

A FINITE ELEMENT, FILTERED EDDY-VISCOSITY METHOD FOR THE NAVIER-STOKES EQUATIONS WITH LARGE REYNOLDS NUMBER

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ABSTRACT

Fluid turbulence is commonly modeled by the Navier-Stokes equations with a large Reynolds number. The simulation of turbulence model is known to be very difficult. We study artificial spectral viscosity models that render the simulation of turbulence tractable. The models introduce several parameters. We show that the models have solutions that converge, in certain parameter limits, to solutions of the Navier-Stokes equations. We also show, using the mathematical analysis, how effective choice for the parameter can be made.

The direct computational simulation of turbulence flow is a formidable task due to the disparate scales that have to be resolved. Turbulence modeling attempts to mitigate this situation by accounting for the effects of small-scale behavior on that at large-scales without explicitly resolving the small scales. One such approach is to add viscosity to the problem; the Smagorinsky and Ladyzhenskaya models and other eddy-viscosity models are examples of this approach. Unfortunately, this approach usually results in over-dampening at the large scales, i.e., large-scale structures are unphysically smeared out. To mitigate this fault of eddy-viscosity modeling, filtered eddy-viscosity methods that add the artificial viscosity only to the high-frequency modes were developed in the context of spectral methods. We apply the filtered eddy-viscosity idea to finite element methods with hierarchical basis functions. We prove the existence and uniqueness of the finite element approximation and its convergence to a weak solution of the Navier-Stokes system.

MODEL PROBLEM

It is generally accepted that incompressible fluid flows, even for high values of the Reynolds number Re , are faithfully modeled by the Navier-Stokes system

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - 2\nu \nabla \cdot D(\mathbf{u}) + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi &= \mathbf{f} && \text{on } (0, T) \times \Omega \\ \nabla \cdot \mathbf{u} &= 0 && \text{on } (0, T) \times \Omega \\ \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega \\ \mathbf{u} &= \mathbf{g} && \text{on } (0, T) \times \Gamma, \end{aligned} \tag{1}$$

where \mathbf{u} and π are the velocity and pressure, $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ is the rate of deformation tensor, \mathbf{f} is a body force per unit mass, \mathbf{u}_0 and \mathbf{g} are given initial and boundary data, and, after appropriate non-dimensionalization, we have that the kinematic viscosity $\nu = 1/Re$.

For values of the Reynolds number above a critical value and for all but sufficiently small data, it is not known if solutions of (1) are globally unique. Also, for such values of the Reynolds number, flows become turbulent, i.e., they feature small eddies. Because of such small-scale behavior, the computational simulation of turbulence flows is a challenging task. In fact, the grid resolution needed to fully resolve eddies renders direct numerical simulation of (1) infeasible, even using the most powerful computers available today or in the foreseeable future.

THE LADYZHENSKAYA AND SMAGORINSKY MODELS

A turbulence model in common use was introduced by Smagorinsky [1] and a generalized version was independently developed by Ladyzhenskaya [2,3]. The Smagorinsky and Ladyzhenskaya models fall into the class of *eddy-viscosity models* in which the viscosity coefficient ν is modified so that in regions where the gradient of the velocity is relatively large, additional viscosity is added. The Smagorinsky and Ladyzhenskaya models have been analyzed by various authors who take advantage of the additional viscosity to show that, e.g., under certain assumptions, the strong solution of Smagorinsky's model exists and is globally unique on a periodic domain [4]. For more detailed discussions and a survey of these models, see, e.g., [5,6] and the references therein.

In [1], Smagorinsky developed a modification of the Navier-Stokes equation based on the assumption that the action of small eddies on large eddies is analogous to a turbulent viscosity so that the energy decay is related to the magnitude of the wave number. Ladyzhenskaya [2,3] independently proposed the more general model

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{u} + \nabla\pi - \nabla \cdot T(D(\mathbf{u})) = \mathbf{f}, \quad (2)$$

where the stress tensor T is a function of $D(\mathbf{u})$. Under certain assumptions on T , she proved the global unique solvability of the boundary value problem for the modification (2) of the Navier-Stokes equation. One choice for $T(D(\mathbf{u}))$ is given by

$$T(D(\mathbf{u})) = 2\left(\nu + \nu_1|D(\mathbf{u})|^{p-2}\right)D(\mathbf{u}), \quad (3)$$

where ν_1 is a positive constant. The Smagorinsky model corresponds to the case $p = 3$ and, if $\nu_1 = 0$, we recover the Navier-Stokes case. Ladyzhenskaya's assumptions for T hold for (3) and the unique solvability holds for $p \geq \frac{5}{2}$.

In this paper, we consider Ladyzhenskaya's model in the form

$$\frac{\partial\mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} + \nabla\pi - \epsilon\nabla \cdot (|\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u}) = \mathbf{f}, \quad (4)$$

where we have used the obvious relation $2\nabla \cdot D(\mathbf{u}) = \Delta\mathbf{u}$. Theoretical results for this model can be found in [2,3]. As briefly mentioned in the introduction, it is known [4] that the strong solution of the Navier-Stokes equations modified according to (4) uniquely exists on a periodic domain when $p \geq \frac{11}{5}$.

Although the additional viscosity added in the Smagorinsky and Ladyzhenskaya models is effective in stabilizing the flow equations, in practice, these models are over diffusive, i.e.,

the large-scale structures are smeared out. This results because those models do not sufficiently differentiate between scales, i.e., artificial viscosity is added at all scales. We study, in a finite element context, the idea of adding eddy viscosity only at the small scales, i.e., only to the high-frequency modes. This idea was used in the context of spectral [7–10], finite element [11,12], and wavelet [13] methods for hyperbolic conservation laws, for spectral methods for the Navier-Stokes equations [14–16], and for the Navier-Stokes equations in a computational implementation using finite element methods [17].

The spectral eddy-viscosity method used in [14–16] is based on Fourier spectral basis functions. We pursue the idea of selectively applying artificial viscosity at only high frequencies in the context of finite element methods. Unlike the Fourier spectral basis functions, standard nodal finite element basis functions are all of the same frequency. Thus, to apply artificial viscosity in a selective manner, we turn to hierarchical finite element basis functions which can be clustered into the groups that have different scales [18,19].

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