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OPTIMALLY TRANSPORTED SCHEMES : ONE DIMENSIONAL CASE

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ABSTRACT. This article reports the first considerations, one dimensional ones, that led us to consider a new to our knowledge class of numerical methods, called "Optimally Transported Schemes". The name "Optimally Transported Schemes" has been given because this method is designed using mainly Optimal Transport Theory and Optimal Quantization considerations. These schemes can be interpreted in the context of Eulerian Lagrangian schemes, Entropic Schemes, kinetic schemes, or also pure particle methods. The basic idea underlying this method is to compute the solution using a moving unstructured mesh, a mesh owning some optimal repartition property.

1. INTRODUCTION

Our motivations to design new numerical methods comes from stochastic considerations arising in Mathematical Finance (see [15]). In this area, valuation of some complex "derivative" financial products sensitive to a huge number of risk sources is an open numerical problem. This problem leads to consider nonlinear diffusion type equations set in high dimensions. Standard numerical Partial Differential Equations (P.D.E.) solvers, that means for us numerical methods based on an Eulerian description of the underlying dynamic, fail considering efficiently these problems. This so-called "curse of dimensions" open numerical problem has been very attractive to us, since it required to design new, innovating numerical methods to be efficiently tackled. Note that innovative and very elegant numerical methods have already be designed to tackle the "Curse of Dimensions". The Partial Differential Equations (PDE) school and the Stochastic school are competing here: Multi scale "sparse tensor product" wavelet analysis has been developed in [13], since a Quantized method, usually considered as a Stochastic approach, can already handle problems up to the first dimensions [5]. A direct multi scale approach using "entropy conservative scheme" may also be found in [1].

To tackle this problem, we designed and promote now a somehow different approach, that is an hybrid PDE / Stochastic method. This method may be considered as a bridge between P.D.E. and Stochastic analysis, since it is a quantized method, that uses mainly P.D.E. arguments. From a continuous point of view, these numerical methods link with Optimal Quantization and Optimal transport theory (see [26] for a review). The Optimal Transport theory seems to be a powerful glue between PDE Theory and Stochastic Analysis. From a discrete point of view, we will be using extensively the notion of entropy schemes techniques. Tadmor's synthesis work ([24]) should be consulted for a modern and brilliant overview of these numerical techniques.

In this paper, we report the fundamentals, self contained, one dimensional ideas, underlying the construction of these "Optimally Transported Schemes". Doing so, we will show that this technique is a general approach that can be used in a variety of different situations. To that aim, this paper fully reviews the one dimensional case, from the continuous to the numerical and algorithmic point of view, for three kind of one dimensional equations :

- A model of diffusive equation : the heat equation

$$(1.1) \quad \partial_t u - \partial_x^2 u = 0$$

- A model of fully non linear hyperbolic equation : the scalar Burger equation

$$(1.2) \quad \partial_t u - \partial_x f(u) = 0$$

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As is well known, the previous equation owns infinitely many weak solutions, and we have to select a unique one. Thus we give some further conditions : there exists a pair of convex entropy - entropy flux $U(u), F(u)$ such that

$$(1.3) \quad \partial_t U(u) - \partial_x F(u) \leq 0$$

in a weak sense.

- A system of nonlinear linear hyperbolic equations : the p system

$$(1.4) \quad \partial_t w = \partial_x z; \partial_t z = \partial_x \sigma(w)$$

This system is also known as the nonlinear wave equation, since it can be written as $\partial_t^2 w - \partial_x (\sigma(\partial_x w)) = 0$. The p -system owns many weak solutions $u := (w, z)$. To select one, we will consider entropic solutions, i.e. solutions that verify weakly

$$(1.5) \quad \partial_t E(u) \leq \partial_x F(u) \text{ with } E(u) := e(w) + \frac{z^2}{2} := \int_0^w \sigma(s) ds + \frac{z^2}{2} \text{ and } F(u) := \sigma(w)z$$

The fact that this numerical technique applies indifferently to these three families of equations indicates that the correct framework seems to be the one given by Conservation Laws, diffusive type equation being part of. Obviously, the interest of this numerical method is not to be found into these academical equations, but in forthcoming studies that deal with industrial applications and higher dimensional case.

We noticed that mesh repartition has been the main concern of several numerical developments during the last decades as Mesh Less methods (see for instance [10]) and force driven meshes (see for instance [20]), or even coupled Eulerian-Lagrangian methods (see for instance [6] for a review). They have been designed to boost algorithmic complexity of standard numerical methods. The framework proposed in this paper may provide a general one able to shed some new lights over their properties.

This method appears to be a new one, even if a deeper look to the existing and wide Mathematic literature is still needed. Note that the pioneering work of G. Russo has already showed off the spins-off of this method for the one dimensional diffusive case, and tackled the two-dimensional one ([8], [9]). We noticed also that quite close ideas may be found within the work of P.G. Lefloch [11] for the one dimensional case, a work that was independently driven. Our contribution may be to have given a general framework for numerical analysis applications, noticing that the Optimal Transport Theory can be used systematically to single out a map, that seems to be the correct generalization to higher dimensions, and noticing also that Optimal Quantization arguments can be used to design schemes satisfying an almost optimal property (see 2.10).

Not that we do not include numerical results in this paper. The numerical results, as well as the C++ code, can be found in open access (see [16]).

2. BACKGROUND MATERIAL

In this preliminary section, we introduce some basic definitions and state some technical results that will be used throughout this paper.

2.1. Equi probable Quantizers. Consider that the solution of our three problems is a positive measure $u(t, x)$, of constant mass, and verifying $\int_{\Omega} u(t, x) dx = 1$. The unstructured mesh that we will be using is simply a collection of N continuous trajectory $\mathbf{S}(t) = \{t \mapsto S_i(t) \in \Omega\}_{i=1, \dots, N}$ that "fit" this measure. This set of trajectory is chosen such that the expectation of every "smooth enough" function φ , with respect to the random variable of density u , is simply approximated by an *equi probable* formula

$$(2.1) \quad \int_{\Omega} \varphi(x) u(t, x) dx \sim \frac{1}{N} \sum_{i=1, \dots, N} \varphi(S_i(t))$$

Remark 2.1. Note that this equi probable approximation is a weaker form of Optimal Quantization. An optimal quantizer formula would yield the approximation

$$(2.2) \quad \int_{\Omega} \varphi u := \int_{\Omega} \varphi(x) u(t, x) dx \sim \sum_{i=1, \dots, N} u_i(t) \varphi(S_i(t))$$

where $u_i(t) = \int_{\Omega_i(t)} u(t, x) dx$, $\sum_i u_i(t) = 1$ and $\bigcup_{i=1, \dots, N} \Omega_i(t) = \Omega$ is the Voronoi diagram induced by the quantizers, i.e.

$$(2.3) \quad \Omega_i(t) = \{x \in \Omega : \|x - S_i(t)\|^2 \leq \|x - S_j(t)\|^2 \text{ for all } j\}$$

The quantizers are selected in order to minimize the quantization error

$$(2.4) \quad E_S(u(t, \cdot)) = \sum_i \int_{\Omega_i(t)} \|x - S_i(t)\|^2 u(t, x) dx$$

that is the Wasserstein distance between the measure $u(t, x) dx$ and the measure $\sum_i u_i(t) \delta_{S_i(t)}$.

Due to these properties, an optimal quantizer of a measure may be its best possible representation by a computer. However, it appeared that they are quite difficult to apprehend, and we do not know efficient methods to compute them, efficiency meaning here CPU-time measurements.

Thus, in this paper, we will restrict ourselves to the computation of these equiprobable quantizers, that should be a correct alternative to Optimal Quantizers, although a little bit less precise, while providing efficient numerical methods. We use this simplification to link easily with Optimal Transport Theory: let us denote S^{-1} the cumulative of u , that is $S^{-1}(t, x) = \int_{\{s \in \Omega; s \leq x\}} u(t, s) ds$. This function takes its value into $x \in \Omega$. We have (with a slight notation abuse) $S^{-1}(t, \Omega) = [0, 1]$. Since u is a positive measure, we can define properly its inverse map $S(t, y)$: this map takes its values into $y \in [0, 1]$, and is uniquely defined through the relation $S(t, S^{-1}(t, x)) = x$. Elementary computations shows that S verifies (and may be defined as) $u(t, x) \partial_y S(t, y) = 1$, for every $x = S(t, y)$. Summarizing this property with a slight notation abuse, we write the one dimensional Jacobian equation

$$(2.5) \quad u(S) \partial_y S \equiv 1$$

S is usually referred as the increasing rearrangement of u . However, from the Optimal Transport Theory point of view, S is the map transporting optimally the uniform measure $1_{[0,1]}(y) dy$ into the measure $u(t, x) dx$. We refer to S as a transport map, because this concept is well defined in higher dimensions, since increasing rearrangements are not. The basic property of this transport map is the following : for every φ "smooth enough", using the change of variable $x = S(t, y)$, the expectation of φ with respect to the measure u , is

$$(2.6) \quad \int_{\Omega} \varphi(t, x) u(t, x) dx = \int_{[0,1]} \varphi(S(t, y)) dy := \int_{[0,1]} \varphi(S)$$

Using this transport map, we have a clear interpretation of equiprobable Quantizers :

$$(2.7) \quad S_i := S_i(t) = N \int_{[\frac{i-1}{N}, \frac{i}{N}]} S(t, y) dy := N \int_{[\frac{i-1}{N}, \frac{i}{N}]} S$$

Each equiprobable quantizer controlling a cell

$$(2.8) \quad \Omega_i(t) = S(t, [\frac{i-1}{N}, \frac{i}{N}])$$

Let us consider now the expectation computation (2.2), together with these "equi probable" quantizer :

$$(2.9) \quad \int_{\Omega} \varphi u \sim \frac{1}{N} \sum_{i=1, \dots, N} \varphi(S_i)$$

Such a formula is a very classical Monte Carlo sampling formula. What is a little bit less classical at our knowledge is the sampling sequence defined by the equi probable quantizers (2.7). This sequence provides a much more accurate formula than the usual sampling sequences : we can prove theoretically, and we also experimented numerically, and it is somehow an optimal property intimated by these "Optimally transported schemes", that the equi probable quantizers (2.7) provides a second order accuracy formula for computing expectations (2.2) in one dimension, in the following sense : for any **convex** function φ ,

$$(2.10) \quad \left| \int_{\Omega} \varphi u - \frac{1}{N} \sum_{i=1, \dots, N} \varphi(S_i) \right| \leq \frac{C(\varphi)}{N^2}$$

This rate of convergence has to be compared to the rate of convergence of other sampling sequences: $O(\frac{C(\varphi)}{\sqrt{N}})$ (pseudo random sequences), $O(\frac{C(\varphi)\ln N}{N})$ (quasi random sequences). We think that only Optimal Quantizers can provide a constant C better than the constant appearing in the error estimation (2.10).

2.2. Change of Variable Formula. As may have been understood in the previous subsection, our strategy is to model numerically the equation after a change of variable given by an Optimal Transport Map. Change of variables for Hyperbolic Conservation Laws is the main topic of P. LeFloch work ([11]). This subsection summarize some results, using materials kindly provided by P. LeFloch.

In this subsection we will consider solutions of system of conservation laws in the general case

$$(2.11) \quad \partial_t u + \partial_x f(u) = 0 \text{ with } (u, f(u)) \equiv \{u_i(t, x), f_i(u)\}_{i=1, \dots, p} \in \Omega^p \times \mathbb{R}^p, \text{ and } x \in \Omega \in \mathbb{R}$$

We will not discuss the technical structure of the non linearity f . For such system, in the geometric approach introduced in ([11]) one seeks for graph-solutions. A graph solution is a solution of ((2.11)) sought as $(t, u(S(t, y)))$, where the continuous parametrized graphs $x = S(t, y)$, where $y \in [0, 1]$ for simplicity, satisfies the monotonicity condition

$$\partial_y S \geq 0,$$

in a suitable sense. Graph-solutions are continuous curves evolving in time, and shock waves are endowed with an ‘‘internal structure’’ determined from an augmented system in which small-scale physical mechanisms (e.g. diffusion) are specified.

For every test function $\varphi(t, x)$, the weak formulation of equation (2.11) is

$$(2.12) \quad \int_{\mathbb{R}^+} \int_{\Omega} u(t, x) \partial_t \varphi(t, x) dt dx + \int_{\mathbb{R}^+} \int_{\Omega} f(u)(t, x) \partial_x \varphi(t, x) dt dx = 0$$

For which an admissible change of variable $x = S(t, y)$ leads to, denoting for short $u(S) \equiv u(t, S(t, y))$ and dropping dependance variable for concision

$$(2.13) \quad \int_{\mathbb{R}^+} \int_{[0,1]} u(S) \partial_y S \partial_t \varphi(S) dt ds + \int_{\mathbb{R}^+} \int_{[0,1]} f(u(S)) \partial_y S \partial_x \varphi(S) dt ds = 0$$

Note that $\frac{d}{dt} \varphi(S) = \partial_t \varphi(S) + \partial_x \varphi(S) \partial_t S$ and also $\partial_y S \partial_x \varphi(S) = \partial_y \varphi(S)$. We deduce thus the formulation

$$(2.14) \quad \int_{\mathbb{R}^+} \int_{[0,1]} u(S) \partial_y S \frac{d}{dt} \varphi(S) dt ds - \int_{\mathbb{R}^+} \int_{[0,1]} u(S) \partial_t S \partial_y \varphi(S) dt ds + \int_{\mathbb{R}^+} \int_{[0,1]} f(u(S)) \partial_y \varphi(S) dt ds = 0$$

This last equation is the weak formulation of the following equation in the *transported* variable $y \in [0, 1]$

$$(2.15) \quad \partial_t (u(S) \partial_y S) - \partial_y (u(S) \partial_t S) + \partial_y f(u(S)) = 0$$

This last equation is a system, and can be rewritten in a closed form involving S and $u(S)$

$$(2.16) \quad \partial_t u(S) \partial_y S + \partial_y f(u(S)) = 0$$

Let us suppose for instance that we use the property of the Optimal Transport Maps (2.5) of the i -th component of u , defining the change of variable $u_i(S) \partial_y S = 1$. Thus the i -th equation of (2.15) is transported, up to a constant, into

$$(2.17) \quad \partial_t S = (\partial_y S) f_i(u(S))$$

Since the others equations of the system (2.16) are transported into

$$(2.18) \quad \{\partial_t u_j(S) + u_i(S) \partial_y f_j(u(S)) = 0\}_{j=1, \dots, p}$$

3. THE HEAT EQUATION

We deal in this section with the Cauchy problem associated to the one dimensional heat equation

$$(3.1) \quad \partial_t u(t, x) - \partial_x^2 u(t, x) = 0$$

with initial conditions $u(0, x) = u_0(x) > 0$ of initial mass $\int u_0 = 1$. In one dimension, we consider the unbounded domain $\Omega = \mathbb{R}$ and $u_0 \in L^1(\mathbb{R}; (1+x^2)dx)$. Mass conservative boundary conditions of type $\partial_x u = 0$ on the boundary of a bounded or half bounded set Ω can easily be included in this analysis. We do not discuss them here to keep the expose as simple as possible.

Our aim in this section is to determine and compute the dynamic of the particle set $\{t \rightarrow S_i(t)\}_i$, i.e. the dynamic of the equi probable quantizers of the map transport of $u(t, \cdot)$. This set of particle is supposed ordered throughout this paper : $\{S_i < S_{i+1}\}_i$. Note that, for an initial Dirac mass $u_0 = \delta_{x_0}$, the equiprobable quantizers can be computed explicitly (see [15]).

Let us recall some basic facts about the heat equation. This is a mass conservative equation, i.e. $\int u(t, \cdot) \equiv 1$, which solution verifies a maximum principle, and verifies the following set of conservation and dissipation properties

- First Moment Balance : $\frac{d}{dt} \int x u(t, x) dx = 0$.
- Second Moment Evolution : $\frac{d}{dt} \int \frac{x^2}{2} u(t, x) dx = \int u_0$.
- Dissipation of the Boltzmann Entropy : $\frac{d}{dt} \int U(u)(t, x) dx \leq 0$, where $U(u) = u \ln u$.

3.1. Optimal transport map of the Density of the heat equation. Let us recall that the optimal transport map $S(t, y)$ is defined, in one dimension, through the Jacobian equation (2.5). In one dimension, the dynamic of this map is fully described in the following proposition :

Proposition 3.1. *Let $0 < u_0(x) \in L^1(\mathbb{R}; (1+x^2)dx)$, and consider a solution $t \mapsto u(t, x)$ of the heat equation (3.1), denote $S(t, y)$ its transport map. Then, S is described by the following equation*

$$(3.2) \quad \partial_t S = -\partial_y \left(\frac{1}{\partial_y S} \right)$$

where the initial condition $S(0, y) \in L^2([0, 1])$ is the increasing rearrangement of $\rho(0, x)$. Equivalently, the transported solution $u(S) = \frac{1}{\partial_y S}$ follows the equation

$$(3.3) \quad \partial_t u(S) = u^2(S) \partial_y^2 u(S)$$

These two equations defines a unique solution verifying

- $S \in C^1([0, +\infty[, L^2([0, 1]))$.
- $u(S) \in C^0([0, +\infty[, L^1([0, 1]))$.

This proposition states basically that the Cauchy problem of the heat equation (3.1) is completely equivalent to its transported version (3.2).

Remark 3.2. *The condition $u(S) \in C^0([0, +\infty[, L^1([0, 1]))$ means that, for **strictly** positive time,*

$$(3.4) \quad \int_{\Omega} u^2(t, x) dx < +\infty.$$

This property would be false at initial time $t = 0$ and a Dirac Mass initial conditions $\rho(0, x) = \delta_0$. However, it would hold for strictly positive time, that is a well know regularization effect of diffusion type equations.

Proof. First let us recall that the heat equation can be rewritten is the following Lagrangian form

$$(3.5) \quad \partial_t u = \partial_x (u \partial_x \ln u)$$

Thus we have the conservative form (2.11) with $f(u) = -u \partial_x \ln u$. Defining the change of variable given by the Jacobian equation (2.5), and using (2.17), we get

$$(3.6) \quad \partial_t S = -\partial_y S u(S) \partial_x \ln u(S) = -\frac{\partial_x u(S)}{u(S)} = -\frac{\partial_y u(S)}{\partial_y S u(S)} = -\partial_y \frac{1}{\partial_y S}$$

Since using (2.18) leads to

$$(3.7) \quad \partial_t u(S) - u(S) \partial_y (u(S) \partial_x \ln u(S)) = \partial_t u(S) - u(S) \partial_y (\partial_x u(S)) = \partial_t u(S) - u^2(S) \partial_y^2 u(S) = 0$$

Note that the basic set of properties of the heat equation are transported in the y variable (mass conservation being implicit) accordingly. Using the equation (3.2), the reader should find easy to prove that the transport map verifies formally

- First moment balance :

$$(3.8) \quad \frac{d}{dt} \int S(t, y) dy = 0.$$

- Second Moment evolution :

$$(3.9) \quad \frac{d}{dt} \int \frac{S(t, y)^2}{2} dy = 1$$

- Dissipation of the Boltzmann Entropy :

$$(3.10) \quad \frac{d}{dt} \int \ln \frac{1}{\partial_y S(t, y)} dy \leq 0.$$

The Second Moment evolution implies straightforwardly local in time existence, and global in time existence of a unique solution

$$S \in \mathcal{C}^1([0, +\infty[, L^2([0, 1])).$$

The dynamic and properties for the transported solution $u(S)$ are also let to the reader. This ends the proof. \square

3.2. Optimally transported Semi Discrete Scheme. Let us build a semi discrete scheme to solve (3.2) : we first choose the optimal quantizer of the uniform distribution in the variable y , that is $\{y_i := \frac{i-1/2}{N}\}_{i=1, \dots, N}$. Then we define $\{S_i(t) \sim N \int_{[\frac{i-1}{N}, \frac{i}{N}]} S\}_{i=1, \dots, N}$. Note that $t \rightarrow \mathbf{S}(t) := \{t \rightarrow S_i(t)\}_{i=1, \dots, N}$ can be interpreted as a set of particles trajectory. We define throughout this paper the following norms

$$\|\mathbf{S}(t)\|_{\ell^p}^p = \sum_{i=1, \dots, N} \frac{|S_i(t)|^p}{N}$$

The following Theorem, unknown at our knowledge, is almost straightforward using the ideas presented in the previous subsection :

Theorem 3.3. *Let consider N ordered particles $\{S_i(t)\}_{i=1, \dots, N}$ satisfying the following semi discrete scheme*

$$(3.11) \quad \frac{d}{dt} S_i = \frac{1}{S_i - S_{i-1}} + \frac{1}{S_i - S_{i+1}}$$

Then :

- *The trajectories are globally defined : $t \rightarrow \mathbf{S}(t) \in \mathcal{C}^1([0, +\infty[, \ell^2)$ with uniform bound $\|\mathbf{S}(t)\|_{\ell^2}^2 = \|\mathbf{S}(0)\|_{\ell^2}^2 + tI$.*
- *It is a second order scheme toward the equi probable quantizers (2.7) of the transport map of the heat equation.*

Remark 3.4. *It is possible to derive closed formula to compute these equi probable quantizer, see ([15]). The purpose here is to show that a numerical scheme can be designed to compute them.*

This semi discrete scheme was discovered at our knowledge first by G. Russo [9] in 1987, as well as some of the properties stated in this subsection. Note that this semi discrete scheme has some interesting properties from a computational point of view :

- This second order semi discrete scheme does not suffer from boundary treatments as standard Eulerian schemes (fixed grid) does.

- A simple time adaptive method is enough to focus on the main dynamic : indeed, we are computing the equi probable quantizer of the solution of the heat equation. It means that this mesh is somehow optimally adapted to the solution in space. Furthermore, this time adaptive technique together with the previous point allows to perform long time computation easily.
- We do not compute the solution $u(t, x)$. If needed the solution may be retrieved using a second order approximation

$$u(t, S_i) \sim \frac{2h}{S_{i+1} - S_{i-1}} = \frac{2}{N} \frac{1}{S_{i+1} - S_{i-1}}$$

with convergence rate $O(\frac{1+t}{N^2})$.

- Higher order accuracy for the spatial discrete operator may be retrieved using the framework developed in [12], that relies on results due to E. Tadmor [23].

We start proving the second order accuracy of the scheme : let us write (3.11) under the following "discrete lagrangian" form

$$(3.12) \quad \frac{d}{dt} S_i = -\delta_{\bar{y}} v_{i,i+1}^* = -\delta_{\bar{y}} \frac{1}{\delta_y S_i} = -\frac{1}{h} \left(\frac{h}{S_{i+1} - S_i} - \frac{h}{S_i - S_{i-1}} \right)$$

where $v_{i,i+1}^*$ is the discrete entropy variable, $\delta_{\bar{y}}$ (resp. δ_y) denotes the backward (resp. forward) finite difference operator $\delta_{\bar{y}} f_i = \frac{f_i - f_{i-1}}{h}$ (resp. $\delta_y f_i = \frac{f_{i+1} - f_i}{h}$). Using this form, the proof of this Proposition is straightforward : it consists in noticing that (3.12) provides a second order approximation of 3.2 using standard finite difference arguments.

Let us state some other interesting properties of this scheme, that will be used during the full discrete in time modeling of the optimal transport of the heat equation.

Proposition 3.5. *Let $\mathcal{S}(t) = \{S_i\}_i$ a set of N trajectories verifying (3.3). Then it enjoys the following set of property that coincide with the continuous ones (mass conservation holds by construction) :*

- *First moment balance*

$$\frac{d}{dt} \bar{\mathcal{S}}(t) := \frac{1}{N} \sum_i \frac{d}{dt} S_i(t) = 0$$

- *Second Moment evolution*

$$\frac{d}{dt} \|\mathcal{S}(t)\|_{\ell_N^2}^2 = \frac{1}{N} \sum_i \frac{d}{dt} \frac{S_i(t)^2}{2} = 1$$

- *Dissipation of the Boltzmann Entropy*

$$\frac{d}{dt} \mathcal{H}(\mathcal{S}(t)) := \frac{1}{N} \sum_i \frac{d}{dt} \psi_{i,i+1} := \frac{1}{N} \sum_{i=2,\dots,N} \frac{d}{dt} \ln \frac{h}{S_{i+1} - S_i} \leq 0$$

The fact that the set of conservation laws of the continuous solution (3.8 - 3.10) are transported at a discrete level indicates that this scheme is accurate.

- The First moment balance is straightforward, noticing that summing over all index i in (3.12) cancels out all terms.
- Second Moment evolution. We compute

$$\begin{aligned} \frac{d}{dt} \|\mathcal{S}(t)\|_{\ell_N^2}^2 &= \frac{1}{N} \sum_i \frac{S_i}{S_i - S_{i-1}} + \frac{S_i}{S_i - S_{i+1}} \\ &= 2 + \frac{1}{N} \sum_i \frac{S_{i-1}}{S_i - S_{i-1}} + \frac{S_{i+1}}{S_i - S_{i+1}} \\ &= 2|\Lambda| - \frac{d}{dt} \|\mathcal{S}(t)\|^2 \end{aligned}$$

Ending the proof.

- Dissipation of the discrete Boltzmann Entropy. Using the notations introduced previously, we compute

$$\frac{d}{dt}\psi_{i,i+1} = \frac{\delta_{\bar{y}}v_{i+1,i+2}^* - \delta_{\bar{y}}v_{i,i+1}^*}{h}v_{i,i+1}^*$$

Introducing the discrete laplacian operator $\delta_{y\bar{y}}f_i = \delta_y\delta_{\bar{y}}f_i = \frac{f_{i+1}-2f_i+f_{i-1}}{h^2}$, we rewrite this as

$$\frac{d}{dt}\psi_{i,i+1} = v_{i,i+1}^*\delta_{y\bar{y}}v_{i,i+1}^*$$

for which a discrete integration by part holds :

$$\sum_i \frac{d}{dt}\psi_{i,i+1} = - \sum_i (\delta_{\bar{y}}v_{i,i+1}^*)^2 \leq 0$$

This set of properties ends the proof of Theorem 3.3.

3.3. Fully discrete Analysis. Let $\{t^n\}_{n \geq 0}$, $t^0 = 0$, $t^n < t^{n+1}$ be a given time stepping. We denote δ_t the forward discrete time derivative operator $\delta_t f^n = \frac{f^{n+1}-f^n}{\tau^n}$, with $\tau^n = t^{n+1} - t^n$, and the backward discrete time derivative operator $\delta_{\bar{t}}f^n = \delta_t f^{n-1}$.

Proposition 3.6. *Let $\mathcal{S}^n = \{S_i^n\}_i$ be a solution of the following scheme*

$$(3.13) \quad \delta_t S_i^n = \frac{1}{S_i^{n+1/2} - S_{i-1}^{n+1/2}} + \frac{1}{S_i^{n+1/2} - S_{i+1}^{n+1/2}}$$

where $S_i^{n+1/2} = \frac{S_i^{n+1} + S_i^n}{2}$. Then it is a fully discrete second order scheme towards the optimal quantizer of the transport map of the heat equation. Furthermore it enjoys the following set of properties :

- First moment balance

$$\overline{\mathcal{S}^n} := \frac{1}{N} \sum_i S_i^n = \mathcal{S}^0$$

- Second Moment evolution.

$$\|\mathcal{S}^n\|_{\ell^2}^2 = \frac{1}{N} \sum_i \frac{(S_i^n)^2}{2} = t^n \mathbf{1}$$

- Dissipation of the discrete Boltzmann Entropy.

$$\delta_t \mathcal{H}(\mathcal{S}^n) := \delta_t \frac{1}{N} \sum_i \psi_{i,i+1}^n := \frac{1}{N} \sum_i \delta_t \ln \frac{h}{S_{i+1}^n - S_i^n} \leq 0$$

Remark 3.7. Here too, fully discrete schemes of arbitrary order can be designed using the framework in [12], relying on [23].

- The First moment balance is straightforward, writing the system under the form

$$(3.14) \quad \delta_t S_i^n = -\delta_{\bar{y}}v_{i,i+1}^{*,n+1/2}$$

and summing over all indexes i .

- Second Moment evolution. We compute, following the same guideline that the semi discrete case

$$\delta_t \|\mathcal{S}^n\|_{\ell^2}^2 = \frac{1}{N} \sum_i S_i^{n+1/2} \delta_t S_i^n = 1 - \delta_t \|\mathcal{S}^n\|^2$$

i.e. $\delta_t \|\mathcal{S}^n\|_{\ell^2}^2 = 1$, ending the proof.

- Dissipation of the discrete Boltzmann Entropy. Using the notations introduced previously, we compute

$$(3.15) \quad \delta_t \psi_{i,i+1}^n = \frac{1}{\tau^n} \ln \frac{v_{i,i+1}^{*,n+1}}{v_{i,i+1}^{*,n}} = \frac{1}{\tau^n} \ln \left(1 + \frac{v_{i,i+1}^{*,n+1} - v_{i,i+1}^{*,n}}{v_{i,i+1}^{*,n+1}} \right)$$

Using $\ln(1+u) \leq u$ we have

$$\delta_t \psi_{i,i+1}^n \leq v_{i,i+1}^{*,n+1/2} \delta_t \frac{1}{v_{i,i+1}^{*,n+1}}$$

From (3.14) we get

$$\delta_t \frac{1}{v_{i,i+1}^{*,n}} = \delta_t \frac{S_{i+1}^n - S_i^n}{h} = -\delta_{\bar{y}} \left(v_{i+1,i+2}^{*,n+1/2} - v_{i,i+1}^{*,n+1/2} \right)$$

Thus

$$\delta_t \psi_{i,i+1}^n \leq \frac{\delta_{\bar{y}} v_{i+1,i+2}^{*,n+1/2} - \delta_{\bar{y}} v_{i,i+1}^{*,n+1/2}}{h} v_{i,i+1}^{*,n+1/2}$$

The proof then follows the same guidelines than the semi discrete case.

An algorithm to solve the scheme 3.13 is proposed in the following Proposition, where we denote $\|f\|_{\ell^\infty} = \sup_i |f_i|$. This algorithm is basically a fixed point one. In the next sections related to Burger's and Wave equation, the same kind of algorithms can be designed to solve the fully discrete schemes. We will not present them, the present analysis providing the material to analyze their properties.

Proposition 3.8. *Let us consider the following algorithm to solve 3.13 :*

$$(3.16) \quad S_i^{n+1,0} = S_i^n$$

$$(3.17) \quad S_i^{n+1/2,k} = \frac{S_i^n + S_i^{n+1,k}}{2}$$

$$(3.18) \quad S_i^{n+1,k+1} = S_i^n + \frac{\tau^n}{S_i^{n+1/2,k} - S_{i-1}^{n+1/2,k}} + \frac{\tau^n}{S_i^{n+1/2,k} - S_{i+1}^{n+1/2,k}}$$

This algorithm is exponentially convergent toward the solution of (3.13) under the C.F.L (Courant Friedrich Levy) condition

$$(3.19) \quad \left\| \frac{\tau^n}{(S_i^n)^2 - (S_{i-1}^n)^2} \right\|_{\ell^\infty} < C$$

Finite difference schemes links naturally with Markov Chains : consider the matrix $\Xi^{n+1/2} = \left(\xi_{i,j}^{n+1/2} \right)_{i,j}$ given by

$$\begin{aligned} \xi_{i,i}^{n+1/2} &= \frac{1}{2} \frac{1}{(S_i^{n+1/2})^2 - (S_{i-1}^{n+1/2})^2} + \frac{1}{(S_i^{n+1/2})^2 - (S_{i+1}^{n+1/2})^2} \\ \xi_{i,i+1}^{n+1/2} &= \frac{1}{2} \frac{1}{(S_i^{n+1/2})^2 - (S_{i+1}^{n+1/2})^2} \\ \xi_{i,i-1}^{n+1/2} &= \frac{1}{2} \frac{1}{(S_i^{n+1/2})^2 - (S_{i-1}^{n+1/2})^2} \end{aligned}$$

Then the fully discrete scheme can be interpreted as a linear system $\mathbf{S}^{n+1} = \Pi^{n+1/2} \mathbf{S}^n$ where the (transition) matrix $\Pi^{n+1/2} = \left(\pi_{i,j}^{n+1/2} \right)_{i,j}$ is given by

$$\Pi^{n+1/2} = (I_d - \tau^n \Xi)^{-1} (I_d + \tau^n \Xi)$$

Thus the convergence rate of our fixed point algorithm depends upon the eigenvalues $\{\lambda_i^{n+1/2}\}_i$ of the matrix $I_d - \Pi^{n+1/2}$. Rough Estimations over these eigenvalues leads to the C.F.L. result.

The optimal value of the C.F.L. constant C is not known precisely. For numerical implementation, we tune dynamically this constant with a simple and efficient time adaptive technique : we start fixing as a guess $C = 1$, that allows us to compute back the value of the next time step t^{n+1} . We stop the computations after a given a stop criteria is reached : $\|S_i^{n+1,k+1} - S_i^{n+1,k}\|_{\ell^2} < \epsilon$ or if the number of iterations is greater than the arbitrary number ten. If the computations has

not converged after these ten first iterations, we divide the C.F.L. constant by two and reset the algorithm. If the number of iterations is less than three, then we multiply the C.F.L. constant by two for the next time step computation.

Note also that $\| \frac{1}{(S_i^n)^2 - (S_{i-1}^n)^2} \|_{\ell^\infty}$ is expected to decrease (at a rate $O(\frac{h}{\sqrt{t^n}})$) due to the Boltzmann Entropy Dissipation 3.15 and the Second Moment evolution.

4. SCALAR'S BURGER EQUATION

Let us recall that the one dimensional Cauchy problem associated to the burger equation is

$$(4.1) \quad \partial_t u - \partial_x f(u) = 0, \quad u, f(u) \in \mathbb{R}, \quad x \in \mathbb{R}, t > 0,$$

together with smooth enough compacted supported initial conditions $0 < u(0, x) = u_0(x) \in L^1(\mathbb{R})$ and scalar entropy conditions (1.3) in a weak sense. Here also, we consider for simplicity the unbounded case $x \in \mathbb{R}$.

To ensure that u can be used to define a transport map, i.e. $u > 0$ for all time, we consider ad hoc hypothesis : the entropy $U(u)$ is supposed to be convex. In this situation, the entropy solution coincides with the limiting Lax viscous solution, and we have $u(t, x) > 0$ if $u_0 > 0$.

Let us recall some basic facts about Burger's scalar equations : they are mass conservative equations, i.e. $\int u(t, \cdot) \equiv 1$. We use also the following set of conservation and dissipation properties

- First Moment Balance : $\frac{d}{dt} \int x u(t, x) dx = - \int f(u)$.
- Entropy Dissipation : $\frac{d}{dt} \int U(u)(t, x) dx \leq 0$.

and as in the previous heat equation case, we denote the initial mass $1 = \int u_0$.

Call $v(u) = \nabla U(u)$ the *entropy variable* associated with the given entropy U . In our case, the entropy is strictly convex, and $v \mapsto v(u)$ is a one-to-one mapping which can be used as a change of variable for convex entropies (Friedrichs and Lax), that is we can set

$$(4.2) \quad g(v) := f(u), \quad G(v) := F(u).$$

Furthermore, in our context, we define the following quantities

$$\tilde{f}(s) := s f\left(\frac{1}{s}\right); \quad \tilde{F}(s) = F\left(\frac{1}{s}\right); \quad \tilde{U}(s) = s U\left(\frac{1}{s}\right)$$

In our context, and in the spirit of definitions (4.2), the following is also well defined,

$$\tilde{v}(s) := \nabla \tilde{U}(s); \quad \tilde{g}(v) := \tilde{f}(s); \quad \tilde{G}(v) = \tilde{F}(u)$$

4.1. Optimal Transport Map dynamic for Burger's equations. As for the heat equation, we define in this subsection the transport map of the density of Burger's equation $S(t, y)$ through the Jacobian equation (2.5). We explicit in the following lemma the dynamic followed by S

Lemma 4.1. *Let $S(t, y)$, with $y \in]0, 1[$, satisfying*

$$(4.3) \quad \partial_t S - \tilde{f}(\partial_y S) = 0$$

Then $u(S) = \frac{1}{\partial_y S}$ is a solution of the Burger equation (1.2). More over, if

$$(4.4) \quad \partial_t \tilde{U}(\partial_y S) - \partial_y \tilde{F}(\partial_y S) \leq 0$$

in a weak sense, then $u(S)$ is the entropy solution of the one dimensional Burger equation (1.2) – (1.3)

Proof. Straightforward application of the change of variable formula 2.17 shows that the dynamic (4.3) holds. Let us now prove that the entropy condition (4.4) is equivalent to the entropy condition (1.3). We expand in the next formula its weak formulation, that is for every smooth $\varphi(y)$

$$\frac{d}{dt} \int_{[0,1]} \tilde{U}(\partial_y S) \varphi + \int_{[0,1]} \tilde{F}(\partial_y S) \partial_y \varphi \leq 0$$

using back the optimal transport change of variable $x = S(t, y)$, with $u(t, S) = \partial_x S^{-1}(t, S)$ we have

$$\frac{d}{dt} \int_{\mathbb{R}} u \tilde{U}\left(\frac{1}{u}\right) \varphi(S^{-1}) + \int_{\mathbb{R}} u \tilde{F}\left(\frac{1}{u}\right) \partial_y \varphi(S^{-1}) \leq 0$$

Notice that $u\partial_y\varphi(S^{-1}) = \partial_x(\varphi(S^{-1}))$ and we recall $u\tilde{U}(\frac{1}{u}) = U(u)$, $\tilde{F}(\frac{1}{u}) = F(u)$. Thus we get

$$\frac{d}{dt} \int_{\mathbb{R}} U(u)\varphi(S^{-1}) + \int_{\mathbb{R}} F(u)\partial_y\varphi(S^{-1}) \leq 0$$

Under the condition $u_0 > 0$, $S(t, y)$ is a isomorphism from $]0, 1[$ into \mathbb{R} , thus the previous formula is equivalent to the weak formulation of the entropy condition (1.3).

Note that the entropy dissipation implies

$$\tilde{U}(\partial_y S) \in L^1 \text{ decreasing in time}$$

And thus $\partial_t S$ also. In fact, the transport map is continuously defined, since u has bounded variations □

4.2. Optimally transported Semi Discrete Scheme. In the context of numerical approximation of conservation laws, it is suggested (see [12] and references therein) to consider entropy conservative schemes, that are schemes verifying (1.3) as an equality

$$\partial_t U(u) - \partial_x F(u) = 0$$

and then add regularization terms. We will follow this idea, looking for a scheme verifying (4.4) as an equality

$$\partial_t \tilde{U}(\partial_y S) - \partial_y \tilde{F}(\partial_y S) = 0$$

To that aim, we recall that the Tadmor fluxes ([23])

$$\tilde{g}^*(v_0, v_1) = \int_0^1 \tilde{g}(v_0 + s(v_1 - v_0)) ds$$

are "conservative" : denoting the numerical entropy as

$$\tilde{G}^*(v_0, v_1) = \frac{\tilde{G}(v_0) + \tilde{G}(v_1)}{2} + \frac{v_0 + v_1}{2} \tilde{g}^*(v_0, v_1) - \frac{1}{2} (v_0 \tilde{g}(v_0) + v_1 \tilde{g}(v_1))$$

we have the formula

$$(4.5) \quad v_0 (\tilde{g}^*(v_0, v_1) - \tilde{g}^*(v_{-1}, v_0)) = \tilde{G}^*(v_0, v_1) - \tilde{G}^*(v_{-1}, v_0)$$

For the space modeling, we use the notation

$$\tilde{U}_{i,i+1} := \tilde{U}\left(\frac{S_{i+1} - S_i}{h}\right); \tilde{v}_{i,i+1} := \tilde{v}\left(\frac{S_{i+1} - S_i}{h}\right)$$

and also

$$\tilde{g}_{i,i+1}^* := \tilde{g}^*(\tilde{v}_{i,i+1}, \tilde{v}_{i+1,i+2}); \tilde{G}_{i,i+1}^* := \tilde{G}^*(\tilde{v}_{i,i+1}, \tilde{v}_{i+1,i+2})$$

We state our result :

Lemma 4.2. *Let consider N ordered particles $\{S_i(t)\}_{i=1,\dots,N}$ satisfying the following semi discrete scheme*

$$(4.6) \quad \frac{d}{dt} S_i - \tilde{g}^*(\tilde{v}_{i-1,i}, \tilde{v}_{i,i+1}) = 0$$

Then :

- *This scheme verifies a discrete entropy conservation*

$$(4.7) \quad \frac{d}{dt} \tilde{U}_{i,i+1} - \delta_y \tilde{G}_{i-1,i}^* = 0$$

- *It is a second order semi discrete scheme toward the equi probable quantizers (see 2.7) of the weak solution of (4.3) with null entropy dissipation.*
- *The trajectories are globally defined : $t \rightarrow \mathbf{S}(t) \in \mathcal{C}^1([0, +\infty[, \ell^1)$.*

Remark 4.3. *Let us make some remarks*

- *These semi-discrete schemes defines global in time strong solutions. It is somehow surprising : actually, without any dissipation terms, solutions of Eulerian type semi-discrete numerical schemes produces spurious oscillations, and a strong solution can not be globally defined. From another point of view, transport map are smoothers than the density they represent.*
- *The trajectories of the particles of this semi-discrete scheme concentrates around shock formations. It is an interesting "shock capturing" property from a numerical point of view.*
- *From a numerical point of view, it is not necessary to add viscosity to these semi-discrete schemes. However this semi discrete scheme does not converges toward the entropy solution without adding viscosity. Actually, would convergence holds without viscosity, then it would imply that the entropy solutions of Burger's equations verify an Energy Conservation :*

$$\frac{d}{dt} \int_{\mathbb{R}} U(u(t, x)) dx = 0$$

that is false.

Proof. Entropy conservation : we compute

$$\frac{d}{dt} \tilde{U}_{i,i+1} = \tilde{v}_{i,i+1} \frac{d}{dt} \frac{S_{i+1} - S_i}{h} = \tilde{v}_{i,i+1} \frac{\tilde{g}_{i,i+1}^* - \tilde{g}_{i-1,i}^*}{h}$$

So that using (4.5) we get

$$\frac{d}{dt} \tilde{U}_{i,i+1} = \frac{\tilde{G}_{i,i+1}^* - \tilde{G}_{i-1,i}^*}{h}$$

that is the result.

Second order accuracy : we recall that the Tadmor's fluxes are second order accurate fluxes. Second order accuracy holds using standard finite difference arguments.

Globally in time defined. Note that the discrete entropy conservation straightforwardly implies that

$$t \rightarrow \| \tilde{U} \|_{\ell^1}$$

is constant. This bounds allows to have a bound over $\frac{h}{S_{i+1} - S_i}$, which allows back to bound $\tilde{g}^*(\tilde{v}_{i-1,i}, \tilde{v}_{i,i+1})$, ending the proof. \square

4.3. Fully discrete analysis. We use the notations in section 3.3. The fully discrete scheme and its properties are summarized in the following Proposition.

Proposition 4.4. *Let $S^n = \{S_i^n\}_{i,n \geq 0}$ be a solution of the following scheme*

$$\delta_t S_i^n = \tilde{g}^*(\tilde{v}_{i-1,i}^{n+1/2}, \tilde{v}_{i,i+1}^{n+1/2})$$

where $S_i^{n+1/2} = \frac{S_i^{n+1} + S_i^n}{2}$. Then

- *This scheme verifies the discrete entropy conservation*

$$\delta_t \tilde{U}_{i,i+1}^n - \delta_y \tilde{G}_{i-1,i}^{*,n+1/2} = 0$$

- *This scheme is a fully discrete second order scheme towards the equi-probable quantizer of the transport map of the entropy conservative burger equation.*

Remark 4.5. *Here too, fully discrete schemes of arbitrary order can be designed using the framework in [12], relying on [23].*

The Hamilton system (4.6) is approximated directly by the non linear system of equations

$$(4.8) \quad \delta_t S_i^n = g^* \left(v_{i-1,i}^{*,n+1/2}, v_{i,i+1}^{*,n+1/2} \right); S_i(0) = S_i^0$$

where

$$(4.9) \quad v_{i,i+1}^{*,n+1/2} = v^* \left(\delta_y S_i^n, \delta_y S_i^{n+1} \right)$$

and $v^*(v_0, v_1)$ is the Tadmor flux associated to v , that is

$$(4.10) \quad v^*(v_0, v_1) = \frac{\int_{v_0}^{v_1} v(s) ds}{v_1 - v_0}$$

The choice of such a scheme is motivated by the fact that it verifies the entropy evolution property

Lemma 4.6. *Let $\{S_i \equiv S_i(t)\}_{i=1, \dots, N}$ be N particles verifying the fully discrete scheme (4.8). Then this set of particles verifies a discrete Energy Conservation*

$$(4.11) \quad \delta_t \sum_i \tilde{U}(\delta_y S_i^n) = 0$$

Proof.

$$(4.12) \quad \delta_t \tilde{U}(\delta_y S_i) = v_{i,i+1}^{*,n+1/2} \delta_t \delta_y S_i^n = v_{i,i+1}^{*,n+1/2} \delta_y g^* \left(v_{i-1,i}^{*,n+1/2}, v_{i,i+1}^{*,n+1/2} \right)$$

Using the property of Tadmor fluxes, we have

$$(4.13) \quad \delta_t \tilde{U}(\delta_y S_i) = \delta_y G(v_{i-1,i}^{*,n+1/2})$$

Summing over all indices i will cancel the right hand term. \square

Thus this scheme suits well to our purposes. To solve the system 4.8, we use the Entropic Scheme fixed point algorithm : first initiate

$$(4.14) \quad S_i^{0,n+1} = S_i^n.$$

then loop for $k \geq 1$:

$$(4.15) \quad v_{i,i+1}^{*,k,n+1/2} := v^* \left(\delta_y S_i^n, \delta_y S_i^{k-1,n} \right); S_i^{k,n+1} = S_i^n + \tau^n g^* \left(v_{i-1,i}^{*,k,n+1/2}, v_{i,i+1}^{*,k,n+1/2} \right)$$

Until a convergence criteria holds, a generic and natural convergence criteria being

$$(4.16) \quad \left\| \frac{\tilde{U}(\delta_y S_i^n) - \tilde{U}(\delta_y S_i^{k,n+1})}{\tau^n} \right\|_{\ell_1} \leq \epsilon \|\tilde{U}(\delta_y S_i^n)\|_{\ell_1}$$

To illustrate numerically, we consider the classical Burger equation with a quadratic Entropy

$$(4.17) \quad \partial_t u + \frac{1}{2} \partial_x u^2 = 0; \frac{1}{2} \partial_t u^2 + \frac{1}{3} \partial_x u^3 \leq 0$$

This corresponds to the particle system

$$(4.18) \quad \partial_t S + \tilde{f}(\partial_y S) = \partial_t S + \frac{1}{2\partial_y S} = 0; \partial_t \frac{1}{2\partial_y S} + \partial_y \frac{1}{3(\partial_y S)^3} \leq 0$$

This yields the expressions

$$(4.19) \quad v(s) = -\frac{1}{2s^2}; g(s) = -\frac{\sqrt{2s}}{2}$$

and the Tadmor fluxes are

$$(4.20) \quad v^*(v_0, v_1) = -\frac{1}{2v_0v_1}; g^*(v_0, v_1) = -\frac{\sqrt{2}}{3} \frac{v_0 + v_1 + \sqrt{v_0v_1}}{\sqrt{v_0} + \sqrt{v_1}}$$

Here, the initial condition correspond to a cauchy data given by a concentrated gaussian $u_0(x) = N(\sigma, x) = \frac{1}{\sqrt{2\sigma}} \exp \frac{-x^2}{\sigma}$, where $\sigma = 0.1$. This leads to initial condition given by $\{S_i^0 = P^{-1}(\sigma, \frac{i+1/2}{N})\}_{i=1, \dots, N-1}$, where $P^{-1}(\sigma, x)$ is the inverse of the gaussian cumulative.

4.3.1. *The quadratic burger equation.* In this section, we drive explicit computations for the quadratic Burger equation. This equation is used for numerical tests.

$$\tilde{v}(u) := -\frac{1}{2u^2}; \tilde{g}(v) := \frac{\sqrt{-2v}}{2}; \tilde{G}(v) = \frac{(-2v)^{3/2}}{3}$$

We compute the Tadmor flux :

$$\tilde{g}^*(v_0, v_1) = -\frac{\sqrt{2}}{3} \frac{v_1^{3/2} - v_0^{3/2}}{v_1 - v_0}$$

Thus the scheme is

$$\frac{d}{dt} S_i - \frac{\sqrt{2}}{3} \frac{\tilde{v}_{i,i+1}^{3/2} - \tilde{v}_{i-1,i}^{3/2}}{\tilde{v}_{i,i+1} - \tilde{v}_{i-1,i}} = 0$$

with $\tilde{v}_{i,i+1} = \tilde{v}(\frac{S_{i+1}-S_i}{h}) = -\frac{h^2}{2(S_{i+1}-S_i)^2}$. Using this last expression, the semi discrete scheme is, with some further computations

$$\frac{d}{dt} S_i - \frac{2}{3} \left(\frac{h}{S_{i+1} - S_i} + \frac{h}{S_i - S_{i-1}} - \frac{h}{S_{i+1} - S_{i-1}} \right) = 0$$

For this last scheme, the energy conservation writes :

$$t \rightarrow \frac{1}{N} \sum_i \frac{h^2}{(S_{i+1} - S_i)^2} = C$$

that implies back the bound $\| \frac{h}{S_{i+1}-S_i} \|_{\ell_N^\infty} \leq \sqrt{NC}$, proving that the scheme is globally defined in time.

5. P-SYSTEMS

Let us recall the Cauchy problem associated to the one dimensional p-system (1.4) :

$$(5.1) \quad \partial_t w = \partial_x z; \partial_t z = \partial_x \sigma(w)$$

where $u(t, x) := (w(t, x), z(t, x))$ and $x \in \Omega$. We consider this Cauchy problem together with compact supported initial Cauchy data conditions $w(0, x) = w_0(x)$, $z(0, x) = z_0(x)$. The domain Ω may be \mathbb{R} or bounded with periodic boundary conditions for simplicity. However, Neumann boundary conditions (mass conservative) could be considered as well. The previous equation owns many weak solutions. To select one, we will consider entropic solutions, i.e. solutions that verifies weakly (1.5). Let us recall some basic fact about entropic solutions of p systems (1.4). We recall the the Riemann problem associated to p systems can be difficult : see for instance ([17]) and references therein.

- Mass conservative :

$$(5.2) \quad \frac{d}{dt} \int_{\Omega} w(t, x) dx = \frac{d}{dt} \int_{\Omega} z(t, x) dx = 0$$

- Entropy dissipation : let

$$(5.3) \quad E(u) = \int_0^w \sigma(s) ds + \frac{z^2}{2} := e(w) + \frac{z^2}{2}$$

be the entropy density, then the p system is entropy dissipative: (1.5) implies

$$(5.4) \quad \frac{d}{dt} \int_{\Omega} E(u) \leq 0$$

- Entropy variable : let $v = (v_1, v_2) = \nabla E(u) = (\sigma(w), z)$ be the *entropy* variables. Then Friedrich and Lax showed that these variables can be used as a change of variable if the entropy functional is convex. In this context, the p system can be linearized :

$$(5.5) \quad \partial_t u = \partial_x g(v) \text{ with } g(v_1, v_2) = (v_2, v_1)$$

Since the equation followed by the entropy variables (v_1, v_2) is, denoting $G(v) := v_1 v_2 = F(u)$,

$$(5.6) \quad \partial_t E(u) - \partial_x G(v) = v_1 \partial_x v_2 + v_2 \partial_x v_1 - \partial_x (v_1 v_2) \leq 0$$

- First momentum balance

$$(5.7) \quad \frac{d}{dt} \int_{\Omega} x E(u) \leq \int_{\Omega} x \partial_x G(v) \leq - \int_{\Omega} v_1 v_2$$

- Second momentum evolution for wave equation ($\sigma(w) = w$, $\partial_t^2 w - \partial_x^2 w = 0$).

$$(5.8) \quad \frac{d^2}{dt^2} \int_{\Omega} \frac{x^2}{2} w = C$$

5.1. Transported equations. We will focus over a optimal transport map constructed over the density w , that is mass conservative. Other transport map may be considered, depending on the quantity of interest for the numerical analyst. Some suggestions could be the velocity density z , or the energy density $E(u)$. For this last quantity, take care that our analysis needs at present time conservation of the density, thus would apply only to Energy conservative p -systems as wave equation ($\partial_t^2 w - \partial_x^2 w = 0$). Modification of this method would be needed to include dissipative densities, as Entropy dissipative p systems.

The change of variable formula (2.15) expresses here as

$$(5.9) \quad \partial_t (w(S) \partial_y S) - \partial_y (w(S) \partial_t S) + \partial_y z(S) = 0 ; \partial_t (z(S) \partial_y S) - \partial_y (z(S) \partial_t S) + \partial_y \sigma(w(S)) = 0$$

Let us define the change of variable $w(S) \partial_y S = 1$, that is S defines the increasing rearrangement of w . We obtain

$$(5.10) \quad -w(S) \partial_t S + z(S) = 0 ; \partial_t (z(S) \partial_y S) - \partial_y (z(S) \partial_t S) + \partial_y (\sigma(w(S))) = 0$$

and we obtain, denoting $\partial_t S = \partial_y V = \frac{z(S)}{w(S)}$,

$$(5.11) \quad \partial_t S = \partial_y V \text{ and } \partial_t V = \frac{(\partial_t S)^2}{\partial_y S} - \sigma\left(\frac{1}{\partial_y S}\right)$$

This is a second order equation $\partial_t^2 S = \partial_y \left(\frac{(\partial_t S)^2}{\partial_y S} - \sigma\left(\frac{1}{\partial_y S}\right) \right)$.

We summarize in the following lemma the main result of this section

Lemma 5.1. *Let consider a solution (S, V) of (5.11) and consider*

$$(5.12) \quad u(S) := (w(S), z(S)) := \left(\frac{1}{\partial_y S}, \frac{\partial_y V}{\partial_y S} \right)$$

Then $u(S)$ is a solution of the p system (1.4). Moreover, if (S, V) verifies weakly

$$(5.13) \quad \partial_t (E(u(S)) \partial_y S) - \partial_y F(u(S)) \leq 0$$

Then $u(S)$ is the entropy solution (1.5) of the p system and S verifies

$$(5.14) \quad S \in \mathcal{C}^2([0, +\infty[, L^2([0, 1])).$$

Proof. We already proved that (5.11) is equivalent to the p system. Let us show that S in Entropy conservative. To that aim, let us consider the weak form of (5.13). This relation means that, for any "smooth" φ

$$(5.15) \quad \frac{d}{dt} \int_{[0,1]} \partial_y S E(u(S)) \varphi(S) + \int_{[0,1]} F(u(S)) \partial_y \varphi(S) = 0$$

Using the transport map $x = S(t, y)$, after expanding $\partial_y \varphi(S) = \partial_y S (\partial_x \varphi)(S)$, this relation is equivalent to

$$(5.16) \quad \frac{d}{dt} \int_{\Omega} E(u) \varphi + \int_{[0,1]} F(u) \partial_x \varphi = 0$$

that is the weak formulation of (1.5). Let us show that $S \in \mathcal{C}^2([0, +\infty[, L^2([0, 1]))$

$$(5.17) \quad \frac{d^2}{dt^2} \int_{[0,1]} \frac{S^2}{2} = \int_{[0,1]} (\partial_t S)^2 + \int_{[0,1]} S \partial_y \left(\frac{(\partial_t S)^2}{\partial_y S} - \sigma\left(\frac{1}{\partial_y S}\right) \right)$$

These computations provides a bound over the L^2 norm of S using the estimation (5.4) :

$$(5.18) \quad \frac{d^2}{dt^2} \int_{[0,1]} \frac{S^2}{2} = \int_{[0,1]} \sigma\left(\frac{1}{\partial_y S}\right) \partial_y S = \int_{[0,1]} \frac{\sigma(w(S))}{w(S)} = \int_{\Omega} \sigma(w) \leq \int_{\Omega} E(u_0)$$

□

5.2. Optimally Transported semi discrete Scheme. We use the same approach as we did for the Burger equation to design our scheme : we design schemes for which the equiprobable quantizers are described by particles trajectories

$$(5.19) \quad \{S_i := S_i(t) \sim N \int_{[\frac{i-1}{N}, \frac{i}{N}]} S\}_{i=1, \dots, N}$$

Let us consider

$$(5.20) \quad \{u_i = (w_i, z_i) = \left(\frac{1}{\delta_y S_i}, \frac{d}{dt} S_i \right)\}_{i=1, \dots, N}$$

As usual, we are first looking for a scheme verifying a discrete version of the Entropy **conservative** condition (5.13). The Entropy functional to be conserved is

$$(5.21) \quad E(u) = E(w, z) = \frac{e(w)}{w} + \frac{(z)^2}{2w}$$

Note that this functional defines the following entropy variables

$$(5.22) \quad v(u) = (v^1, v^2)(u) = \nabla E(u) = \left(-\frac{e(w)}{w^2} + \frac{\sigma(w)}{w} - \frac{(z)^2}{2w^2}, \frac{z}{w} \right)$$

Thus viscosity of shape

$$(5.23) \quad \delta_{y\bar{y}} v(u_i) = \delta_{y\bar{y}} \left(-\frac{e(w_i)}{w_i^2} + \frac{\sigma(w_i)}{w_i} - \frac{(z_i)^2}{2w_i^2}, \frac{z_i}{w_i} \right)$$

should be used to define entropy decreasing schemes for numerical applications. We denote as usual $h := \frac{1}{N}$. We state our result :

Proposition 5.2. *Let consider N ordered particles $\{S_i(t)\}_{i=1, \dots, N}$ satisfying the following semi discrete scheme*

$$(5.24) \quad \frac{d}{dt} S_i = \delta_{y\bar{y}} V_i \text{ and } \frac{d}{dt} V_i = \frac{\left(\frac{d}{dt} S_i\right)^2}{\delta_y S_i} - \sigma\left(\frac{1}{\delta_y S_i}\right)$$

Then :

- This semi discrete scheme computes the equiprobable quantizer of the density $w(t, x)$ of the weak solution of the Entropy **Conservative** p system at second order accuracy.
- The trajectories are globally defined : $t \rightarrow \mathbf{S}(t) \in \mathcal{C}^1([0, +\infty[, \ell^2)$ with uniform bound $\|\mathbf{S}(t)\|_{\ell^2}^2 = \|\mathbf{S}(0)\|_{\ell^2}^2 + \frac{t^2}{2} E$.

We do not know any reference for such a scheme. As for the heat equation, second accuracy holds using standard finite difference arguments.

As for the heat equation, we state some other interesting properties of this scheme, that will be also used also during the full discrete in time modeling.

Lemma 5.3. *Let $\mathcal{S}(t) = \{S_i\}_i$ a set of N trajectories verifying (5.24). Then it enjoys the following set of property that coincide with the continuous ones (Mass conservation holds by construction) :*

$$(5.25) \quad \frac{d^2}{dt^2} \|\mathcal{S}(t)\|_{\ell_N^2}^2 \leq E(u_0)$$

Here too, we think that the fact that the set of conservation laws of the continuous solution are transported at a discrete level indicates that this scheme is accurate.

Proof. Entropy conservation. We drop the index i for readability. From (5.21) we compute

$$(5.26) \quad \frac{d}{dt} E(u) = v^1(u) \frac{d}{dt} w + v^2(u) \frac{d}{dt} z$$

We compute also, recalling that $w = \frac{1}{\delta_y S}$ and $z = w \frac{d}{dt} S = w \delta_{\bar{y}} V$

$$(5.27) \quad \frac{d}{dt} w = -w^2 \delta_y \frac{d}{dt} S \text{ and } \frac{d}{dt} z = w \delta_{\bar{y}} \frac{d}{dt} V - w^2 \frac{d}{dt} S \delta_y \frac{d}{dt} S$$

Note that $\frac{d}{dt} V = w \left(\frac{d}{dt} S \right)^2 - \sigma(w)$.

Second Moment evolution. We compute, using discrete integration by parts

$$\begin{aligned} \frac{1}{2} \frac{d^2}{dt^2} \|\mathcal{S}(t)\|_{\ell_N^2}^2 &= \frac{1}{N} \sum_i \left(\frac{d}{dt} S_i \right)^2 + \frac{1}{N} \sum_i S_i \delta_{\bar{y}} \left(\frac{\left(\frac{d}{dt} S_i \right)^2}{\delta_y S_i} - \sigma\left(\frac{1}{\delta_y S_i} \right) \right) \\ &= \frac{1}{N} \sum_i \delta_y S_i \left(\sigma\left(\frac{1}{\delta_y S_i} \right) \right) \\ &\leq E(u_0) \end{aligned}$$

Ending the proof. □

To end the proof of the main Proposition, we use the same guidelines as the heat equation : local in time existence of a solution holds. Global existence follows using the Second Moment evolution.

5.3. Fully Discrete Analysis. We use the same notations than section 3.3.

Proposition 5.4. *Let $\mathcal{S}^n = \{S_i^n\}_{i,n \geq 0}$ be a solution of the following scheme*

$$(5.28) \quad \delta_t S_i^n = \delta_{\bar{y}} V_i^{n+1/2} \text{ and } \delta_t V_i^n = \frac{\left(\delta_{\bar{y}} V_i^{n+1/2} \right)^2}{\delta_y S_i^{n+1/2}} - \sigma\left(\frac{1}{\delta_y S_i^{n+1/2}} \right)$$

with the notations $S_i^{n+1/2} = \frac{S_i^n + S_i^{n+1}}{2}$. Then

- This scheme verifies the discrete Second Moment evolution

$$(5.29) \quad \delta_{t\bar{t}} \|\mathcal{S}^n\|_{\ell_N^2}^2 = E$$

- This scheme is a fully discrete second order scheme towards the equiprobable quantizer of the transport map of the entropy conservative burger equation.

This scheme is solved by an iterative procedure, exactly as for the heat equation. To have an Entropy decreasing scheme, one have to add viscosity terms in the entropy variables of (5.21).

Remark 5.5. *Here too, fully discrete schemes of arbitrary order can be designed using the framework in [12], relying on [23].*

Proof. Second order accuracy holds using standard finite difference schemes arguments. Let us concentrate to the proof of the Second Moment evolution.

The following discrete finite difference derivation rules holds :

$$\delta_t (u^n v^n) = u^{n+1/2} \delta_t v^n + v^{n+1/2} \delta_t u^n$$

We deduce

$$\delta_{t\bar{t}} \frac{(S_i^n)^2}{2} = \delta_{\bar{t}} \left(S_i^{n+1/2} \delta_t S_i^n \right) = S_i^{\pm} \delta_{t\bar{t}} S_i^n + \delta_{\bar{t}} S_i^{n+1/2} (\delta_t S_i^n)^{n-1/2}$$

We check $\delta_{\bar{t}} S_i^{n+1/2} = \delta_t S_i^{n-1/2}$. Using the fully discrete scheme 5.28, we have thus

$$\delta_{t\bar{t}} \|\mathcal{S}^n\|_{\ell_N^2}^2 = \|\mathbf{V}^n + \mathbf{V}^{n-}\|_{\ell_N^1} - \frac{E}{N} \sum_i S_i^{\pm} \delta_{\bar{y}} \left(\frac{1}{\delta_y S_i^{\pm}} (1 - V_i^{n+} V_i^{n-}) \right)$$

The proof follows now the same guidelines that the semi discrete case. □

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