{h}$ is the so-called Langmuir isotherm (see [20]),

$$h_i(\mathbf{c}) = \frac{k_i c_i}{D}. \tag{4.2}$$

The k_i's given here are numbers $0 < k_1 < k_2 < \cdots < k_N$ and $D = 1+c_1+c_2+\cdots+c_N$. Function $\mathbf{h}$ is defined for $D > 0$, which contains the "physical domain" $\{c_i \geq 0, 1 \leq i \leq N\}$. We set in the following $\mathbf{c}(x,t) = (c_1(x,t), \ldots, c_N(x,t))$.

System (4.1) of partial differential equations has been treated by Rhee, Aris, and Admundson in [24] for chromatography, which corresponds to $v = 0$, and in [25] for a countercurrent model of chromatography, which is very close to the system we deal with. Canon and James also studied both systems [5], [6], respectively, for distillation and chromatography. Serre [26] studied a variant of this system, which emphasizes the structure of the function $\mathbf{h}$. On the same variant, a kinetic formulation was obtained in [14], which led to L^∞ estimates and strong convergence properties for bounded sequences of solutions, even though system (4.1) is not hyperbolic on the whole physical domain. The entropies we are about to use are very similar to those in [14], and before defining them, we recall without proof some fundamental algebraic properties of $\mathbf{h}$ (see [5], [24], [26]).

LEMMA 4.1. (i) *If $c_i \geq 0$ for $1 \leq i \leq N$, then $A(\mathbf{c}) = \nabla_{\mathbf{c}}\mathbf{h}(\mathbf{c})$ has N real eigenvalues $\mu_i(\mathbf{c})$, and $w_i \stackrel{\text{def}}{=} D\mu_i$ satisfies*

$$0 < w_1 \leq k_1 \leq w_2 \leq k_2 \leq \cdots \leq k_{N-1} \leq w_N \leq k_N;$$

(ii) *w_i is a strong i-Riemann invariant, in the sense that $\nabla_{\mathbf{c}} w_i$ is a left eigenvector of $A(\mathbf{c})$;*

(iii) $$D = \prod_{i=1}^{N} \frac{k_i}{w_i};$$

(iv) $$c_i \prod_{j\neq i}\left(1 - \frac{k_i}{k_j}\right) = -\prod_{j=1}^{N}\left(1 - \frac{k_i}{w_j}\right);$$

(v) $$\sigma_0 \stackrel{\text{def}}{=} \prod_{i=1}^{N} k_i, \qquad \sigma_j(\mathbf{c}) \stackrel{\text{def}}{=} \sum_{1\leq i_1,\ldots,i_j\leq N} \frac{1}{w_{i_1}\ldots w_{i_j}} \quad \textit{for } 1 \leq j \leq N,$$

are $N+1$ independent affine functions of $(c_1, \ldots, c_N)$.

These properties are very strong. (i) and (ii) give the so-called richness (Serre [26]): system (4.1) admits a diagonal form for smooth solutions, namely,

$$(1 + \mu_i)\partial_t w_i + (u + v\mu_i)\partial_x w_i = 0. \tag{4.3}$$

Moreover, this system also belongs to the Temple class [28] for which some existence and uniqueness results are known in BV when they are strictly hyperbolic (see [26], [13]).

Remark 4.1. Let us point out an important point (see [6] for further details). Property (i) allows a degeneracy of the system (two equal eigenvalues). This can happen only for $w_i = w_{i+1} = k_i$, then $\mu_i = \mu_{i+1}$, and $c_i = 0$. It requires that, initially, $w_i^0(x) = w_{i+1}^0(x) = k_i$ for some $x \in \mathbb{R}$.

Section 4.1 is devoted to some technical devices to generalize the kinetic entropies of [14] for system (4.1). Next, we establish some invariants regions in section 4.2. In particular, we prove that the domain $\{c_i \geq 0\}$ is invariant. Finally, we prove strong convergence results in section 4.3. In the following, we shall say that a vector $\mathbf{z}$ is nonnegative, $\mathbf{z} \in \mathbb{R}^N_+$ (respectively, nonpositive, $\mathbf{z} \in \mathbb{R}^N_-$), if all its components are nonnegative (respectively, nonpositive). We denote by w_1 (respectively, $w_i^{\mathbf{a}}$, w_i^0) the i-Riemann invariant associated by Lemma 4.1(ii) with $\mathbf{c}^1_\varepsilon$ (respectively, with the data on $x = 0$, with the initial data).

4.1. Some specific entropies. Now, we define a first trivial (i.e., affine) diphasic entropy for system (4.1), from which we shall build a specific family of nontrivial (i.e., convex) diphasic entropies. This set of entropies was already mentioned by Serre [26]. For $\xi \in \mathbb{R}_+$ and $\mathbf{c}^1 \in \mathbb{R}^N_+$, we set

$$E_0(\xi; \mathbf{c}^1) = \prod_{i=1}^N \left(1 - \frac{\xi}{w_i^1}\right), \qquad \gamma(\xi) = E_0(\xi; 0) = \prod_{i=1}^N \left(1 - \frac{\xi}{k_i}\right),$$

where w_i^1 are the Riemann invariants corresponding to $\mathbf{c}^1$.

LEMMA 4.2. *The function E_0 is affine with respect to $\mathbf{c}^1$. Let $\nabla_{\mathbf{c}} E_0(\xi)$ denote its gradient. We now define, for $\xi \in \mathbb{R}^+$ and $\mathbf{c}^2 \in \mathbb{R}^N$,*

$$F_0(\xi; \mathbf{c}^2) = \nabla_{\mathbf{c}} E_0(\xi) \cdot \mathbf{c}^2 + \xi\gamma(\xi). \tag{4.4}$$

Then the pair of functions (E_0, F_0) defines a diphasic entropy for (4.1), and we have

$$F_0(\xi; \mathbf{h}(\mathbf{c}^1)) = \frac{\xi E_0(\xi; \mathbf{c}^1)}{D}. \tag{4.5}$$

Proof. First notice that, if E_0 is affine and F_0 is given by (4.4), then obviously the pair (E_0, F_0) defines a diphasic entropy, since $\nabla_{\mathbf{c}^2} F_0(\xi; \mathbf{h}(\mathbf{c}^1)) = \nabla_{\mathbf{c}} E_0(\xi)$.

We are going to prove that E_0 satisfies

$$\begin{cases} E_0(k_i; \mathbf{c}^1) &= \beta_i c_i^1, \qquad \text{where} \quad \beta_i = \prod_{j \neq i} (1 - k_i/k_j), \\ E_0(\xi; \mathbf{c}^1) &= -\gamma(\xi) \left[\sum_{i=1}^N \frac{k_i c_i^1}{k_i - \xi} - D\right] \\ &= -\gamma(\xi) \left[\xi \sum_{i=1}^N \frac{c_i^1}{k_i - \xi} - 1\right] \quad \text{for } \xi \neq k_i, \end{cases} \tag{4.6}$$

so that for $\xi \neq k_i$, $\nabla_{\mathbf{c}} E_0(\xi) = -\xi\gamma(\xi)(\frac{1}{k_i - \xi})_{1 \leq i \leq N}$. To prove (4.6), recall that the Riemann invariants w_i^1 are the roots of the algebraic equation $\varphi(\xi) = 0$, where

$$\varphi(\xi) = \sum_{i=1}^N \frac{k_i c_i}{k_i - \xi} - D = \xi \sum_{i=1}^N \frac{c_i}{k_i - \xi} - 1. \tag{4.7}$$

But φ is also a rational fraction with poles k_i and roots w_i^1; thus an easy computation gives

$$\varphi(\xi) = -D \prod_{i=1}^N \frac{\xi - w_i^1}{\xi - k_i} = -\prod_{i=1}^N \frac{\frac{\xi}{w_i^1} - 1}{\frac{\xi}{k_i} - 1} = -\frac{E_0}{\gamma(\xi)} \tag{4.8}$$

by Lemma 4.1(iii) and the definitions of E_0 and $\gamma(\xi)$. Putting together (4.7) and (4.8) gives (4.6). Finally, (4.5) is obtained by playing with the two definitions of E_0, since

$$\nabla_{\mathbf{c}} E_0(\xi) \cdot \mathbf{h}(\mathbf{c}^1) = -\xi\gamma(\xi) \sum_{i=1}^N \frac{k_i c_i^1}{D} \frac{1}{k_i - \xi} = \frac{\xi}{D} E_0(\xi; \mathbf{c}^1) - \xi\gamma(\xi),$$

and this completes the proof. □

Remark 4.2. We state here a few useful properties of F_0. First, it is a polynomial of degree $N+1$ in ξ, very similar to E_0: if $c_i^2 \in \mathbb{R}_+^N$, it has roots $0, w_1^2, \dots, w_N^2$, with $0 < w_1^2 \le k_1 \le \cdots \le w_N^2 \le k_N$. We easily obtain also that, for any $\mathbf{z} \in \mathbb{R}^N$,

$$F_0(k_i; \mathbf{z}) = \nabla_{\mathbf{c}} E_0(k_i) \cdot \mathbf{z} = \beta_i z_i, \qquad 1 \le i \le N. \tag{4.9}$$

A crucial point now is to remark that $E_0(\xi; \mathbf{c})$ and $F_0(\xi; \mathbf{h}(\mathbf{c}))$ vanish simultaneously for $\xi = w_i^1 = w_i^2$, $1 \le i \le N$. Thus, taking the convention $w_0^j = 0$ and $w_{N+1}^j = +\infty$, we easily deduce the following.

Corollary 4.3. *For $0 \le i \le N$, $\mathbf{c}^1 \in \mathbb{R}_+^N$, $\mathbf{c}^2 \in \mathbb{R}_+^N$, let w_i^1 (respectively, w_i^2) be the roots of E_0 (respectively, the nonzero roots of F_0). Define*

$$\begin{aligned} \chi_i^1(\xi; \mathbf{c}^1) &= |E_0(\xi; \mathbf{c})| \mathbb{1}_{\{\xi \in]w_i^1, w_{i+1}^1[\}}, \\ \chi_i^2(\xi; \mathbf{c}^2) &= |F_0(\xi; \mathbf{c}^2)| \mathbb{1}_{\{\xi \in]w_i^2, w_{i+1}^2[\}}. \end{aligned} \tag{4.10}$$

Then the pair (χ_i^1, χ_i^2) defines a diphasic entropy for (4.1).

Notice that χ_i^1 (respectively, χ_i^2) is actually convex with respect to $\mathbf{c}^1$ (respectively, to $\mathbf{c}^2$), as the absolute value of an affine function. Thus the function $\eta_i(\xi; \mathbf{c}) \stackrel{\text{def}}{=} \chi_i^1(\xi; \mathbf{c}) + \chi_i^2(\xi; \mathbf{h}(\mathbf{c}))$ is indeed a nontrivial convex diphasic entropy for (4.1).

The class of entropies we consider now is defined as follows. Set, for $j = 1, 2$, $\mathbf{c}^j \in \mathbb{R}_+^N$, and a fixed $0 \le i \le N$,

$$S^j(\mathbf{c}^j) = \int_{\mathbb{R}_+} g(\xi) \chi_i^j(\xi; \mathbf{c}^j)\, d\xi, \qquad j = 1, 2.$$

The functions $S(\mathbf{c}) = S^1(\mathbf{c}) + S^2(\mathbf{h}(\mathbf{c}))$ are diphasic entropies for (1.2), for any nonnegative function g such that $g\chi_i^j$ is integrable at $+\infty$ in ξ (recall that χ_i^j is a polynomial in ξ). The corresponding entropy flux is $Q(\mathbf{c}) = uS^1(\mathbf{c}) + vS^2(\mathbf{h}(\mathbf{c}))$. We have to complement these functions by using for g a Dirac mass, $g(\xi) = \delta_{\xi^*}(\xi)$. To justify this, consider a sequence of nonnegative g's which converge to such a Dirac mass. These entropies will appear in the proof of the maximum principle below. Let us denote by $\mathcal{E}$ the set of all these entropies for $0 \le i \le N$.

Remark 4.3. The entropies in $\mathcal{E}$ are defined only on $\mathbb{R}_+^N$ and therefore cannot be used to prove the invariance of $\mathbb{R}_+^N$. But it is easily checked that the pairs $([c_i^1]^-, [c_i^2]^-)$, where r^- is the negative part of $r \in \mathbb{R}$, define diphasic entropies on the domain $D > 0$.

4.2. Invariant regions. In this subsection, we shall prove that the solution $(\mathbf{c}_\varepsilon^1, \mathbf{c}_\varepsilon^2)$ to (4.1) is bounded in L^∞ uniformly in ε, thus giving rise to a weakly convergence subsequence. In the next subsection, we prove that this subsequence actually converges almost everywhere to a solution in the sense of (4.14) below.

Theorem 4.4. *Assume $\mathbf{c}^0 \in L^1 \cap L^\infty(]0,1[)^N$, $\mathbf{a} \in L^\infty(\mathbb{R}_+)^N$, $\mathbf{b} \in L^\infty(\mathbb{R}_+)^N$, $\mathbf{c}^0, \mathbf{a}$ nonnegative, and $\mathbf{b}$ nonpositive. Let $0 < w^- \le k_1$ satisfy $w^- \le w_1^{\mathbf{a}}(t), w_1^0(x) \le k_1$ for all (t, x). Define*

$$\psi(\xi) \stackrel{\text{def}}{=} \nabla_{\mathbf{c}} E_0(\xi) \cdot \mathbf{b} + (u + v\xi)\gamma(\xi),$$

and assume that

$$\xi^\star \stackrel{\text{def}}{=} \inf\{\xi \le k_1; \exists \xi' \le \xi, \psi(\xi') \le 0\} > w^-. \tag{4.11}$$

Let $(\mathbf{c}_\varepsilon^1, \mathbf{c}_\varepsilon^2)$ *be a solution of* (1.1). *Then there exists a constant* C *independent of* ε *such that* $0 \le \mathbf{c}_\varepsilon^i(x,t) \le C$, $1 \le i \le N$, $\forall (t,x) \in [0,T] \times [0,1]$.

Remark 4.4. Once again, one can choose $\xi_0 = w^-$ only if w^- satisfies $\psi(w^-) \le 0$.

Remark 4.5. The existence of ξ^* relies on the nonpositivity of the polynomial ψ on $[0, k_1]$ (one has $\psi(0) = u > 0$ and $\psi(k_1) = -k_1 b_1 \prod_{i>1}(k_i - k_1) \ge 0$, so this is not trivially satisfied). This leads to a condition on $u, v, \mathbf{b}$, and k_1, which is actually not very explicit, except for $N = 1$ (see Remark 3.1). However, one can rewrite things as follows. For $0 < \xi < k_1$, define $\mathbf{c}(\xi) \in \mathbb{R}_+^N$ by

$$c_i(\xi) = \frac{k_i - \xi}{N\xi}, \qquad 1 \le i \le N.$$

Then a few easy algebraic computations prove

$$\nabla_{\mathbf{c}} E_0(\xi) \cdot \mathbf{c}(\xi) = -\gamma(\xi), \quad \nabla_{\mathbf{c}} E_0(\xi) \cdot \mathbf{h}(\mathbf{c}(\xi)) = -\xi\gamma(\xi),$$

so that $\psi(\xi) \le 0$ rewrites $\nabla_{\mathbf{c}} E_0(\xi) \cdot [\mathbf{b} - (u\mathbf{c}(\xi) + \xi\gamma(\xi))] \le 0$. Thus condition (4.11) can be compared to (3.4) in a more consistent way. Notice that this can also be read as an entropy inequality, since $\nabla_{\mathbf{c}} E_0(\xi) \cdot [\mathbf{b} - (u\mathbf{c}(\xi) + \xi\mathbf{h}(\mathbf{c}(\xi)))] = E_0(\xi; \mathbf{b}) - E_0(\xi; u\mathbf{c}(\xi) + \xi\mathbf{h}(\mathbf{c}((\xi))) = F_0(\xi; \mathbf{b}) - F_0(\xi; u\mathbf{c}(\xi) + \xi\mathbf{h}(\mathbf{c}((\xi)))$.

Proof of Theorem 4.1. To lighten the notations a bit, we omit the index ε in this proof. First notice that w^- exists since $\mathbf{a}$ and $\mathbf{c}^0$ are nonnegative and uniformly bounded.

Let us prove first that for a given index i, if $a_i \ge 0$, $b_i \le 0$, and $c_i^0 \ge 0$, then $c_i^j \ge 0$ for $j = 1, 2$. For this purpose we make use of the entropy introduced in Remark 4.3. Inequality (2.4) can be rewritten here as

$$\frac{d}{dt}\int_0^1 \left([c_i^1(x,t)]^- + [c_i^2(x,t)]^-\right) dx \le [c_i^1(0,t)]^- + v[c_i^2(0,t)]^- - u[c_i^1(1,t)]^- - v[c_i^2(1,t)]^-.$$

Now, as in the scalar case, we notice that $v < 0$ and $(c_i^j)^- \ge 0$, so $u[c^1(0,t)]^- + v[c^2(0,t)]^- \le u[a_i(t)]^-$ by the boundary condition at $x = 0$. Since $a_i(t) \ge 0$ for $1 \le i \le N$, $[a_i(t)]^- = 0$, and the same occurs for the initial data.

For $x = 1$, we have to prove that $F \overset{\text{def}}{=} -u[c^1(1,t)]^- - v[c^2(1,t)]^- \le 0$. This clearly occurs if $(b_i - uc_i^1(1,t))/v \ge 0$. If this is not the case, we have $0 \ge b_i(t) \ge uc_i(t)$ since $v < 0$ so that $F = b_i(t)|\beta_i| \le 0$. Hence the following differential inequality holds:

$$\frac{d}{dt}\int_0^1 \left([c^1(x,t)]^- + [c^2(x,t)]^-\right) dx \le 0.$$

The conclusion now follows easily: the components of $\mathbf{c}_\varepsilon^1$ and $\mathbf{c}_\varepsilon^2$ remain nonnegative for any $t > 0$.

We turn now to the proof of the upper bound. For simplicity, we assume the nonnegativity. In view of formula (iv) in Lemma 4.1, we have to prove that there exists $\xi_0 > 0$ such that $w_1^j \ge \xi_0$ for all (t,x). We consider the diphasic entropy (S^1, S^2),

$$S^1(\mathbf{c}^1) = \int_{w_1^1}^{w_2^1} |E_0(\xi; \mathbf{c}^1)| g(\xi) d\xi, \quad S^2(\mathbf{c}^2) = \int_{w_1^2}^{w_2^2} |F_0(\xi; \mathbf{c}^2)| g(\xi) d\xi.$$

The usual trick of convexity of S^1 and S^2 leads to

$$(4.12)\quad \frac{d}{dt}\int_0^1 [S^1(\mathbf{c}^1(x,t)) + S^2(\mathbf{c}^2(x,t))]\,dx \le -\left[uS^1(\mathbf{c}^1(x,t)) + vS^2(\mathbf{c}^2(x,t))\right]\Big|_{x=0}^{x=1}.$$

Set $H_0 = uS^1(\mathbf{c}^1(0,t)) + vS^2(\mathbf{c}^2(0,t))$ and $H_1 = -[uS^1(\mathbf{c}^1(1,t)) + vS^2(\mathbf{c}^2(1,t))]$. We have

$$H_0 = \int_{w_1^1}^{w_2^1} u|E_0(\xi;\mathbf{c}^1)|g(\xi)d\xi + \int_{w_1^2}^{w_2^2} v|F_0(\xi;\mathbf{c}^2)|g(\xi)d\xi \le \int_{w_1^{\mathbf{a}}}^{w_2^{\mathbf{a}}} u|E_0(\xi;\mathbf{a})|g(\xi)d\xi,$$

since $v < 0$. For any $\xi_0 \le w^-$, choosing $g = \delta_{\xi_0}$ cancels the right-hand side of the preceding inequality.

Concerning H_1, we want to take $g = \delta_{\xi_0}$ for a carefully chosen $\xi_0 \le w^-$ such that

$$(4.13)\quad H_1 = -\int_{w_1^1}^{w_2^1} u|E_0(\xi;\mathbf{c}^1)|\delta_{\xi_0}(\xi)d\xi - \int_{w_1^2}^{w_2^2} v\left|F_0\left(\xi;\frac{1}{v}[\mathbf{b} - u\mathbf{c}^1]\right)\right|\delta_{\xi_0}(\xi)d\xi \le 0.$$

We know, since everything is nonnegative, that $0 < w_1^1, w_1^2 \le k_1 \le w_2^1, w_2^2$, so that necessarily $\xi_0 \le w_2^1, w_2^2$. Now, if $\xi_0 < w_1^2$, then $H_1 \le 0$ by (4.13). If $\xi_0 \ge w_1^2$, we have by construction $F_0(\xi_0;(\mathbf{b} - u\mathbf{c}^1)/v) \le 0$ (indeed one can check that $F_0(\xi = 0) = 0$ and $\partial_\xi F_0(\xi = 0) \ge 0$). On the other hand, an easy computation shows

$$vF_0\left(\xi;\frac{1}{v}[\mathbf{b} - u\mathbf{c}^1]\right) = \psi(\xi_0) - uE_0(\xi_0;\mathbf{c}^1).$$

Since $w^- \ge \xi^\star$, one can choose any $\xi^\star \le \xi_0 \le w^-$ such that $\psi(\xi_0) \le 0$. The preceding equality therefore gives $E_0(\xi_0;\mathbf{c}) \le 0$, so that $\xi_0 \in [w_{2p+1}^1, w_{2p+2}^1]$ for some $p \ge 1$, by assertion (iv) in Lemma 4.1. Since $\xi_0 \le k_1 \le w_2^1$, necessarily $\xi_0 \in [w_1^1, w_2^1]$ so that finally, H_1 can be rewritten, by simple consideration of sign on E_0 and F_0,

$$H_1 = uE_0(\xi_0;\mathbf{c}^1) + vF_0\left(\xi_0;\frac{1}{v}[\mathbf{b} - u\mathbf{c}^1]\right) = \psi(\xi_0) \le 0.$$

The preceding choice of ξ_0 cancels the right-hand side of (4.12). When integrating in t, we introduce the initial data, but the choice of $g = \delta_{\xi_0}$ for $\xi_0 \le w^-$ gives also $S^1(\mathbf{c}^0(x)) = S^2(\mathbf{h}(\mathbf{c}^0(x)) = 0$, so finally (4.12) gives

$$\int_0^1 [S^1(\mathbf{c}^1(x,t)) + S^2(\mathbf{c}^2(x,t))]\,dx \le 0,$$

which leads to $S^1(\mathbf{c}^1(x,t)) = S^2(\mathbf{c}^2(x,t)) = 0$, $\forall t > 0$. A simple contradiction argument then gives $w_1^1(x,t) \ge \xi_0$ and $w_1^2(x,t) \ge \xi_0$ for a.e. x, $\forall t > 0$. ∎

4.3. Strong convergence. The L^∞ estimate leads obviously to the following weak convergence result: $\mathbf{c}_\varepsilon^2 - h(\mathbf{c}_\varepsilon^1)$ tends to 0 in $\mathcal{D}'(\Omega)^N$ when ε tends to 0. We actually have a stronger convergence result.

LEMMA 4.5. *Under the above assumptions ensuring the L^∞ bounds on the solution, $\mathbf{c}_\varepsilon^2 - h(\mathbf{c}_\varepsilon^1)$ tends to 0 in $L^2_{\mathrm{loc}}(\Omega)^N$.*

From (1.1), we can obtain the following inequality for the entropies (χ_i^1, χ_i^2):

$$\partial_t\left(\chi_i^1(\xi;\mathbf{c}_\varepsilon^1) + \chi_i^2(\xi;\mathbf{c}_\varepsilon^2)\right) + \partial_x\left(u\chi_i^1(\xi;\mathbf{c}_\varepsilon^1) + v\chi_i^2(\xi;\mathbf{c}_\varepsilon^2)\right) \le 0.$$

The negative sign holds since $\nabla_{\mathbf{c}}\chi_i^2(\xi;\cdot)$ is a monotone operator, as before. Now, multiply this inequality by any nonnegative $\varphi \in \mathcal{D}(\overline{\Omega})$, integrate by parts, and treat the boundary conditions as in the above proof. One obtains

$$\begin{aligned}
&-\int_0^T\int_0^1 \left[\partial_t\varphi\left(\chi_i^1(\xi;\mathbf{c}_\varepsilon^1(x,t))+\chi_i^2(\xi;\mathbf{c}_\varepsilon^2(x,t))\right)\right.\\
&\qquad\left.+\partial_x\varphi\left(u\chi_i^1(\xi;\mathbf{c}_\varepsilon^1(x,t))+v\chi_i^2(\xi;\mathbf{c}_\varepsilon^2(x,t))\right)\right]\,dx\,dt\\
&\le \int_0^T \varphi(0,t)u\chi_i^1(\xi;\mathbf{a}(t))\,dt-\int_0^1\varphi(x,0)S(\mathbf{c}^0(x))\,dx\\
&\quad-\int_0^T\left[u\chi_i^1(\xi;\mathbf{c}_\varepsilon^1(1,t))+v\chi_i^2(\xi;\mathbf{c}_\varepsilon^2(1,t))\right]\varphi(1,t)\,dt.
\end{aligned}$$

Once again, some considerations of sign allow us to prove that for the boundary term on $x=1$, we have for any ξ, since $\mathbf{c}_\varepsilon^2(1,t)=(\mathbf{b}(t)-u\mathbf{c}_\varepsilon^1(1,t))/v$,

$$u\chi_i^1(\xi;\mathbf{c}_\varepsilon^1(1,t))+v\chi_i^2(\xi;\mathbf{c}_\varepsilon^2(1,t))\le|\nabla_{\mathbf{c}}E_0(\xi)\cdot\mathbf{b}(t)-(u+v\xi)\gamma(\xi)|\stackrel{\text{def}}{=}B(t).$$

The resulting entropy estimate is analogous to (3.7). Now, following the lines of [14], we can apply compensated compactness to obtain the following result of strong convergence.

THEOREM 4.6. *We make the same assumptions as in Theorem* 4.1. *Then there exists a subsequence of solutions to* (1.1), *still denoted by* $\mathbf{c}_\varepsilon^1$, *which converges almost everywhere and strongly in* $]0,1[\times]0,T[$ *to* $\mathbf{c}\in L^\infty(]0,T[;L^1(]0,1[))^N$. *Moreover,* $\mathbf{c}$ *satisfies, for any* $\varphi\in\mathcal{D}(\overline{\Omega})$, $\varphi\ge 0$, $\xi>0$,

$$\begin{aligned}
&-\int_0^T\int_0^1[S(\mathbf{c})\partial_t\varphi+Q(\mathbf{c})\partial_x\varphi]\,dx\,dt\\
&\quad\le\int_0^T u\chi_i^1(\mathbf{a}(t))\varphi(0,t)\,dt-\int_0^T B(t)\varphi(1,t)\,dt-\int_0^1 S(\mathbf{c}^0(x))\varphi(x,0)\,dx,
\end{aligned}\tag{4.14}$$

with $B(t)=|\nabla_{\mathbf{c}}E_0(\xi)\cdot\mathbf{b}(t)-(u+v\xi)\gamma(\xi)|$, *for* $S(\mathbf{c})=\chi_i^1(\mathbf{c})+\chi_i^2(\mathbf{h}(\mathbf{c}))$, $Q(\mathbf{c})=u\chi_i^1(\mathbf{c})+v\chi_i^2(\mathbf{h}(\mathbf{c}))$, *and* χ_i^j *being defined by* (4.10).

Proof of Lemma 4.3. Consider the pair of entropies (η_1,η_2) obtained by choosing, for a given i, $g=\mathbb{1}_{[0,k_i]}$. Their gradients are given by

$$\begin{aligned}
\nabla_{\mathbf{c}}\eta_1(\mathbf{c}^1)&=\int_{w_i^1}^{k_i}\operatorname{sign}(E_0(\xi;\mathbf{c}^1))\nabla_{\mathbf{c}}E_0(\xi)d\xi,\\
\nabla_{\mathbf{c}}\eta_2(\mathbf{c}^2)&=\int_{w_i^2}^{k_i}\operatorname{sign}(F_0(\xi;\mathbf{c}^2))\nabla_{\mathbf{c}}E_0(\xi)d\xi.
\end{aligned}$$

Omitting here the dependence in ε, we take the scalar product of the two equations in (1.1), respectively, by $\nabla_{\mathbf{c}}\eta_1(\mathbf{c}^1)$ and $\nabla_{\mathbf{c}}\eta_2(\mathbf{c}^2)$, sum the two equations, and integrate $dx\,dt$ with a nonnegative test function $\varphi\in\mathcal{D}(\overline{\Omega})$. We obtain, after integration by parts and multiplication by ε,

$$A^\varepsilon \stackrel{\text{def}}{=} \varepsilon \int_0^T \int_0^1 \left(\partial_t\varphi(x,t)\big[\mathbf{c}^1(x,t)+\mathbf{c}^2(x,t)\big] + \partial_x\varphi(x,t)\big[u\mathbf{c}^1(x,t)+v\mathbf{c}^2(x,t)\big]\right) dx\, dt$$

$$= -\int_0^T \int_0^1 \left[\int_{w_i^1}^{k_i} \operatorname{sign}(E_0(\xi;\mathbf{c}^1))\nabla_{\mathbf{c}} E_0(\xi)d\xi - \int_{w_i^2}^{k_i} \operatorname{sign}(F_0(\xi;\mathbf{c}^2))\nabla_{\mathbf{c}} E_0(\xi)d\xi\right] \cdot (\mathbf{c}^2 - \mathbf{h}(\mathbf{c}^1))dx\, dt.$$

Notice that $A^\varepsilon \geq 0$ by the second equality and the convexity of η_i. Obviously, since $\mathbf{c}^1$ and $\mathbf{c}^2$ are bounded in L^∞, A^ε tends to 0 when ε goes to zero. We have to work from now on with

$$P(x,t) \stackrel{\text{def}}{=} -\left[\int_{w_i^1}^{k_i} \operatorname{sign}(E_0(\xi;\mathbf{c}^1))\nabla_{\mathbf{c}} E_0(\xi)d\xi - \int_{w_i^2}^{k_i} \operatorname{sign}(F_0(\xi;\mathbf{c}^2))\nabla_{\mathbf{c}} E_0(\xi)d\xi\right] \cdot(\mathbf{c}^2 - \mathbf{h}(\mathbf{c}^1)).$$

It is easy to check that $\operatorname{sign} E_0(\xi;\mathbf{c}^1) = \operatorname{sign} F_0(\xi,\mathbf{h}(\mathbf{c}^1)) = \operatorname{sign} F_0(\xi;\mathbf{c}^2)$ for $\xi \in [w_i^1, k_i] \cap [w_i^2, k_i]$. We are thus left with an integral over $[\min(w_i^1, w_i^2), \max(w_i^1, w_i^2)]$. Let us assume that $w_i^1 \leq w_i^2$; the computations are the same if the converse holds. We have, by considerations of sign on F_0,

$$P(x,t) = 2\int_{w_i^1}^{w_i^2} \big[|F_0(\xi;\mathbf{c}^2)| + |F_0(\xi;\mathbf{h}(\mathbf{c}^1))|\big]\, d\xi.$$

Now, we write for $N \geq 4$,

$$|F_0(\xi;\mathbf{c}^2)| = (\xi - w_{i-1}^2)(\xi - w_i^2)(w_{i+1}^2 - \xi)\frac{\prod_{j\notin\{i-1,i,i+1\}} |w_j^2 - \xi|}{\prod_{i=1}^N w_j^2}.$$

The fourth term is greater than some $K > 0$ (K depending on $k_1, \dots, k_N$ and ξ_0), since either $w_j^2 \leq k_{i-2}$ or $w_j^2 \geq k_{i+1}$, and $k_{i-1} \leq \xi \leq k_i$. For the first three terms, we simply write $(\xi - w_{i-1}^2)(\xi - w_i^2)(w_{i+1}^2 - \xi) \geq (\xi - w_i^2)^2(w_{i+1}^2 - w_i^2)$, which leads by integration to

$$\int_{w_i^1}^{w_i^2} |F_0(\xi;\mathbf{c}^2)|d\xi \geq \frac{K}{3}(w_i^1 - w_i^2)^3(w_{i+1}^2 - w_i^2) \geq \frac{K}{3}(w_i^1 - w_i^2)^4.$$

For $N = 3$, we have a similar estimate, since the fourth term reduces to $K/(w_1^2 w_2^2 w_3^2)$. Because the same holds for $|F_0(\xi;\mathbf{h}(\mathbf{c}^1))|$, we have finally that for some $C > 0$, depending only on $k_1, \dots, k_N$,

$$\int_0^T \int_0^1 |w_i^1(x,t) - w_i^2(x,t)|^4 \varphi(x,t)dx\, dt \leq C\int_0^T \int_0^1 P(x,t)\varphi(x,t)dx\, dt = A^\varepsilon$$

tends to zero. Thus $|w_i^1 - w_i^2|$ tends to 0 in $L^4_{\text{loc}}(\Omega)$ and therefore in $L^2_{\text{loc}}(\Omega)$. For $N = 2$, the same computations lead to convergence in $L^3_{\text{loc}}(\Omega)$ and hence in $L^2_{\text{loc}}(\Omega)$. Finally, if $N = 1$, we directly obtain $L^2_{\text{loc}}(\Omega)$. Since the function $(w_1, \dots, w_N) \mapsto (c_1, \dots, c_N)$ is Lipschitz continuous, we are done. ∎

Proof of Theorem 4.2. We merely give the sketch of the proof, referring to [14] for the detailed computations, which are identical. Summing the equations for $0 \leq i \leq N$, we obtain, with the same notations as in the scalar case,

$$T^\varepsilon(\xi) = \partial_t[G_0(\xi, \mathbf{c}^1) + H_0(\xi, \mathbf{h}(\mathbf{c}^1))] + \partial_x[uG_0(\xi, \mathbf{c}^1) + vH_0(\xi, \mathbf{h}(\mathbf{c}^1))] = \mu^\varepsilon(\xi) + g^\varepsilon(\xi), \tag{4.15}$$

with $G_0 = |E_0|$ and $H_0 = |F_0|$. Since η_1 and η_2 are convex, the usual computation proves that $\mu^\varepsilon(\xi)$ is a nonpositive measure. By Lemma 4.3, $g^\varepsilon(\xi)$ is compact in $H^{-1}_{\mathrm{loc}}(\Omega)$; thus, again applying Murat's lemma, we can apply the compensated compactness lemma to (4.15), for two different values ξ and ξ'. We obtain, after some easy simplifications,

$$\overline{G_0(\xi)}\ \overline{\xi' G_0(\xi')/D} - \overline{G_0(\xi')}\ \overline{\xi G_0(\xi)/D} \ = \ (\xi' - \xi)\overline{G_0(\xi)G_0(\xi')/D}.$$

Dividing by $\overline{G_0(\xi)}\ \overline{G_0(\xi')}\ (\xi' - \xi)$ and letting ξ' go to ξ, we get

$$\partial_\xi \frac{\xi\overline{G_0(\xi)/D}}{\overline{G_0(\xi)}} = \frac{\overline{G_0(\xi)^2/D}}{\overline{G_0(\xi)}^2}. \tag{4.16}$$

Of course (4.16) has to be justified at points where $\overline{G_0(\xi)} = 0$. This occurs when $G_0(\xi_0, w) = 0$ for all w in the support of ν, that is, $w_j = \xi_0$ for some j. If w_j is a simple eigenvalue, the formula is justified by applying l'Hospital's rule in a neighborhood of ξ_0 to G_0', which is not zero since the root is simple. When we have a double root, that is, $\xi_0 = k_j$, the same technique can be used with G_0'', which in no case can be zero, since the root cannot be triple.

Equation (4.16) is not completely satisfactory because its right-hand side does not vanish when $d\nu$ is a Dirac mass. Therefore, we again apply compensated compactness to (4.15) for a given ξ and

$$\partial_t \left(D + \frac{\alpha}{D}\right) + \partial_x \left(uD + v\frac{\alpha}{D}\right) = 0, \qquad \alpha = k_1 u_1 + \cdots + k_N u_N.$$

This yields $\overline{G_0(\xi)}\ \overline{\alpha/D} - \xi\overline{G_0(\xi)/D}\ \overline{D} = \overline{G_0(\xi)\alpha/D} - \xi\overline{G_0(\xi)}$. After dividing it by $\overline{D}\ \overline{G_0(\xi)}$, we can combine it with the left-hand side of (4.15) to get

$$-\partial_\xi \frac{\overline{G_0(\xi)\alpha/D}}{\overline{G_0(\xi)}} = \frac{\overline{G_0(\xi)^2/D}}{\overline{G_0(\xi)}^2} - \frac{1}{\overline{D}} \ \geq 0, \tag{4.17}$$

this last inequality being just Cauchy–Schwarz. Inequality (4.17) is the keystone to proving that $d\nu$ is in fact a Dirac mass.

Indeed, if in (4.17) the inequality is strict at some point, we obtain a contradiction by comparing the values of the nonincreasing function

$$\frac{\overline{G_0(\xi)\alpha/D}}{\overline{G_0(\xi)}}$$

at the points $\xi = 0$ and $\xi = +\infty$, then $\xi = k_i$ and $\xi = \infty$. This means that the inequality in (4.17) is just an equality for all $\xi \geq 0$, so that the equality case in Cauchy–Schwarz applies. We obtain the existence of a function $\lambda(\xi)$ such that, for all $\xi \geq 0$, $G_0(\xi, \mathbf{c}^1) = \lambda(\xi)D(\mathbf{c}^1)$ a.e. in the support of $d\nu(\mathbf{c}^1)$. From this we deduce that, for two possible elements $\mathbf{c}^1$, $\mathbf{c}^{'1}$ of the support of $d\nu(\mathbf{c}^1)$, we have necessarily $G_0(\xi,\mathbf{c}^1) = G_0(\xi,\mathbf{c}^{'1})$. Thus $\sigma_j(\mathbf{c}^1) = \sigma_j(\mathbf{c}^{'1})$, and by Lemma 4.1 (v) this proves that $\mathbf{c}^1 = \mathbf{c}^{'1}$ and the support of ν is a single point. ∎

Remark 4.6. Notice that formula (4.14) is exactly the kinetic formulation obtained in [14], but the boundary terms forbid us to write it in the usual way, with some nonnegative measure on the right-hand side.

5. Boundary conditions. So far, we have defined in Theorems 3.2 and 4.2 kinds of weak solutions. The aim of this section is to prove that these solutions are actually solutions to (1.2) in the sense of distributions, and to give a meaning to the reflux boundary condition at $x = 1$. It seems that we lose the Dirichlet-like boundary condition at $x = 0$ when passing to the limit. This is not really surprising, since we pass from $2N$ equations to N equations: the system becomes overdetermined.

Before precisely stating our results, we need to introduce some material. Indeed, we want to precisely state the meaning of the boundary conditions. But we deal with L^∞ functions, which usually do not have any trace on the boundary. The following result, which we state as a lemma, follows easily by choosing the test functions $\varphi \in \mathcal{D}(\Omega)$ in (3.5) or (4.14).

LEMMA 5.1. *Let (η_1,η_2) be any pair of convex functions defining a diphasic entropy. Let $\mathbf{c} \in L^\infty(\Omega)$ be a weak solution as in Theorems* 3.2 *or* 4.2. *Then the vector-valued function $\psi = (\psi_1,\psi_2) \stackrel{\text{def}}{=} (\eta_1(\mathbf{c}) + \eta_2(\mathbf{h}(\mathbf{c})), u\eta_1(\mathbf{c}) + v\eta_2(\mathbf{h}(\mathbf{c})))$ is in $L^\infty(\Omega)$, and $\operatorname{div}\psi = \partial_t\psi_1 + \partial_x\psi_2$ is a nonnegative measure in Ω.*

We are thus in a position to apply a result by Anzellotti [1, Theorems 1.2 and 1.9], which essentially states that ψ has a trace on $\partial\Omega$, in some sense. We recall this result here without proof.

THEOREM 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with locally Lipschitz boundary $\partial\Omega$. Set $X(\omega) = \{\psi \in L^\infty(\Omega;\mathbb{R}^n); \operatorname{div}\psi$ is a bounded measure in $\Omega\}$. Then there exists a trace operator*

$$\gamma : X(\Omega) \to L^\infty(\partial\Omega),$$

such that, for any $\varphi \in BV(\Omega)\cap L^\infty(\Omega)\cap C^0(\Omega)$,

$$\int_\Omega \varphi \operatorname{div}\psi\, dx + \int_\Omega (\psi,\varphi)\, dx = \int_{\partial\Omega} \gamma\psi\,\varphi\, d\sigma, \tag{5.1}$$

where σ is the superficial measure on $\partial\Omega$.

In this result, (ψ,φ) has to be defined as a measure (Definition 1.4 in [1]). We denote by γ^0 the trace on $]0,1[\times\{0\}$, by γ_0 and γ_1 the traces, respectively, on $\{0\}\times]0,T[$ and $\{1\}\times]0,T[$. Since $\gamma\psi$ is, by construction, a weak trace on $\partial\Omega$ of the normal component of ψ, we have

at $t = 0$, $\quad \gamma\psi = \gamma^0[\eta_1(\mathbf{c}) + \eta_2(\mathbf{h}(\mathbf{c}))]$,

at $x = 0, 1$, $\quad \gamma\psi = \gamma_{0,1}[u\eta_1(\mathbf{c}) + v\eta_2(\mathbf{h}(\mathbf{c}))]$.

In particular, for the trivial entropies, we recover the conservative variables so that, for $1 \le i \le N$, $c_i + h_i(\mathbf{c})$ has a trace on $t = 0$, and $uc_i + vh_i(\mathbf{c})$ has traces on $x = 0$ and $x = 1$. Notice that this trace is attained in a weak sense (see [18]), in contrast with the traces of BV functions, which are attained in L^1.

THEOREM 5.3. (i) *Let $\mathbf{c}$ be a solution as in Lemma* 5.1. *Then it is a solution to* (1.2) *in $\mathcal{D}'(\Omega)$, and we have, for $1 \le i \le N$,*

$$\begin{cases} \gamma_1[uc_i + vh_i(\mathbf{c})] & = b_i, \quad a.e.\ t \in]0,T[; \\ \gamma^0[c_i + h_i(\mathbf{c})] & = c_i^0 + h_i(\mathbf{c}^0), \quad a.e.\ x \in]0,1[. \end{cases} \tag{5.2}$$

(ii) *For any pair* (η_1, η_2) *denoting the Kružkov entropies in the scalar case, the kinetic entropies for the Langmuir system, define* ψ *as in Lemma* 5.1. *Then the following entropy inequalities hold for a.e.* $t \in]0, T[$:

$$\begin{cases} \gamma_0[u\eta_1(\mathbf{c}) + v\eta_2(\mathbf{h}(\mathbf{c}))] & \leq u\eta_1(\mathbf{a}), \\ \gamma_1[u\eta_1(\mathbf{c}) + v\eta_2(\mathbf{h}(\mathbf{c}))] & \leq B(t), \end{cases} \tag{5.3}$$

where $B(t) = |b(t) - f(k)|$ *in the scalar case and is defined in Theorem* 4.2 *for the Langmuir system.*

Remark 5.1. This theorem shows that the initial condition and the reflux boundary condition are satisfied in a strong sense (in $L^\infty(\partial\Omega)$, actually). We have no information about the input boundary condition at $x = 0$, except for the entropy inequalities (5.3). Notice that, even for the conservative variables themselves, we loose some information. Indeed, we know that there is a trace for $u\mathbf{c} + v\mathbf{h}(\mathbf{c})$ at $x = 0$, but this function is not one-to-one, so we cannot compare $\mathbf{c}$ to $\mathbf{a}$. Moreover, even if $u\mathbf{c} + v\mathbf{h}(\mathbf{c})$ is one-to-one, a boundary layer phenomenon will very likely occur here, as the following easy computation shows.

Consider a stationary solution to (1.1) in the scalar case, for a linear function $f(c) = (u + vk)c$, with $k > u/|v|$. The system boils down to the single ordinary differential equation

$$\frac{dc}{dx} = \frac{1}{\varepsilon uv}[b - f(c)], \qquad c(0) = a.$$

There exists a unique equilibrium point c^* such that $f(c^*) = b$, and it is attractive. The solution c_ε is computed explicitly:

$$c_\varepsilon(x) = \frac{b}{u + kv} + \left(a - \frac{b}{u + kv}\right)\exp\left(-\frac{u + kv}{\varepsilon uv}x\right) = c^* + (a - c^*)\exp\left(-\frac{u + kv}{\varepsilon uv}x\right).$$

Obviously, the trace of the limit solution is c^*, which has no reason to coincide with a. We do not wish to investigate this boundary layer now, and leave it for future work.

Proof of Theorem 5.2. To prove part (i) of the theorem, we sum the two equations in (1.1), which gives the conservation of matter, and proceed exactly as in the proof of the convergence theorems. Provided we choose a test function $\varphi \in \mathcal{D}(]0, 1] \times [0, T[)$, that is, if the test function does not see the boundary condition at $x = 0$, we obtain a weak formulation with an equality sign:

$$\begin{aligned} & - \int_0^1 \int_0^T [(\mathbf{c} + \mathbf{h}(\mathbf{c}))\partial_t\varphi + (u\mathbf{c} + v\mathbf{h}(\mathbf{c}))\partial_x\varphi] dx\, dt \\ & \qquad = \int_0^1 [\mathbf{c}^0 + \mathbf{h}(\mathbf{c}^0)]\varphi(x, 0)dx - \int_0^T \mathbf{b}(t)\varphi(1, t)dt, \end{aligned} \tag{5.4}$$

since the boundary condition at $x = 1$ is satisfied exactly.

As a first consequence, we obtain, by taking $\varphi \in \mathcal{D}(\Omega)$, that $\mathbf{c}$ is actually a solution to (1.2) in $\mathcal{D}'(\Omega)$. Therefore we can apply (5.1) with $\psi = (c_i + h_i(\mathbf{c}), uc_i + vh_i(\mathbf{c}))$, $1 \leq i \leq N$, and $\varphi \in \mathcal{D}(]0, 1] \times [0, T[)$. The left-hand side of (5.4) is exactly $\int_\Omega (\psi, \varphi)$, and $\operatorname{div} \psi = 0$, so we are left with

$$\begin{aligned} & \int_0^1 \gamma^0[c_i + h_i(\mathbf{c})]\varphi(x, 0)dx - \int_0^T \gamma_1[uc_i + vh_i(\mathbf{c})]\varphi(1, t)dt \\ & \qquad = \int_0^1 [c_i^0 + h_i(\mathbf{c}^0)]\varphi(x, 0)dx - \int_0^T b_i(t)dt. \end{aligned}$$

Since this holds for any φ, we obtain (5.2).

Now, (5.3) follows from (3.5) or (4.14). By Lemma 5.1, for any pair (η_1, η_2), $\psi = (\eta_1(\mathbf{c}) + \eta_2(\mathbf{h}(\mathbf{c})), u\eta_1(\mathbf{c}) + v\eta_2(\mathbf{h}(\mathbf{c})))$ satisfies that $\operatorname{div}\psi$ is a nonnegative measure. Thus we can apply (5.1) in both formulae, with $\varphi \in \mathcal{D}([0,1] \times [0,T[)$, and obtain

$$\int_0^1 \gamma^0[\eta_1(\mathbf{c}) + \eta_2(\mathbf{h}(\mathbf{c}))]\varphi(x,0)dx - \int_0^T \gamma_1[u\eta_1(\mathbf{c}) + v\eta_2(\mathbf{h}(\mathbf{c}))]\varphi(1,t)dt \leq \int_0^1 [\eta_1(\mathbf{c}^0) + \eta_2(\mathbf{h}(\mathbf{c}^0))]\varphi(x,0)dx - \int_0^T B(t)dt.$$

Since this holds for any φ, we obtain (5.3). □

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