

On the spectral viscosity method in multidomain Chebyshev discretizations

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Abstract

This paper describes how one can use the spectral vanishing viscosity method proposed by Tadmor in multidomain solution of hyperbolic systems. Interface conditions are derived using a variational approach, and open boundary conditions are derived using the approach used in [9] for incomplete parabolic systems.

1 Introduction

Filtering of the solution is a very common technique when using spectral methods on problems with solutions of limited regularity. The main reason for using filtering is to prevent the buildup of large components of high spatial frequency, and hence to stabilize the solution. There are many variants of filtering described in the literature, see e.g. [8] for techniques to handle discontinuities, and [6] for a general overview. We will here concentrate on problems where we don't have to deal explicitly with shocks or discontinuities, but where we filter to stabilize the smooth solution.

We consider a quasi-linear hyperbolic system

$$u_t + \sum_{i=1}^d A_i(u, x, t) u_{x_i} = b(u, x, t), \quad u, b \in \mathbb{R}^m, \quad A_i \in \mathbb{R}^{m \times m}. \quad (1)$$

Chebyshev spectral collocation will be used to discretize the PDE system at least in one direction, and we consider the solution of (1) in multiple subdomains, i.e. the domain is given by $\Omega = \bigcup_{i=1}^n \Omega_i$. The interfaces between the subdomains are denoted by Γ_{ij} and the outer boundaries by $\partial\Omega_i$ (which may be empty). We will therefore have to find interface conditions at Γ_{ij} for the numerical method to work properly. In addition we want open or transparent boundary conditions, since we are interested in wave-like solutions where our boundary is just an artificial one. Open boundary conditions for hyperbolic systems are described in many works, but here we consider the method using characteristic variables described in e.g. [11], [16] and [1].

We will use the spectral viscosity method for Chebyshev discretizations proposed by Tadmor, [14], and discussed in detail in [2]. Other filtering or artificial viscosity methods are described in [6].

The rest of the paper is organized as follows: In section 2 we describe the spectral viscosity in detail and interface conditions are derived using a variational technique. Section 3 is devoted to the derivation of open boundary conditions.

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2 Spectral viscosity and interface conditions

Spectral viscosity or more precisely, spectral vanishing viscosity was introduced by Tadmor in the Fourier case in [15], and he showed that this method converged to the entropy solution of a system of conservation laws. The spectral viscosity (SV) method is a spectral filter which acts only on frequencies higher than a certain threshold, and hence should leave the low frequency (smooth) components intact.

For the Chebyshev case, Tadmor have suggested the following family of SV filters, see [14] and [16]:

$$\frac{\varepsilon_N}{w(x)^p} \frac{\partial}{\partial x} \left(R_N \star \frac{u_x^N}{w(x)^q} \right), \quad (2)$$

where ε_N is the "viscosity" coefficient, $w(x) = (1 - x^2)^{1/2}$ the Chebyshev weight, $p, q \in \mathbb{Z}$. The filtering function is defined as:

$$R_N \star \frac{u_x^N}{w(x)^q} = \sum_{m_N < l < N} \hat{R}_l \tilde{b}_l T_l(x) \quad (3)$$

where \hat{R}_l are filter coefficients and \tilde{b}_l are the Chebyshev coefficients of the weighted first derivative $1/w(x)^q u_x^N$. The parameters suggested by Tadmor are:

$$\varepsilon_N \sim N^{-1}, \quad m_N \sim N^{1/2}, \quad \hat{R}_k \geq 1 - \left(\frac{m_N}{k} \right)^2, \quad k > m_N.$$

The parameters p and q were set to 1 and 0 respectively in the numerical experiments reported in [16], because one wanted to keep the hyperbolic boundary and interface conditions. These values have also been chosen in larger experiments, see e.g. [3]. If we don't impose the restriction of using the hyperbolic boundary and interface conditions, but rather derive boundary conditions based on the PDE system with the viscosity term, we are free to choose values for p and q . The simplest choice is to set both to zero, such that it resembles the Fourier variant.

Consider now the PDE system (1) with the SV term added:

$$u_t + \sum_{i=1}^d A_i u_{x_i} = \varepsilon \sum_{i=1}^d P^{(ii)} \frac{1}{w(x)^p} (R_i \star u_{x_i})_{x_i} + b \quad (4)$$

where we have assumed that the matrix $P^{(ii)}$ is diagonal, and that ε represents a small parameter. Note that the SV method is defined in discrete form, and here we assume the existence of operators R_i for which R_{N_i} is the discrete approximation.

To derive the interface conditions, we will use the variational method applied in [9] for incomplete parabolic systems. In order to perform this we have to freeze the coefficients at the interface, i.e. to linearize locally around the solution and the position. For (4), this implies that the matrices A_i will be constant. Let (\cdot, \cdot) denote the L_2 inner product within a domain, and denote by $\langle \cdot, \cdot \rangle$ the L_2 inner product on the boundary. For sufficiently smooth functions, we will use the following Green's formulas:

$$(A_i u_{x_i}, v) + (A_i u, v_{x_i}) = \langle A_i u, v n_i \rangle \quad (5a)$$

$$\left(\sum_i (R \star u_{x_i})_{x_i}, v \right) + \left(\sum_i (R \star u_{x_i}), v_{x_i} \right) = \left\langle \sum_i (R_i \star u_{x_i}), v \right\rangle \quad (5b)$$

We will use an antisymmetric term for the first derivative terms, and from the Green's formula, we have immediately that:

$$(A_i u_{x_i}, v) = \frac{1}{2} [(A_i u_{x_i}, v) - (A_i u, v_{x_i})] + \frac{1}{2} \langle A_i u, v n_i \rangle$$

Hence we can write the PDE system for $p = q = 0$ in variational form:

$$\begin{aligned} (u_t, v) + \frac{1}{2} \sum_i [(A_i u_{x_i}, v) - (A_i u, v_{x_i})] + \varepsilon \sum_i P^{(ii)} ((R_i \star u_{x_i}), v_{x_i}) \\ \frac{1}{2} \langle \sum_i A_i u, v n_i \rangle + \varepsilon \sum_i P^{(ii)} \langle (R_i \star u_{x_i}), v \rangle + (b, v) \end{aligned} \quad (6)$$

For $p = 1$ we get a similar result, but now the boundary term for the SV vanishes because of the form of the weight function. Since this will not give us other interface conditions than in the hyperbolic case, we concentrate on the case $p = 0$ from now on.

If we introduce bilinear forms:

$$\begin{aligned} a(u, v) &= \frac{1}{2} \sum_i [(A_i u_{x_i}, v) - (A_i u, v_{x_i})] \\ s(u, v) &= \left(\sum_i (R_i \star u_{x_i}), v_{x_i} \right) \end{aligned}$$

we can write the variational form:

$$(u_t, v) + a(u, v) + s(u, v) = \langle \mathcal{E} u, v \rangle + (b, v) \quad (7)$$

where

$$\mathcal{E} u = \sum_i \left(\varepsilon P^{(ii)} (R_i \star u_{x_i}) + \frac{1}{2} A_i u \right)$$

We now proceed exactly as done in [9] and introduce two subdomains Ω^+ and Ω^- and bilinear forms a^+ and so on defined over the respective subdomain. We have that:

$$\begin{aligned} a(u, v) &= a^+(u, v) + a^-(u, v) \\ s(u, v) &= s^+(u, v) + s^-(u, v) \end{aligned}$$

So by writing the equation (7) for Ω^+ and Ω^- , we can add these equations and subtract (7). We then obtain

$$\langle \mathcal{E} u, v \rangle_{\Omega^+} + \langle \mathcal{E} u, v \rangle_{\Omega^-} = 0 \quad (8)$$

where the superscripts indicate the subdomains to which the quantity belongs. If we assume that the test function v is compactly supported in Ω , then the boundary inner products reduces to that of the interface. The transmission conditions are:

$$(\mathcal{E} u)^- = (\mathcal{E} u)^+ \quad (9a)$$

$$u^- = u^+ \quad (9b)$$

where the superscripts "+" and "-" refers to the values of the quantity taken in Ω^+ and Ω^- respectively. So in particular if Ω^+ and Ω^- are half-spaces with the x_1 -axis as the interface, we obtain the following conditions:

$$\varepsilon P^{(11)} (R_1^+ \star u_{x_1}^+) = \varepsilon P^{(11)} (R_1^- \star u_{x_1}^-) \quad (10a)$$

$$u^+ = u^- \quad (10b)$$

Note that the explicit dependence on A_i disappears because we have to assume that this matrix is non-singular. Alternatively, we can of course use the condition $A_i u^+ = A_i u^-$, or $TA_i u^+ = TA_i u^-$ where T is the left eigenvector matrix of A_i . The latter condition expresses that the characteristic variables should be continuous at the interface. Hence for each component of u , we have apart from the continuity of the variable itself, a relation of the form:

$$\varepsilon^+ (R_1^+ \star u_{x_1}^+) = \varepsilon^- (R_1^- \star u_{x_1}^-) \quad (11)$$

where u now denotes one component. If we now take the discrete version of these conditions and use the definition (3) of the SV filter,

$$\varepsilon^+ \sum_{m_{N_+} < l < N_+} \hat{R}_l^+ \tilde{b}_l^+ T_l(x) = \varepsilon^- \sum_{m_{N_-} < l < N_-} \hat{R}_l^- \tilde{b}_l^- T_l(x), \quad (12)$$

we see that if the number of gridpoints in the two domains are equal, and therefore the parameters in the SV method, the conditions reduces to require that the Chebyshev coefficients in the two subdomains should be equal for $m_N \leq l \leq N$. In the much more interesting case where the number of grid points are not equal, we see from the definition that the coefficients of the expansion of the filtered values must be matched. In particular. So for the case where $N_+ > N_-$, and hence $m_{N_+} > m_{N_-}$, we have the conditions:

$$\varepsilon^- \hat{R}_l^- \tilde{b}_l^- = 0, \quad m_{N_-} \leq l \leq m_{N_+} \quad (13a)$$

$$\varepsilon^+ \hat{R}_l^+ \tilde{b}_l^+ = \varepsilon^- \hat{R}_l^- \tilde{b}_l^-, \quad m_{N_+} \leq l \leq N_- \quad (13b)$$

$$\varepsilon^+ \hat{R}_l^+ \tilde{b}_l^+ = 0, \quad N_- \leq l \leq N_+ \quad (13c)$$

These conditions are quite different from the interface conditions in [9], and the reason is that spectral viscosity is defined in spectral space, and hence we get matching conditions on the Chebyshev spectra in each domain.

It is fairly obvious that these conditions can be generalized to work for an interface of arbitrary (but smooth) shape.

3 Boundary conditions

Again we consider the PDE system (4) with $p = 0$:

$$u_t + \sum_{i=1}^d A_i u_{x_i} = \varepsilon \sum_{i=1}^d P^{(ii)} (R_i \star u_{x_i})_{x_i} + b \quad (14)$$

We are now interested in imposing correct boundary conditions on Ω_i . We know from numerical experiments, see e.g. [5] and [12], that indirect imposition of the boundary conditions seems to work well. This procedure goes as follows: Assume that we have a viscous term of the form

$$\frac{\partial}{\partial x} \left(\varepsilon \frac{\partial u}{\partial x} \right)$$

and a boundary condition of the form

$$\alpha \frac{\partial u}{\partial n} + \beta u = \gamma$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. At the boundary we now solve for $\frac{\partial u}{\partial n}$ in the boundary condition and insert this expression in the viscous term. Then we perform the second differentiation. These modified second derivatives are only computed at the boundary, elsewhere we compute the term as usual. If our viscous term is of the SV type, we can use the same procedure, but slightly modified: We solve for the derivative in the boundary condition as before, but now we expand the result (a linear function of u) in Chebyshev series and filter these coefficients before performing the second differentiation.

The situation is different if we want open or transparent boundary conditions since it is not always possible to express such conditions as mixed type of boundary conditions. It is well known that the ideal open boundary conditions are global both in time and space, and therefore local approximations have to be made. There are many methods suggested for a Navier-Stokes type of equations, but here we will use the theory developed in [9]. Applications of this theory and discussion of the discrete case is given in [12]. The starting point for the derivation of the open boundary conditions in [9] are the interface conditions between two subdomains. The theory relies on the fact that the linearized incomplete parabolic systems has solutions of the normal mode type:

$$\hat{u} = \sum_{i=1}^{r+p} \lambda_i e^{\xi_i x_1} \Phi_i.$$

For the explanation of the symbols, see [9]. If one will use this theory also for the SV case, we have to show that such solutions also exist for this case.

3.1 The advection-diffusion equation

In order to gain some insight into the existence of normal mode solutions for our type of SV equation, consider an advection equation with SV in 1D:

$$v_t + cv_x = \varepsilon \frac{\partial}{\partial x} (R \star v_x) \quad (15)$$

in the half-space $\Omega^+ = (0, \infty)$. If we perform a Laplace transform in t and assume zero initial conditions we get

$$s\hat{v} + c\hat{v}_x = \varepsilon \frac{\partial}{\partial x} (R \star \hat{v}_x).$$

Now inserting a solution $e^{\xi x}$, we get after canceling the common $e^{\xi x}$ -term:

$$s + c\xi = \varepsilon \xi^2 \hat{R}(\xi) \quad (16)$$

where $\hat{R}(\xi)$ is the Laplace transform of the filtering function R . This means that we can solve (16) for ξ (at least in principle), and then find a normal mode for our PDE. The equation (16) is the same as the basic equation [9, (1.1)], and hence we have for the general case that

$$Q(\xi, i\eta) = A_1 \xi + \sum_{j=2}^d A_j i\eta_j + \varepsilon P^{(11)} \xi^2 \hat{R}_1(\xi) - \sum_{j=2}^d \varepsilon P^{(jj)} \eta^2 \hat{R}_j(i\eta) \quad (17)$$

To allow for different filtering parameters for the individual equations, we let $\hat{R}_1(\xi) = \text{diag}(\hat{R}_{11}(\xi), \dots, \hat{R}_{1n}(\xi))$, and similarly for the other coordinate directions.

If we now go back to (16), we see that we should have an explicit expression for $\hat{R}(\xi)$, and we have not specified the operator itself yet, only indicated that it should be so constructed that the discrete filtering operator R_N is a good approximation to it. If we consider R as a distribution we know that $R = \delta$ will give the usual advection-diffusion equation, because we have that $\delta * v_x = v_x$ and $\mathcal{L}\delta = 1$ (\mathcal{L} denotes the Laplace transform of a distribution). Open boundary conditions for the advection-diffusion equation are derived in [13].

We can express the filtering distribution in terms of a type of summability kernels used in harmonic analysis, see e.g. [10]. An example of such a kernel is the De la Vallée Poussin kernel

$$V_\lambda = 2K_{2\lambda} - K_\lambda$$

where K_λ is the Féjer kernel, see again [10]. The Fourier transform of the De la Vallée Poussin kernel is shown in figure 1. We see that a possible filtering operation may be expressed as

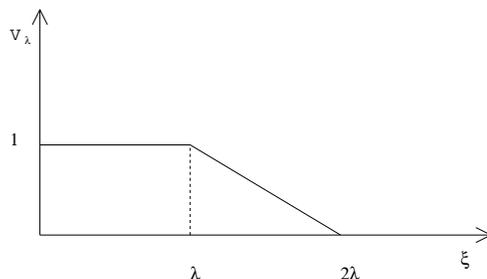


Figure 1: Fourier transform of the De la Vallée Poussin kernel

$R * v_x = v_x - V_\lambda * v_x$ for a suitable λ , hence the filtering distribution can be written $R = \delta - V_\lambda$. We are seeking the Laplace transform of this distribution, and that is different from the Fourier representation since we have the relation

$$\mathcal{L}(f(x)) = \mathcal{F}(f(x) e^{-\sigma x}), \quad \xi = \sigma + i\omega,$$

where ω is the variable in Fourier space. We will not be using the De la Vallée Poussin kernel, but a kernel with Fourier transform matching the form of the spectral viscosity. Let $V(\xi)$ be the Laplace transform of such a kernel. Then we can write (16)

$$s + c\xi = \varepsilon \xi^2 (1 - V(\xi)). \quad (18)$$

The distribution $V(\xi)$ should then tend to zero as $|\xi| \rightarrow \infty$. An interesting kernel is the function of Riesz-mean type discussed in [4, p.34]. In Fourier space it is given by

$$K(\omega) = \begin{cases} 1 - |\omega|^2 & \omega \leq 1 \\ 0 & \omega > 1 \end{cases}$$

Note that the Riesz-mean kernel does not have the same Fourier representation as the SV method calls for, but the characteristics are very similar. We will see below that there are analytical results available for the Riesz-mean, which give us insight into how the filtering affects the construction of the open boundary conditions. If $0 < a < b$ the following kernel has the wanted properties:

$$V_{ab}(\omega) = \alpha K\left(\frac{\omega}{b}\right) - (\alpha - 1)K\left(\frac{\omega}{a}\right), \quad \alpha = \frac{b^2}{b^2 - a^2}$$

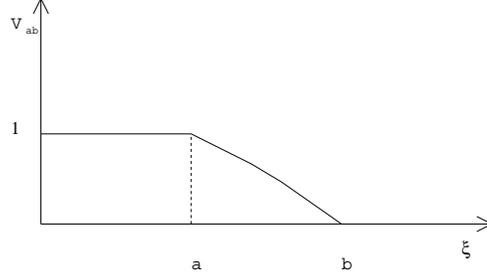


Figure 2: Fourier representation of the kernel V_{ab}

The Fourier representation of this function is given in figure 2. The inverse transform of $K(\omega)$ is also given in [4]:

$$\mathcal{F}^{-1}(K(\omega)) = x^{-3/2} J_{3/2}(x),$$

hence

$$\mathcal{F}^{-1}V_{ab} = \alpha b^{-1/2} J_{3/2}(bx) - (\alpha - 1) a^{-1/2} J_{3/2}(ax)$$

that goes as $\mathcal{O}(x^{-2})$ as $x \rightarrow \infty$. Hence it is a well-behaved function. The Laplace transform of $\mathcal{F}^{-1}(K(\omega))$ is possible to calculate and it is expressed as a hypergeometric function.

$$\mathcal{L}\left(x^{-3/2} J_{3/2}(\beta x)\right) = \frac{\beta^{3/2}}{2^{3/2} \Gamma\left(\frac{5}{2}\right) \xi} F\left(\frac{1}{2}; 1; \frac{5}{2}; -\frac{\beta^2}{\xi^2}\right).$$

The hypergeometric function is the expression

$$F\left(\frac{1}{2}; 1; \frac{5}{2}; -\frac{\beta^2}{\xi^2}\right) = \sum_{m=0}^{\infty} \frac{1 \cdot 3}{(2m+1)(2m+3)} \left(-\frac{\beta^2}{\xi^2}\right)^m$$

It is easy to see that the transform behaves like $\mathcal{O}(\xi^{-1})$ as $\xi \rightarrow \infty$, hence have the wanted behavior.

The Laplace transform of the filtering distribution can now be written

$$V(\xi) = 1 - C_1 \frac{1}{\xi} - C_2 \frac{1}{\xi^3} + \mathcal{O}(\xi^{-5}) \quad (19)$$

for $|\xi|$ sufficiently large, and where $C_1, C_2 \in \mathbb{R}$. To simplify further, we may assume $|\xi|$ so large that the $\frac{C_2}{\xi^3}$ -term can be ignored, and we are left with the $\frac{C_1}{\xi}$ -term. The constant C_1 is easily calculated from the expression for the Laplace transform:

$$C_1 = \frac{1}{2^{3/2} \Gamma\left(\frac{5}{2}\right)} (\alpha b - (\alpha - 1)a) = \frac{1}{2^{3/2} \Gamma\left(\frac{5}{2}\right)} \frac{a^2 + ab + b^2}{a + b}$$

So the "characteristic" equation (16) now becomes

$$s + c\xi = \varepsilon \xi^2 \left(1 - \frac{C_1}{\xi}\right) \quad (20)$$

Consider now a kernel that has a Fourier representation that corresponds to the SV method. Let

$$\mathcal{K}(\omega) = \begin{cases} 1 & 0 \leq \omega \leq a \\ \frac{a^2}{\omega^2} & \omega > a \end{cases} \quad (21)$$

We want to compute the inverse Fourier transform for this distribution:

$$\mathcal{F}^{-1}(\mathcal{K}) = \int_0^a e^{i\omega t} dt + \int_a^\infty e^{i\omega t} \frac{a^2}{\omega^2} dt.$$

The first integral, denoted by I_1 , is elementary:

$$I_1 = \frac{1}{it}(e^{iat} - 1) \quad (22)$$

The second integral is more difficult, but the inverse transform of $\text{Pf} \frac{1}{\omega^2}$ is known. Furthermore, the following relation holds:

$$\mathcal{F}^{-1}(\text{Pf} \frac{1}{\omega^2}) = \mathcal{F}^{-1}(1_+(\omega) \frac{1}{\omega^2}) + \mathcal{F}(1_+(\omega) \frac{1}{\omega^2}),$$

where $1_+(\omega)$ is the unit step function. The last term can be computed by using the result in [7, p.177]:

$$\mathcal{F}(1_+(\omega) \frac{1}{\omega^2}) = a_0^{(2)} t - a_{-1}^{(2)} t \ln(t - i0)$$

where $a_0^{(2)} = i$, $a_{-1}^{(2)} = i(1 + \Gamma'(1) + i\frac{\pi}{2})$. Using the expression for $\text{Pf} \frac{1}{\omega^2}$ in [17, p.204], we have the wanted inverse transform:

$$I_2 = \frac{t}{2} - 1_+(t) t - a_0^{(2)} t + a_{-1}^{(2)} t \ln(t - i0) \quad (23)$$

We will now compute the Laplace transform of I_1 and I_2 , and again we are only interested in the asymptotic properties of these transforms. We have

$$\mathcal{L}(I_1) = i \log \frac{\xi}{\xi - ia}$$

but expanding I_1 in power series we get the asymptotic result:

$$\mathcal{L}(I_1) = \frac{a}{\xi} + \frac{a^2}{\xi^2} + \mathcal{O}(\xi^{-3}) \quad (24)$$

The Laplace transform of I_2 has the following asymptotic expansion:

$$\mathcal{L}(I_2) = \frac{C}{\xi^2} + \mathcal{O}(\xi^{-3}), \quad C \in \mathbb{R} \quad (25)$$

Hence by assuming that $|\xi|$ is sufficiently large, we may include only the ξ^{-1} -term, and this gives the same "characteristic equation as (20), but where the constant C_1 is now substituted by a .

The open boundary condition comes from the transmission condition where we have inserted the normal mode solution in the outer domain:

$$R * \varepsilon v_x = \varepsilon \xi \hat{R}(\xi) v, \quad (26)$$

cfr. [9, Thm.2.2], with $\hat{R}(\xi) = 1 - V(\xi)$. The exact form of the boundary condition now depends on the behavior of $V(\xi)$ as $\varepsilon \rightarrow 0$ for the obtained values of ξ .

Let us first consider the case where all $V(\xi)$ can be ignored, i.e. $|\xi|$ is sufficiently large for this to be satisfied within the required accuracy for the open boundary conditions. Then (16) reduces to the ordinary advection-diffusion equation:

$$s + c\xi = \varepsilon\xi^2$$

which has roots:

$$\frac{c}{\varepsilon} + \frac{s}{c} + \mathcal{O}(\varepsilon), \quad \text{and} \quad -\frac{s}{c} + \varepsilon \left(\frac{s^2}{c^3} \right) + \mathcal{O}(\varepsilon^2)$$

This gives the following first order open boundary conditions:

$$\varepsilon(R \star v_x) = 0, \quad \text{on outflow} \quad (27a)$$

$$\varepsilon(R \star v_x) = c v, \quad \text{on inflow} \quad (27b)$$

The second order (and time-dependent) conditions are:

$$\varepsilon(R \star v_x) = -\frac{\varepsilon}{c}v_t, \quad \text{on outflow} \quad (28a)$$

$$\varepsilon(R \star v_x) = c v + \frac{\varepsilon}{c}v_t, \quad \text{on inflow} \quad (28b)$$

These conditions make sense since for $R = \delta$ they are identical to the conditions for the advection-diffusion equation obtained in [13]. The interpretation of the conditions in the discrete case is via the Chebyshev expansion both sides of the equations. So for the first order conditions, the interpretation is as follows:

outflow :

$$\begin{aligned} 0 \leq l \leq m_N & : b_l \text{ unchanged} \\ m_N < l \leq N & : \hat{R}_l b_l = 0 \end{aligned}$$

inflow :

$$\begin{aligned} 0 \leq l \leq m_N & : a_l = 0 \\ m_N < l \leq N & : \hat{R}_l b_l = c a_l \end{aligned}$$

where a_l are the Chebyshev coefficients for the unknown u .

Similar interpretations are valid for the second order conditions. An alternative is to use indirect imposition of the boundary conditions, i.e. substitute for $\varepsilon(R \star v_x)$ at the boundary before performing the second differentiation in the viscous term.

Now consider the equation (20) again, and this has the following roots expanded in ε -series:

$$\frac{c}{\varepsilon} + \frac{s}{c} + C_1 + \mathcal{O}(\varepsilon), \quad \text{and} \quad -\frac{s}{c} + \varepsilon \left(\frac{s^2}{c^3} + \frac{sC_1}{c^2} \right) + \mathcal{O}(\varepsilon^2)$$

With $C_1 = a = m_N = \varepsilon^{-1/2}$ we have that $\varepsilon \xi \hat{R}(\xi) = \varepsilon \xi - \varepsilon^{1/2} + \mathcal{O}(1)$, hence with the roots above this becomes:

$$c + \frac{s}{c}\varepsilon + \mathcal{O}(\varepsilon^{3/2}), \quad \text{and} \quad -\frac{s}{c}\varepsilon + \frac{s}{c^2}\varepsilon^{3/2} + \mathcal{O}(\varepsilon^2)$$

This gives the following open boundary conditions: For the outflow case we get the same conditions as above, but now the terms omitted are $\mathcal{O}(\varepsilon^{3/2})$. For the inflow case we also get the same conditions because the terms of order $\varepsilon^{1/2}$ in $\varepsilon \xi \hat{R}(\xi)$ cancel.

If we include the second order term in the Laplace transform, we get the same first root, but the ε -term in the second root, which does not influence the boundary condition, now becomes

$$\frac{s^2}{c^3} + a\frac{s}{c^2} + \frac{a^2}{c}.$$

If we include the third order term in the Laplace transform, and not the second order term. we get an entirely similar result: The ε -term of the second root becomes:

$$\frac{s^2}{c^3} + C_1\frac{s}{c^2} + \frac{C_2}{s}.$$

We see that in this case we obtain an integro-differential relation, which is not local.

3.2 The incomplete parabolic system

We now return to the incomplete parabolic system. From the above results for the advection-diffusion equation we may infer that the conditions to be used here are the same as the ones derived in e.g. [12], but where the left hand side of the conditions now are of the form:

$$\nu(R \star u_{i,x}).$$

However, this has to be justified (at least partially) because the theory developed in [9] is based upon certain assumptions which we now have to check. We will therefore follow the derivation in [9] in broad terms. In the following we will assume that the filtering function $\hat{R}(\xi)$ will have the form $1 - V(\xi)$ for all the variables. Moreover, the matrices $P^{(jj)}$ in the SV-term will be assumed to be diagonal. The latter assumption should not cause problems for the systems we will consider below. We will consider the problem in the half-space $\Omega^- = \{x : x_1 < 0\}$ bounded by the boundary $\Gamma = \{x : x_1 = 0\}$. The half-space $\Omega^+ = \{x : x_1 > 0\}$ then represents the outer domain.

The first point to check is if the normal modes ξ have the same behavior as stated in [9, Thm.1.1] as $\varepsilon \rightarrow 0$. In essence, r values should tend to infinity and n values should have a finite limit. The proof in [9] can be used almost as it stands, but we have to check that

$$\left| \varepsilon P^{(11)} \xi^2 \hat{R}(\xi) - sI \right| = 0$$

has r roots with negative real part and r roots with positive real part. Using the assumption that the matrix $P^{(11)}$ is diagonal and positive definite, and that $\hat{R}(\xi)$ has the required form, we can easily find that the theorem holds.

Now we can assume a solution in the outer domain of the normal mode type:

$$\hat{u} = \sum_{i=1}^{r+p} \lambda_i e^{\xi_i x_1} \Phi_i.$$

This solution will be used to derive the open boundary condition in the first form. We start with the transmission condition

$$R * \varepsilon P^{(11)} \hat{u}_x = R * \varepsilon P^{(11)} i \hat{u}_x^+ \quad (29)$$

and compute \hat{u}_x^+ and $R * \hat{u}_x^+$. We get

$$R * \hat{u}_x^+ = \sum_{i=1}^{r+p} \lambda_i \xi_i \hat{R}(\xi_i) e^{\xi_i x_1} \Phi^i$$

and hence the transmission condition is

$$R * \varepsilon P^{(11)} \hat{u}_x = \varepsilon P^{(11)} \sum_{i=1}^{r+p} \lambda_i \xi_i \hat{R}(\xi_i) \Phi^i.$$

Now using the derivation in the proof of [9, Thm.2.2], we obtain an equivalent result:

Lemma 1 *The open boundary condition at Γ for the half-space Ω^- is:*

$$R * \varepsilon P^{(11)} \hat{u}_x = \varepsilon P^{(11)} \sum_{i=1}^{r+p} \sum_{j=1}^{r+p} \xi_i \hat{R}(\xi_i) M_{ij}^{-1} \hat{u}_j \Phi^i \quad (30)$$

■

We have used the same notation as in [9] and [12]. From the assumptions we have

$$\varepsilon \xi_i \hat{R}(\xi_i) = \varepsilon \xi_i - \varepsilon^{1/2} + \mathcal{O}(1).$$

The next step depends on the asymptotic properties of ξ . It is relatively straightforward to show using the proof of [9, Thm.1.1] that we have the following expressions:

$$\xi_j = \begin{cases} \alpha + \mathcal{O}(\varepsilon^{1/2}) & 1 \leq j \leq m \\ \frac{\theta}{\varepsilon} + \frac{\chi}{\varepsilon^{1/2}} + \mathcal{O}(1) & m < j \leq r+p \end{cases} \quad (31)$$

where the quantities α and θ are found as in [9], and where χ is found from another generalized eigenproblem which we don't give here because it will not be used in the following. Hence we see that the asymptotic properties of $\varepsilon \xi_i \hat{R}(\xi_i)$ are the same as in [9], but now we have terms of order $\varepsilon^{1/2}$ in both expressions. So we can now construct boundary conditions of half orders.

By performing the limiting process, we obtain a parallel to the conditions [9, (2.10a), (2.10b)]:

$$R * P^{(11)} \hat{u}_x = P^{(11)} \sum_{i=m+1}^{r+p} \sum_{i=1}^{r+p} \zeta_i N_{ij}^{-1} \hat{u}_j \Psi^i \quad (32a)$$

$$\hat{u}_k = \sum_{i=1}^{r+p} \sum_{i=1}^{r+p} N_{ij}^{-1} \hat{u}_j \Psi_k^i, \quad r+p+1 \leq k \leq n \quad (32b)$$

where now

$$\zeta_i = \begin{cases} \theta_i + \chi_i \varepsilon^{1/2} + \mathcal{O}(\varepsilon) & m+1 \leq i \leq r+p \\ \frac{s}{\lambda_i} + \mathcal{O}(\varepsilon^{1/2}) & 1 \leq i \leq m \end{cases}$$

hence the first order conditions are the same as those obtained in e.g. [12], whereas the second order conditions are now in fact conditions of order $1\frac{1}{2}$.

3.3 Applications to the Euler equations for an atmosphere

In this section we will be applying the results obtained above to the Euler equations used to simulate gravity waves in the atmosphere, for the physics see e.g. [3]. The governing equations are as follows:

$$\rho \frac{d\mathbf{u}}{dt} = \nabla p + \rho \mathbf{g} + \mathbf{F} \quad (33a)$$

$$\frac{dp}{dt} + \gamma p \nabla \cdot \mathbf{u} = 0 \quad (33b)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (33c)$$

Here \mathbf{u} is the velocity vector, ρ is the density, p is the pressure, \mathbf{g} is the acceleration of gravity, and γ is the ratio of specific heats. The open boundary conditions for this system of PDEs is given e.g. in [2]. The open boundary conditions for the incomplete parabolic system for this PDE system is not given explicitly in the literature, but we can easily infer what the conditions are from the results for a closely related system, namely the equations for nonlinear acoustic propagation in sea water, given in [12].

we include a SV-term in these PDE system and assume that $P^{(11)} = \text{diag}(1, 1, 1, \kappa/\varepsilon, 0)$, and also that the filtering function (using the parameters m_N and N) is the same for all variables. Hence the pressure equation can have a different amount of damping from the momentum equations. It is straightforward to extend the results to the case where $P^{(11)}$ can have arbitrary positive elements.

Since the first order conditions are identical to the ones obtained for the usual incomplete parabolic system except that the left hand side is the filtered derivative, we can write down the results directly from the formulas given in [12]. Consider first the right interface and the inflow case. We then get:

$$R * \varepsilon \frac{\partial \bar{u}}{\partial x} = \frac{\theta_1}{\hat{\rho}(\hat{a} + \hat{u} - \theta_1)} (\hat{\rho} \hat{a} \bar{u} - \bar{p}) \quad (34a)$$

$$R * \varepsilon \frac{\partial \bar{v}}{\partial x} = \hat{u} \bar{v} \quad (34b)$$

$$R * \varepsilon \frac{\partial \bar{w}}{\partial x} = \hat{u} \bar{w} \quad (34c)$$

$$R * \varepsilon \frac{\partial \bar{p}}{\partial x} = \frac{\theta_1(\theta_1 - \hat{u})}{(\hat{a} + \hat{u} - \theta_1)} (\hat{\rho} \hat{a} \bar{u} - \bar{p}) \quad (34d)$$

Here the hatted quantities are the frozen coefficients used in the derivation of the characteristic variables, \bar{u} and so on represents the variables with homogeneous initial conditions, a is the sound speed, and

$$\theta_1 = \frac{1}{2\kappa} \left(\hat{u}(1 + \kappa) - \sqrt{4(\hat{a}^2 - \hat{u}^2) + \hat{u}^2(1 + \kappa)^2} \right).$$

In the inflow case there is also a hyperbolic part which is an old friend:

$$\rho - \frac{p}{\hat{a}^2} = \rho_0 - \frac{p_0}{\hat{a}^2} \quad (35)$$

where the quantities with the zero subscript refers to the values exterior to the domain.

For the outflow case we don't have any hyperbolic part of the boundary conditions, and the parabolic part, again from the results in [12] is:

$$R * \varepsilon \frac{\partial \bar{u}}{\partial x} = \frac{\theta_1}{\hat{\rho}(\hat{a} + \hat{u} - \theta_1)}(\hat{\rho}\hat{a}\bar{u} - \bar{p}) \quad (36a)$$

$$R * \varepsilon \frac{\partial \bar{v}}{\partial x} = 0 \quad (36b)$$

$$R * \varepsilon \frac{\partial \bar{w}}{\partial x} = 0 \quad (36c)$$

$$R * \varepsilon \frac{\partial \bar{p}}{\partial x} = \frac{\theta_1(\theta_1 - \hat{u})}{(\hat{a} + \hat{u} - \theta_1)}(\hat{\rho}\hat{a}\bar{u} - \bar{p}) \quad (36d)$$

We see that these conditions contains the outgoing fast characteristic for the hyperbolic part of the system, so that the conditions reduce to specifying the incoming characteristics in the $\varepsilon \rightarrow 0$ limit.

For the left boundary we have a different incoming characteristic which will enter in the expression. Again from [12] we get for the inflow case:

$$R * \varepsilon \frac{\partial \bar{u}}{\partial x} = \frac{\theta_2}{\hat{\rho}(\hat{a} + \hat{u} - \theta_2)}(\hat{\rho}\hat{a}\bar{u} + \bar{p}) \quad (37a)$$

$$R * \varepsilon \frac{\partial \bar{v}}{\partial x} = \hat{u}\bar{v} \quad (37b)$$

$$R * \varepsilon \frac{\partial \bar{w}}{\partial x} = \hat{u}\bar{w} \quad (37c)$$

$$R * \varepsilon \frac{\partial \bar{p}}{\partial x} = \frac{\theta_2(\theta_2 - \hat{u})}{(\hat{a} + \hat{u} - \theta_2)}(\hat{\rho}\hat{a}\bar{u} + \bar{p}) \quad (37d)$$

We also have a hyperbolic part of the boundary conditions and this identical to (35).

For the outflow case we get in a similar way

$$R * \varepsilon \frac{\partial \bar{u}}{\partial x} = \frac{\theta_2}{\hat{\rho}(\hat{a} + \hat{u} - \theta_2)}(\hat{\rho}\hat{a}\bar{u} + \bar{p}) \quad (38a)$$

$$R * \varepsilon \frac{\partial \bar{v}}{\partial x} = 0 \quad (38b)$$

$$R * \varepsilon \frac{\partial \bar{w}}{\partial x} = 0 \quad (38c)$$

$$R * \varepsilon \frac{\partial \bar{p}}{\partial x} = \frac{\theta_2(\theta_2 - \hat{u})}{(\hat{a} + \hat{u} - \theta_2)}(\hat{\rho}\hat{a}\bar{u} + \bar{p}) \quad (38d)$$

and there is no hyperbolic part here

The interpretation of these conditions in the discrete case is again via the Chebyshev coefficients, exactly as shown for the advection-diffusion case. The practical implementation can be done in several ways, and in [12] the numerical experiments show that the indirect imposition method works best. This procedure can of course also be applied here, but then the direct correspondence between the Chebyshev coefficients is somewhat hidden.

Note that for frequencies (and ξ -values) below the filtering threshold, we have a hyperbolic system with its corresponding open boundary conditions. In practice that means that we have to impose both types of boundary conditions, those belonging to the hyperbolic system, and viscous ones like those derived above.

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