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L^∞-STABILITY OF CONTINUOUS SHOCK WAVES IN A RADIATING GAS MODEL*

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Abstract. In the present article, we study the asymptotic stability in L^∞-topology of shock waves in a model system of radiating gases. It is known that the system admits discontinuous shock waves if the shock strength is strictly above a threshold value of $\sqrt{2}$, while if it is below (subcritical case) or equal to (critical case) $\sqrt{2}$, shock waves are continuous [Kawashima and Nishibata, *SIAM J. Math. Anal.*, 30 (1998), 95–117]. We prove that all subcritical shock waves are stable to piecewise smooth perturbations of small amplitude. The stability of subcritical shock waves is robust in the sense that it is not affected by possible collisions of discontinuities contained in initial data and the solutions converge to shock waves beyond such events. Sufficient conditions for occurrence and nonoccurrence of collision of discontinuities are both given as well. In the meantime, the critical shock wave blows up the first order derivative if certain types of perturbations are added however small the perturbations may be. Some conditional stability results are also addressed which are applicable for both subcritical and critical cases. The results imply an optimality of a blowup criterion given by Kawashima and Nishibata [*Math. Models. Methods Appl. Sci.*, 9 (1999), pp. 69–91].

Key words. hyperbolic-elliptic system, shock wave, blowup, collision of discontinuities

AMS subject classifications. 35B30, 35B40, 76N15

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1. Introduction. The present paper deals with an initial value problem to a simplified model system of radiating gases proposed by Hamer [7]:

$$u_t + uu_x + q_x = 0, \tag{1.1a}$$
$$-q_{xx} + q + u_x = 0, \tag{1.1b}$$

where $u(t,x)$ and $q(t,x)$ are real-valued functions for $t \geq 0$ and $x \in \mathbb{R}$ with

$$u(0,x) = u^0(x), \tag{1.2a}$$
$$u^0(x) \to u_\pm,\ q(t,x) \to 0 \quad \text{as} \quad x \to \pm\infty. \tag{1.2b}$$

Notation. For a constant $p \in [1,\infty]$, $|f|_p$ denotes the L^p norm of a function f. For nonnegative integer n and a domain Ω in $\mathbb{R}$, we denote by $B^n(\Omega)$ a subspace of $C^n(\Omega)$ with derivatives being bounded up to nth order. Equipped with the norm of $\|f\|_{B^n(\Omega)} := \sum_{k=0}^n \sup_{x\in\Omega} |f^{(k)}(x)|$, $B^n(\Omega)$ is a Banach space. Functions in $B^n(\mathbb{R})$ do not have discontinuities. If Ω is not $\mathbb{R}$ and $n \geq 1$, functions in $B^n(\Omega)$ are called piecewise smooth functions. For a function f which is continuous except for discontinuities of the first kind, $[\![f]\!](x)$ denotes the jump amplitude in the spatial direction of a function f at x, i.e.,

$$[\![f]\!](x) := f(x-0) - f(x+0)$$

and $\overline{f}(x)$ denotes the mean of the left and the right limit of a function f at x, i.e.,

$$\overline{f}(x) := (f(x-0) + f(x+0))/2.$$

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In the case when f depends also on t, $[\![f]\!]$ and $\overline{f}$ are defined similarly. Finally, c and C denote generic positive constants.

We are particularly concerned with the stability of shock waves or traveling wave solutions to the Hamer model, which are expressed in the form of

$$(u,q)(t,x) = (U,Q)(\eta), \quad \eta = x - st$$

for a certain constant s. Substituting this into (1.1) and (1.2b), we obtain

$$-sU' + UU' + Q' = 0, \tag{1.3a}$$

$$-Q'' + Q + U' = 0, \tag{1.3b}$$

$$U(x) \to u_\pm, \; Q(x) \to 0 \quad \text{as} \;\; x \to \pm\infty. \tag{1.3c}$$

Conditions for the existence of solutions to (1.3) and their properties were obtained in [8] and are summarized as follows.

Proposition 1.1. *Suppose the "shock strength" satisfies*

$$\delta_S := u_- - u_+ \in (0, \sqrt{2}].$$

Then there exists a solution $(U,Q) \in B^1(\mathbb{R}) \times B^2(\mathbb{R})$ *to* (1.3) *satisfying*

$$U(\eta) \to u_\pm, \quad Q(\eta) \to 0 \quad \textit{as} \;\; \eta \to \pm\infty \quad \textit{and} \;\; s = (u_- + u_+)/2.$$

The solution is unique up to a shift and satisfies

$$|U(\eta) - u_\pm| \le \frac{1}{2}\delta_S e^{-c|\eta|}, \quad 0 > U'(\eta) \ge U'(0) = \left(-1 + \sqrt{1 - \delta_S^2/2}\right)/2 \tag{1.4}$$

for an arbitrary $\eta \in \mathbb{R}$*, where* c *is a positive constant depending only on* δ_S*. If the shift is chosen so that* $U(0) = (u_- + u_+)/2$ *holds, then*

$$U(\eta) + U(-\eta) = u_- + u_+ \;\; \textit{and} \;\; Q(\eta) = Q(-\eta).$$

U *is infinitely differentiable except at* $\eta = 0$ *and in particular*

$$\operatorname{sgn}(\eta)\, U''(\eta) > 0 \; \textit{for} \; \eta \neq 0. \tag{1.5}$$

Moreover, if

$$\delta_S < 2\sqrt{2n}/(n+1)$$

holds for a certain integer $n(\ge 2)$*, then* $(U,Q) \in B^n(\mathbb{R}) \times B^{n+1}(\mathbb{R})$.

Remark 1.2. In the case $\delta_S > \sqrt{2}$, by introducing the notion of an admissible traveling wave solution as in [8], there exists a solution to (1.3) which is discontinuous at one point. Setting the point of discontinuity at $x = 0$, $U'(0-0) = U'(0+0) = -1$ holds. An abrupt change in the profile across $\delta_S = \sqrt{2}$ is evident comparing with (1.4).

The mathematical study of this system was initiated by Schochet and Tadmor [21], inspired by Rosenau's regularization of the Chapman–Enskog expansion for hydrodynamics [20]. They gave sufficient conditions for the existence of small and smooth shock waves. Kawashima and Nishibata [8] posed thorough criteria on the regularity of shock waves as in Proposition 1.1. Subsequently, studies pursuing conditions for the existence of shock waves have been extended to small and smooth shocks for a complete set of equations for radiating gases [14], to possibly discontinuous shocks for a generalized system [13], and recently to discontinuous shocks for the full radiating gas system [5, 16].

Regarding the stability of shock waves, L^1-stability by Serre [22, 23] is known. It is powerful enough to be applicable to shock waves with arbitrary strength and arbitrary large perturbations in L^1. On the other hand, the pointwise behavior of solutions is also a major concern in view of weaker diffusivity of the system than usual diffusion terms represented by Laplacian. In fact, Kawashima and Nishibata [9] for (1.1) and later Liu and Tadmor [15] for a general equation provided sufficient conditions for occurrence and nonoccurrence of finite time blowups of the first order derivatives by the method of characteristic curves. A uniform convergence toward shock waves with $\delta_S < \sqrt{6}/2$ was shown using L^2 energy method in [8], with initial perturbations small in $L^1 \cap H^2$. See [6] and the references therein for many studies in this direction. For more general scalar equations and for systems, detailed results are obtained using the Evans function techniques in [12, 17], at the price of stronger assumptions on the shock strength and the regularity of perturbations at least for the Hamer model.

The first objective of the present paper is to show that all subcritical shock waves ($\delta_S < \sqrt{2}$) are stable to piecewise B^1 as well as $L^1 \cap L^2$ perturbations with small amplitude, while the critical shock wave ($\delta_S = \sqrt{2}$) blows up the first order derivative in a finite time if a certain type of perturbation in the same space is added however small they may be. In this sense, this work relates the emergence of discontinuous shock waves for $\delta_S > \sqrt{2}$ elucidated by phase plane analysis in [8] to the difference in the behaviors between subcritical and critical shock waves. The methodology basically follows [8, 9].

We also aim to observe the behavior of discontinuities when there are more than two. Nishibata [18] proved that traveling waves with $\delta_S \le 1/2$ are stable to small perturbations having discontinuity at one point. He also showed that the location of the discontinuity converges to the center of mass of the initial data and the amplitude of the discontinuity diminishes exponentially fast in time. These results were extended to shock waves with $\delta_S < \sqrt{6}/2$ by the author [19]. So the natural question arises whether discontinuities, if there are several, keep away each other or collide, or whether blowups happen before that. If ever discontinuities collide, the possibility to extend the solution beyond that is also a concern. Although shock formation is known to occur in several parabolic or weakly dissipative systems [1, 2, 3, 24], behaviors or interactions of multiple discontinuities are not well studied unlike the situations in purely hyperbolic systems [4]. Investigation into behaviors of discontinuities will be beneficial for the future study of supercritical shock waves. Other than these purely mathematical motivations, studying evolutions of fluids driven by convection and diffusion with multiple discontinuities is physically important as well.

In order to observe the pointwise behavior of solutions, data under consideration are smooth functions except at a finite number of discontinuities. We denote by K a fundamental solution to $-\partial_x^2 + 1$, that is,

$$K(x) := \frac{1}{2} e^{-|x|}. \tag{1.6}$$

When $u(t, \cdot)$ is a piecewise B^1 function with discontinuities located at $d_j(t)(j = 1, \ldots, J)$, q is explicitly written down as

$$q(t,x) = -K * u_x(t,x) + \frac{1}{2} \sum_{j=1}^{J} [\![u]\!](t, d_j(t)) \exp\left(-|x - d_j(t)|\right) = -K' * u(t,x), \tag{1.7}$$

where the second argument of u_x and u to compute the convolution $K * u_x$ and $K' * u$ takes values in $\mathbb{R} \setminus \{d_j(t) | j = 1, \ldots, J\}$. Noting the definition of K in (1.6), the derivative of q in the sense of distribution is

$$q_x = u - K * u. \tag{1.8}$$

Substitution of (1.8) in (1.1a) rewrites the system (1.1) into a single equation of

$$u_t + uu_x + u - K * u = 0. \tag{1.9}$$

At $x = d(t)$, a point of discontinuity in u at time t, the Rankine–Hugoniot conditions

$$[\![q]\!] = 0, \tag{1.10}$$

$$[\![u - q_x]\!] = 0, \tag{1.11}$$

$$\dot{d}(t) = \frac{1}{2}\left(u(t, d(t) - 0) + u(t, d(t) + 0)\right) = \overline{u}(t, d(t)) \tag{1.12}$$

hold, where $\dot{d}(t)$ is a derivative of $d(t)$.

To treat data with discontinuities, we interpret the solutions as an admissible solution in the sense of Kružkov in [10].

Definition 1.3 (see [9, 21]). *We define an admissible solution $(u, q)(t, x)$ to (1.1) and (1.2) in the weak sense by a set of functions $(u, q) \in L^\infty([0, T) \times \mathbb{R})$ which satisfies*

$$\int_0^T \int_{\mathbb{R}} \left\{ |u - k| f_t + \operatorname{sign}(u - k)\left(\frac{1}{2}u^2 - \frac{1}{2}k^2\right) f_x - \operatorname{sign}(u - k)(u - K * u) f \right\} dxdt \geq 0$$

for an arbitrary nonnegative function $f \in C_0^\infty((0, T) \times \mathbb{R})$ and an arbitrary constant $k \in \mathbb{R}$,

$$\int_{\mathbb{R}} (-q g_{xx} + q g - u g_x)\, dx = 0$$

for an arbitrary $g \in \mathcal{S}(\mathbb{R})$, and the initial condition

$$u(0, x) = u^0(x) \ \ a.e.\ x \in \mathbb{R}.$$

It is well known that a pair of piecewise smooth functions (u, q) provides an admissible solution if it satisfies (1.1) almost everywhere and the Rankine–Hugoniot conditions almost everywhere along each discontinuity curve.

To state our main results precisely, we define some quantities and functions.

Suppose u^0 is a piecewise B^1 function with a finite number of discontinuities at $\{d_j^0 \in \mathbb{R} | j = 1, \dots, J\}$ with $d_1^0 < \cdots < d_J^0$:

$$u^0 \in B^1(\Omega_0), \quad \text{where} \quad \Omega_0 := \mathbb{R} \setminus \cup_{j=1}^J \{d_j^0\}. \tag{1.13}$$

For later convenience, for $t \geq 0$, let $\Omega_t \subset \mathbb{R}$ be a complement of a set of locations of discontinuities of $u(t, \cdot)$. At each discontinuity d_j^0 $(j = 1, \dots, J)$, we assume the entropy condition

$$[\![u^0]\!](d_j^0) = u^0(d_j^0 - 0) - u^0(d_j^0 + 0) > 0. \tag{1.14}$$

If u^0 is continuous, (1.13) should be understood as $u^0 \in B^1(\mathbb{R})$ and (1.14) is not imposed. If a discontinuity satisfies $[\![u^0]\!](d_j^0) < 0$, it will immediately vanish for $t > 0$

emanating rarefaction waves. Since our interest is in the study of the behavior of discontinuities, such as exponential decay of jump amplitude, collisions of them, and survival of the solution beyond collisions, we treat only the case of $[\![u^0]\!](d_j^0) > 0$.

When

$$u^0 - u_S \in L^1, \text{ where } u_S(x) := u_\pm \ (\pm x > 0) \tag{1.15}$$

is satisfied, we define the "center of mass" of u^0 in the following way. Setting the shift of the traveling wave solution so that $U(0) = (u_- + u_+)/2$ holds, the center of mass x_0 of the initial data u^0 is given by

$$x_0 := \frac{1}{u_- - u_+} \int_{-\infty}^{\infty} (u^0(x) - U(x))\, dx,$$

which is equivalent to

$$\int_{-\infty}^{\infty} (u^0(x) - U(x - x_0))\, dx = 0.$$

Letting $s_0 = (u_- + u_+)/2$, we change variables as

$$\hat{u}(t, \hat{x}) = u(t, \hat{x} + x_0 + s_0 t) - s_0, \quad \hat{q}(t, \hat{x}) = q(t, \hat{x} + x_0 + s_0 t),$$

so that we may assume $s = 0$, i.e., $u_- + u_+ = 0$ without loss of generality. In the new coordinate, the center of mass of the initial data is located at the origin and the traveling wave does not translate. We denote new variables $\hat{u}, \hat{q}, \hat{x}$ simply by u, q, x, respectively. Hereafter we fix U such that $U(0) = 0$ holds.

The initial perturbation is

$$\phi^0(x) := u^0(x) - U(x),$$

and we define its potential, or, in other words, antiderivative, by

$$\Phi^0(x) := \int_{-\infty}^{x} \phi^0(y)\, dy.$$

$\Phi^0(x)$ is well-defined by (1.4) and (1.15). Note

$$\int_{-\infty}^{\infty} \phi^0(x)\, dx = 0$$

holds by its definition. We define two constants a_1 and b_1 by

$$a_1 := \min\left\{ \inf_{x\in\Omega_0} u_x^0(x), \frac{-1/2 + U'(0)}{2} \right\}, \quad b_1 := \sup_{x\in\Omega_0} u_x^0(x).$$

See (1.4) for $U'(0)$. Now we are ready to state our main results.

THEOREM 1.4. *In the case* $\delta_S \in (0, \sqrt{2})$, *assume* (1.13)–(1.15),

$$\Phi^0, \ \phi^0 \in L^2(\mathbb{R}), \tag{1.16}$$

and

$$\inf_{x\in\Omega_0} u_x^0(x) > \left(-1 - \sqrt{1 - \delta_S^2/2}\right)/2. \tag{1.17}$$

If $|\Phi^0|_2 + |\phi^0|_2$ *is sufficiently small, the initial value problem* (1.1)–(1.2) *has a unique piecewise* B^1 *solution* (u, q) *globally in time. The solution satisfies maximal principle type estimates:*

$$\inf_{x\in\Omega_0} u^0(x) \le u(t,x) \le \sup_{x\in\Omega_0} u^0(x),$$
$$-1 < a_1 \le u_x(t,x) \le b_1$$

for an arbitrary $t \ge 0$ *and* $x \in \Omega_t$. *The solution converges uniformly to the shock wave:*

$$\sup_{x\in\Omega_t} |u(t,x) - U(x),\ q(t,x) - Q(x)| \to 0 \quad \text{as} \quad t \to \infty. \tag{1.18}$$

When u^0 *is discontinuous, every discontinuity diminishes and its position converges to the center of mass:*

$$[\![u]\!](t, d_j(t)) \to 0, \quad d_j(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{1.19}$$

Remark 1.5. If there are more than two discontinuities, several of them actually collide in some cases, as shown in the next theorem. This theorem holds regardless of such events.

THEOREM 1.6. *Under the same conditions as in Theorem* 1.4, *consider the case* $J \ge 2$:

(i) *If additionally* $a_1 > -1/2$ *and*

$$\max_{1\le j\le J-1} \frac{[\![u^0]\!](d_j^0) + [\![u^0]\!](d_{j+1}^0)}{2(d_{j+1}^0 - d_j^0)} \le 1 + 2a_1 \tag{1.20}$$

are satisfied, then

$$d_{j+1}(t) - d_j(t) \ge ce^{-(1+a_1)t} \quad j = 1, \ldots, J-1$$

holds for an arbitrary t.

(ii) *If additionally*

$$\max_{1\le j\le J-1} \frac{[\![u^0]\!](d_j^0) + [\![u^0]\!](d_{j+1}^0)}{2(d_{j+1}^0 - d_j^0)} > 1 + 2b_1 \tag{1.21}$$

is satisfied, there exists a finite value $T_*(> 0)$ *such that*

$$\min_{1\le j\le J-1} (d_{j+1}(t) - d_j(t)) \to 0 \quad \text{as } t \to T_* - 0.$$

In contrast to the theorems above, the critical shock wave, which is still continuous, blows up the first order derivative in a finite time if certain types of perturbations are added, however small they may be.

THEOREM 1.7. *Let* (U, Q) *be a traveling wave solution in the case* $\delta_S = \sqrt{2}$. *For an arbitrary* $\phi^0(\not\equiv 0) \in B^1(\mathbb{R})$ *which satisfies*

$$\phi^0(-x) = -\phi^0(x) \quad \text{and} \quad \phi^0(x) \le 0 \quad \text{for} \quad x \ge 0,$$

the solution to (1.1) *and* (1.2) *with*

$$u^0(x) = U(x) + \phi^0(x)$$

blows up in a finite time, i.e.,

$$\inf_{x\in\mathbb{R}} u_x(t,x) \to -\infty \quad as \quad t \to T_* - 0$$

for a certain finite value $T_*(>0)$.

The next theorem presents another kind of blowup set. It shows that passing δ_S formally to $\sqrt{2}$ in Theorem 1.4 is not valid.

THEOREM 1.8. *Consider the case* $\delta_S = \sqrt{2}$. *For an arbitrary positive constant* ε, *there exists an initial data* $u^0 \in B^1(\mathbb{R})$ *satisfying* (1.15), (1.16), $|\Phi^0|_2 + |\phi^0|_2 < \varepsilon$, *and*

$$\inf_{x\in\mathbb{R}} u^0_x(x) > -1/2$$

such that the solution to (1.1) *and* (1.2) *blows up the first order derivative in a finite time.*

One might expect from these two theorems that in the critical case, $\inf_{x\in\mathbb{R}} u^0_x(x) < -1/2$ would surely lead to a blowup of u_x in a finite time. It is, however, denied in the next theorem. Note that neither Theorem 1.7 nor 1.8 excludes the possibility of L^∞-stability of the critical shock wave, as inferred from [11] or [22].

THEOREM 1.9. *Let* (U, Q) *be a traveling wave solution in the case* $\delta_S \in (0, \sqrt{2}]$. *For an arbitrary* $\varepsilon \in (0, 1/2)$, *there exists an initial data* $u^0 \in B^1(\mathbb{R})$ *with*

$$\inf_{x\in\mathbb{R}} u^0_x(x) = -1 + \varepsilon$$

such that (1.1) *and* (1.2) *have a global solution which converges uniformly to* (U, Q) *as in* (1.18).

Remark 1.10. The initial data $u^0 \in B^1(\mathbb{R})$ with $\inf_{x\in\mathbb{R}} u^0_x(x) \geq 0$ always has a global solution as shown in Theorem 5 of [9]. For the case with $\inf_{x\in\mathbb{R}} u^0_x(x) \in [-1/2, 0)$, traveling waves themselves are the desired initial data.

Remark 1.11. Recall Theorem 3 in [9], which claims, if restricted to nonincreasing initial data $u^0 \in B^1(\mathbb{R})$, that $\inf_{x\in\mathbb{R}} u^0_x < -1$ inevitably leads to a finite time blowup of the first order derivative. This theorem gives the optimality of their blowup criterion in terms of $\inf_{x\in\mathbb{R}} u^0_x(x)$ for nonincreasing smooth initial data, except for the marginal case of $\inf_{x\in\mathbb{R}} u^0_x(x) = -1$.

Outline of the paper. In section 2, we prove the existence of a local solution uniquely in the piecewise B^1 space and see the factors to determine the existence time. We also show that the perturbation from the traveling wave is in L^1 and L^2 if the initial data is so. In section 3, assuming the smallness of the initial perturbation in a suitable norm, we give a bound to $|(\Phi, \phi)|_2$ and a lower bound to u_x both uniformly in time. Then we show that the solution can be extended beyond collisions of discontinuities, completing the proof of Theorem 1.4. Theorem 1.6 is also proved. The first part of section 4 is devoted to the proofs of Theorems 1.7 and 1.8. Finally, Theorem 1.9 is proved in the second part of that section based on another conditional stability result of Theorem 4.1 which corresponds to Theorem 1.7.

2. Local solvability.

We suppose $J \geq 2$ throughout this section since the problems with $J = 0$ or $J = 1$ are easier than the problem with multiple discontinuities. In fact, the case with $J \leq 1$ is handled simply by skipping treatments concerning multiple discontinuities in the following arguments. Taking account of quasilinearity, nonlocality of the governing equations, and the existence of moving discontinuities, we construct local solutions to (1.9) iteratively. The basic strategy is to construct a set of "spatially local solutions" around each discontinuity not to violate

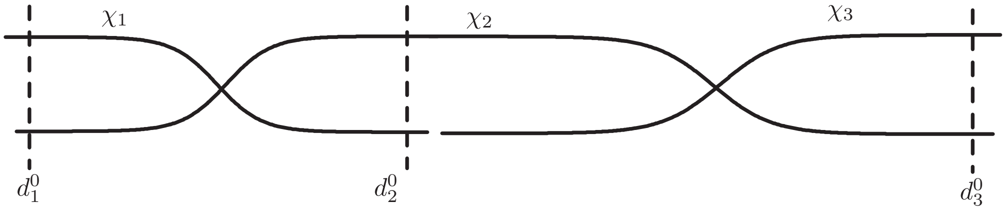

FIG. 1. *Partition of unity* $\{\chi_j\}_{j=1}^J$ *subject to* $\{d_j^0\}_{j=1}^J$.

the entropy condition, and patch them altogether. To be precise, let a set of functions $\chi_j(x) \in C^\infty(\mathbb{R}),\ j = 1, \dots, J$ be a partition of unity satisfying $\chi_j(x) \in [0,1]$, $\sum_{j=1}^J \chi_j(x) \equiv 1$ for an arbitrary $x \in \mathbb{R}$ and following properties:

$$\chi_1(x) := \begin{cases} 1, & x \le (2d_1^0 + d_2^0)/3, \\ 0, & x \ge (d_1^0 + 2d_2^0)/3, \end{cases}$$

$$\chi_j(x) := \begin{cases} 1, & x \in [(d_{j-1}^0 + 2d_j^0)/3, (2d_j^0 + d_{j+1}^0)/3] \\ 0, & x \notin [(2d_{j-1}^0 + d_j^0)/3, (d_j^0 + 2d_{j+1}^0)/3] \end{cases} \qquad \text{for } 1 < j < J,$$

$$\chi_J(x) := \begin{cases} 0, & x \le (2d_{J-1}^0 + d_J^0)/3, \\ 1, & x \ge (d_{J-1}^0 + 2d_J^0)/3. \end{cases}$$

Each function χ_j is localized around d_j^0 and does not contain any other discontinuities in its support (see Figure 1). Note that we can choose χ_j $(j = 1, \dots, J)$ such that

$$\max_{1 \le j \le J} |\chi_j'|_\infty \le C \left(\min_{1 \le j \le J-1} \{d_{j+1}^0 - d_j^0\} \right)^{-1} \tag{2.1}$$

holds. Define constants C_i $(i = 0, 1, 2)$ and T_i $(i = 0, 1, 2, 3)$ by

$$\begin{aligned} &C_0 := \|u^0\|_{B^0} + 1, \\ &C_1 := \max\left\{ \|u^0\|_{B^0} \sum_{j=1}^J |\chi_j'|_\infty + 1,\ \ \|u^0\|_{B^0}^2 \sum_{j=1}^J |\chi_j'|_\infty + 2\|u^0\|_{B^0} + 1 \right\}, \\ &C_2 := 2(J+1)(\|u^0\|_{B^1} + 1)^2, \end{aligned} \tag{2.2}$$

and

$$T_1 := \frac{1}{2C_1(C_2+2)}, \quad T_2 := \frac{1}{8C_2} \min_{1 \le j \le J} [\![u^0]\!](d_j^0), \quad T_3 := \frac{1}{72C_0} \min_{1 \le j \le J-1} \{d_{j+1}^0 - d_j^0\},$$

$$T_0 := \min\{T_1, T_2, T_3\}.$$

Let domains $D_1, \dots, D_J$ in $\mathbb{R}^2$ be defined by

$$D_1 := \left\{ (\tau, \xi) \,|\, \tau \in [0, T_0],\ \xi \in \left(-\infty, \frac{3}{4}(d_2^0 - d_1^0) - 2C_0\tau \right),\ \xi \ne 0 \right\},$$

$$D_j := \left\{ (\tau, \xi) \,|\, \tau \in [0, T_0],\ \xi \in \left(-\frac{3}{4}(d_j^0 - d_{j-1}^0) + 2C_0\tau, \right.\right.$$
$$\left.\left. \frac{3}{4}(d_{j+1}^0 - d_j^0) - 2C_0\tau \right),\ \xi \ne 0 \right\}$$

for $1 < j < J$ and

$$D_J := \left\{ (\tau, \xi) \,|\, \tau \in [0, T_0],\ \xi \in \left(-\frac{3}{4}(d_J^0 - d_{J-1}^0) + 2C_0\tau, +\infty \right),\ \xi \neq 0 \right\}.$$

In the first iteration, for each $j = 1, \dots, J$ we set

$$\tilde{u}_j^{(0)}(\tau, \xi) = u^0(\xi + d_j^0) \text{ for } (\tau, \xi) \in D_j. \tag{2.3}$$

For $n \geq 1$ and each $j = 1, \dots, J$, assume that we have a piecewise B^1 function $\tilde{u}_j^{(n-1)}$ over D_j which satisfies

$$[\![\tilde{u}_j^{(n-1)}]\!](\tau, 0) > 0 \text{ for an arbitrary } \tau \in [0, T_0] \tag{2.4}$$

and

$$\inf_{x \in \Omega_0} u^0(x) \leq \tilde{u}_j^{(n-1)}(\tau, \xi) \leq \sup_{x \in \Omega_0} u^0(x) \text{ for an arbitrary } (\tau, \xi) \in D_j. \tag{2.5}$$

Then set

$$d_j^{(n)}(\tau) := d_j^0 + \int_0^\tau \overline{\tilde{u}_j^{(n-1)}}(s, 0) ds \text{ for } \tau \in [0, T_0]. \tag{2.6}$$

Note that

$$d_j^{(n)}(\tau) \in \chi_j^{-1}(1)$$

for all j and $\tau \leq T_0$ by the definitions of χ_j and T_3 and hence discontinuities never collide with each other. For $(t, x) \in [0, T_0] \times \mathbb{R} \setminus \cup_{i=1}^J \{(\tau, d_i^{(n)}(\tau)) \,|\, \tau \in [0, T_0]\}$, set

$$u^{(n-1)}(t, x) := \sum_{i=1}^J \tilde{u}_i^{(n-1)}(t, x - d_i^{(n)}(t))\chi_i(x), \tag{2.7}$$

where $\tilde{u}_i^{(n-1)}$ is extended to be zero outside D_i. Here we see that the set of arguments in (2.7) satisfies

$$\{(\tau, x - d_i^{(n)}(\tau)) \mid \tau \in [0, T_0],\ x \in \text{supp}\chi_i\} \subset D_i \tag{2.8}$$

for all i. In fact, letting p_i the left edge of $\text{supp}\chi_i (i \geq 2)$, for example, the second argument of $\tilde{u}_i^{(n-1)}$ in (2.7) is bounded below for $\tau \leq T_0 (\leq T_3)$ as

$$p_i - d_i^{(n)}(\tau) \geq p_i - d_i^0 - C_0\tau \geq -\frac{2}{3}(d_i^0 - d_{i-1}^0) - C_0\tau \geq -\frac{3}{4}(d_i^0 - d_{i-1}^0) + 2C_0\tau$$

by the definitions of $d_i^{(n)}$, C_0, χ_i, and T_3, where the last term is the left edge of D_i at time τ. In this way, we have (2.8).

Let $\tilde{D}_j$ be the set of points contained in D_j through which a unique integral curve generated by the vector field $(1, u^{(n-1)}(\tau, \xi + d_j^{(n)}(\tau)) - \dot{d}_j^{(n)}(\tau))$ over $(\tau, \xi) \in D_j$ passes. Then we define $\tilde{u}_j^{(n)}(\tau, \xi)$ for $(\tau, \xi) \in \tilde{D}_j$ by a solution to a linear initial value problem:

$$\partial_\tau \tilde{u}_j^{(n)}(\tau, \xi) + \left(u^{(n-1)}(\tau, \xi + d_j^{(n)}(\tau)) - \dot{d}_j^{(n)}(\tau) \right) \partial_\xi \tilde{u}_j^{(n)}(\tau, \xi) + \tilde{u}_j^{(n)}(\tau, \xi) = \left(K * u^{(n-1)} \right)(\tau, \xi + d_j^{(n)}(\tau)), \tag{2.9a}$$

$$\tilde{u}_j^{(n)}(0, \xi) = u^0(\xi + d_j^0). \tag{2.9b}$$

Now we prove that $\tilde{D}_j = D_j$ and $\tilde{u}_j^{(n)}$ is well-defined over D_j for all j and n. Due to (2.5)–(2.7), we have

$$\left| u^{(n-1)}(\tau, \xi + d_j^{(n)}(\tau)) - \dot{d}_j^{(n)}(\tau) \right| < 2C_0. \tag{2.10}$$

From (2.4) and the restriction on T_0 by T_2, we see for $\tau \le T_0$ and for all j that

$$[\![\tilde{u}_j^{(n)}]\!](\tau, 0) > 0. \tag{2.11}$$

By (2.9) and $|K|_1 = 1$, we easily see that for all j and $(\tau, \xi) \in \tilde{D}_j$,

$$\min\left\{ \inf_{x\in\Omega_0} u^0(x), \min_{1\le j\le J} \inf_{(\tau,\xi)\in D_j} \tilde{u}_j^{(n-1)}(\tau,\xi) \right\} \le \tilde{u}_j^{(n)}(\tau,\xi)$$
$$\le \max\left\{ \sup_{x\in\Omega_0} u^0(x), \max_{1\le j\le J} \sup_{(\tau,\xi)\in D_j} \tilde{u}_j^{(n-1)}(\tau,\xi) \right\}.$$

Using (2.5), we obtain

$$\inf_{x\in\Omega_0} u^0(x) \le \tilde{u}_j^{(n)}(\tau,\xi) \le \sup_{x\in\Omega_0} u^0(x)$$

for all j and $(\tau,\xi) \in \tilde{D}_j$. Let $\tilde{T}_1 := 1/\{2C_1(C_2+1)\}(> T_1)$ and define a function $Z(\tau)$ for $\tau \in [0, \tilde{T}_1]$ by

$$\frac{d}{d\tau} Z(\tau) = C_1(Z(\tau)+1)^2, \quad Z(0) = C_2.$$

Note that $Z(\tau)$ is explicitly represented by $Z(\tau) = 1/\{(C_2+1)^{-1} - C_1\tau\} - 1$, where the blowup time of the right-hand side is $2\tilde{T}_1$. Following standard arguments as in section 3.8 of [4], we have uniform bounds for $\partial_\xi \tilde{u}_j^{(n)}$ and $\partial_\tau \tilde{u}_j^{(n)}$ as

$$\left|\partial_\xi \tilde{u}_j^{(n)}(\tau,\xi)\right|, \left|\partial_\tau \tilde{u}_j^{(n)}(\tau,\xi)\right| \le Z(\tau) \quad \text{for} \quad (\tau,\xi) \in \tilde{D}_j.$$

In particular, it holds that

$$\left|\partial_\xi \tilde{u}_j^{(n)}(\tau,\xi)\right|, \left|\partial_\tau \tilde{u}_j^{(n)}(\tau,\xi)\right| \le 2C_2 \quad \text{for} \quad (\tau,\xi) \in \tilde{D}_j \tag{2.12}$$

since $T_1 < \tilde{T}_1$ and $Z(\tilde{T}_1) \le 2C_2$. Estimates (2.10)–(2.12) imply $\tilde{D}_j = D_j$, which means $\tilde{u}_j^{(n)}(\tau,\xi)$ is well-defined over D_j for all n. Based on these, the unique existence of the local solution is stated in the following way.

Lemma 2.1. *Assuming* (1.13) *and* (1.14), *there exists a positive constant* T_0 *depending on* $\|u^0\|_{B^1}$, $\min_{1\le j\le J}\{[\![u^0]\!](d_j^0)\}$, *and* $\min_{1\le j\le J-1}\{d_{j+1}^0 - d_j^0\}$ *such that* $\tilde{u}_j^{(n)}(\tau,\xi)$ *is defined as a unique piecewise* B^1 *solution to* (2.6) *and* (2.9) *over* D_j *iteratively in* $n(\ge 1)$ *with the first step given by* (2.3). *Moreover,* $\{\tilde{u}_j^{(n)}(\tau,\xi)\}_{n=0}^\infty$ *converges uniformly in* $(\tau,\xi) \in D_j$ *to a certain piecewise* B^1 *function* $\tilde{u}_j(\tau,\xi)$, *while* $\{d_j^{(n)}(\tau)\}_{n=0}^\infty$ *converges uniformly in* $\tau \in [0, T_0]$ *to a certain* B^1 *function* $d_j(\tau)$. *Extending* $\tilde{u}_i$ *to be zero outside* D_i *and defining*

$$u(t,x) := \sum_{i=1}^J \tilde{u}_i(t, x - d_i(t))\chi_i(x)$$

for $(t,x) \in D_0 := [0,T_0]\times\mathbb{R}\setminus\cup_{j=1}^{J}\{(t,d_j(t)) \mid t\in[0,T_0]\}$, *u is a piecewise* B^1 *function having separate discontinuities at* $\{(t,d_j(t)) \mid t\in[0,T_0]\}$, $j=1,\dots,J$, *each of which obeys* (1.12) *and the entropy condition*

$$\llbracket u(t,d_j(t))\rrbracket > 0,\ \ t\in[0,T_0]. \tag{2.13}$$

Defining q by (1.7), (u,q) *is a unique admissible solution to* (1.1)–(1.2) *satisfying* (1.10) *and* (1.11), *and* u_x *is a broad solution (section* 3.4 *of* [4]*) to*

$$\partial_t u_x + u\partial_x u_x + u_x^2 + u_x - \partial_x(K*u) = 0. \tag{2.14}$$

The solution also satisfies

$$\inf_{x\in\Omega_0} u^0(x) \le u(t,x) \le \sup_{x\in\Omega_0} u^0(x), \tag{2.15}$$

$$u_x(t,x) \le \sup_{x\in\Omega_0} u_x^0(x). \tag{2.16}$$

Proof. The convergence of $\{\tilde{u}_j^{(n)}\}_{n=0}^{\infty}$, $\{d_j^{(n)}\}_{n=0}^{\infty}$ to $\tilde{u}_j$, d_j, respectively, and their B^1 regularity are shown in a standard way as in section 3.8 of [4]. Letting $n\to\infty$ in (2.9), we have

$$\begin{aligned}\partial_\tau\tilde{u}_j(\tau,\xi) + \Big(u(\tau,\xi+d_j(\tau)) - \dot{d}_j(\tau)\Big)\,\partial_\xi\tilde{u}_j(\tau,\xi) + \tilde{u}_j(\tau,\xi)&\\ = (K*u)\,(\tau,\xi+d_j(\tau))\ \ \text{for } (\tau,\xi)\in D_j&,\\ \tilde{u}_j(0,\xi) = u^0(\xi+d_j^0)\ \ \text{for } \xi \text{ with } (0,\xi)\in D_j&.\end{aligned}$$

Introducing new variables $u_j(t,x) := \tilde{u}_j(t,x-d_j(t))$ for $j=1,\dots,J$, this is rewritten as

$$\begin{aligned}\partial_t u_j(t,x) + u(t,x)\partial_x u_j(t,x) + u_j(t,x) = (K*u)(t,x)&\\ \text{for } (t,x)\in E_j := \{(t,x) | (t,x-d_j(t))\in D_j\}&,\end{aligned} \tag{2.17a}$$

$$u_j(0,x) = u^0(x)\ \ \text{for } x \text{ with } (0,x)\in E_j. \tag{2.17b}$$

Since $u_i(t,x) = u_j(t,x)$ for $(t,x)\in E_i\cap E_j$ is deduced from this, multiplying (2.17) by $\chi_j(x)$ and summing up for $j=1,\dots,J$ concludes that u satisfies

$$\begin{aligned}\partial_t u(t,x) + u(t,x)\partial_x u(t,x) + u(t,x) = (K*u)\,(t,x)\ \ \text{for } (t,x)\in D_0,&\\ u(0,x) = u^0(x)\ \ \text{for } x\in\Omega_0.&\end{aligned}$$

The uniqueness of the solution in L^∞ is assured in [11, 23]. Since the characteristic curves and the locations of discontinuities are already obtained and $\partial_x(K*u) = K'*u$ holds, it is easy to show (2.14). The definition of q by (1.7) concludes (1.10) and (1.11). The upper bound (2.16) for u_x is shown as in [9]. ∎

In order to evaluate temporal evolutions of integrals of certain time-dependent variables over domains which also change with time, we frequently use the so-called Reynolds transport theorem stated as follows. The proof is elementary for one-dimensional cases.

LEMMA 2.2. *Suppose $a(t)$ and $b(t)$ are C^1 functions defined over $t \in [t_0, t_1]$ which satisfy $a(t) < b(t)$. If f is a C^1 function defined over $\{(t,x)\,|\,t \in [t_0,t_1], x \in (a(t), b(t))\}$, it holds that*

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)dx = \int_{a(t)}^{b(t)} f_t(t,x)dx + f(b(t))\dot{b}(t) - f(a(t))\dot{a}(t).$$

We now define variables for the perturbation by

$$(\phi,\psi)(t,x) := (u,q)(t,x) - (U,Q)(x).$$

From (1.1) and (1.3), the system of governing equations for (ϕ,ψ) reads

$$\phi_t + (U+\phi)\phi_x + U'\phi + \psi_x = 0, \tag{2.18a}$$
$$-\psi_{xx} + \psi + \phi_x = 0 \tag{2.18b}$$

with the initial data given by

$$\phi(0,x) = \phi^0(x) = u^0(x) - U(x). \tag{2.19}$$

Corresponding to (1.7) and (1.8), it holds that

$$\psi = -K' * \phi, \tag{2.20a}$$
$$\psi_x = \phi - K * \phi. \tag{2.20b}$$

LEMMA 2.3. *Assuming* (1.13)–(1.15), *the perturbation $\phi(t,\cdot)$ obtained in Lemma* 2.1 *belongs to $L^1(\mathbb{R})$ and satisfies*

$$|\phi(t)|_1 \le |\phi^0|_1, \quad t \in [0, T_0]. \tag{2.21}$$

Moreover, the potential or the antiderivative of ϕ defined by

$$\varPhi(t,x) := \int_{-\infty}^{x} \phi(t,y)dy \tag{2.22}$$

satisfies

$$\varPhi_t + U\varPhi_x + \frac{1}{2}\varPhi_x^2 + \psi = 0. \tag{2.23}$$

Proof. The integrability of $\phi(t,\cdot)$ and the L^1-contraction (2.21) are proved by similar arguments to those in [8]. Reference [11] is also helpful. Here we concentrate on the proof of (2.23). Define intervals by

$$I_0(t) := (-\infty, d_1(t)),$$
$$I_j(t) := (d_j(t), d_{j+1}(t)) \ (j = 1, \dots, J-1),$$
$$I_J(t) := (d_J(t), \infty)$$

and suppose $x \in I_k(t)$ for certain $t \in [0,T]$ and $k \in \{0,1,\dots,J\}$. Applying Lemma 2.2

to (2.22), we obtain

$$\begin{aligned}
\partial_t \Phi(t,x) &= \partial_t \left(\sum_{j=0}^{k-1} \int_{I_j(t)} \phi(t,y)dy + \int_{d_k(t)}^{x} \phi(t,y)dy \right) \\
&= \sum_{j=0}^{k-1} \int_{I_j(t)} \phi_t(t,y)dy + \int_{d_k(t)}^{x} \phi_t(t,y)dy + \sum_{j=1}^{k} [\![\phi]\!](t,d_j(t))\dot{d}_j(t) \\
&= -\sum_{j=0}^{k-1} \int_{I_j(t)} \partial_y \left(U\phi + \frac{\phi^2}{2} + \psi \right) dy - \int_{d_k(t)}^{x} \partial_y \left(U\phi + \frac{\phi^2}{2} + \psi \right) dy \\
&\quad + \sum_{j=1}^{k} [\![\phi]\!](t,d_j(t))\dot{d}_j(t) \\
&= -\sum_{j=1}^{k} \left(U[\![\phi]\!] + \frac{1}{2}[\![\phi^2]\!] \right)(t,d_j(t)) - \left(U\phi + \frac{\phi^2}{2} + \psi \right)(t,x) \\
&\quad + \sum_{j=1}^{k} [\![\phi]\!] \left(U + \overline{\phi} \right)(t,d_j(t)) \\
&= -\left(U\phi + \frac{\phi^2}{2} + \psi \right)(t,x) = -\left(U\Phi_x + \frac{1}{2}\Phi_x^2 + \psi \right)(t,x).
\end{aligned}$$

Differentiation under the integral sign from the second to the third lines for $j = 0$ is justified by noting $\lim_{x\to-\infty}(\phi,\psi)(t,x) = (0,0)$ obtained with the help of (2.20a), where the convergence is locally uniform in t owing to the boundedness of ϕ_t as assured in Lemma 2.1. We use (2.18a), (1.10), (1.11), and (1.12) afterwards. ∎

LEMMA 2.4. *Assuming* (1.13)–(1.16), $\Phi(t,\cdot)$ *and* $\phi(t,\cdot)$ *obtained in Lemmata* 2.1 *and* 2.3 *belong to* $L^2(\mathbb{R})$ *and* $\psi(t,\cdot), \psi_x(t,\cdot) \in L^2(\mathbb{R})$ *at each time* $t \in [0,T_0]$.

Proof. Multiply (2.18a) by ϕ and (2.23) by Φ, respectively, to get

(2.24)
$$\partial_t \left(\frac{1}{2}\Phi^2 \right) + \partial_x \left\{ \frac{U}{2}\Phi^2 + (\Phi+\psi)(\psi_x - \phi) \right\} + \frac{-U'}{2}\Phi^2 + \left(1 + \frac{\Phi}{2} \right)\phi^2 - (\psi^2 + \psi_x^2) = 0,$$

$$\partial_t \left(\frac{1}{2}\phi^2 \right) + \partial_x \left\{ \frac{U}{2}\phi^2 + \frac{1}{3}\phi^3 - \psi(\psi_x - \phi) \right\} + \frac{U'}{2}\phi^2 + (\psi^2 + \psi_x^2) = 0. \tag{2.25}$$

Multiply (2.25) by two and add to (2.24) and integrate the result over $[0,t]\times[-M,M]$, where $t \in [0,T_0]$ is arbitrary and M is a sufficiently large positive number so that $d_1(t), d_J(t) \in (-M,M)$ holds for $t \in [0,T_0]$. Using Lemma 2.2, we have

(2.26)
$$\begin{aligned}
&\int_{-M}^{M} \left(\frac{1}{2}\Phi^2 + \phi^2 \right)(t,x)dx + \int_0^t \int_{-M}^{M} \frac{-U'(x)}{2}\Phi^2(s,x)dxds \\
&\quad + \int_0^t \int_{-M}^{M} \left(1 + U'(x) + \frac{\Phi(s,x)}{2} \right)\phi^2(s,x)dxds + \int_0^t \int_{-M}^{M} \left(\psi^2 + \psi_x^2 \right)(s,x)dxds \\
&= \int_{-M}^{M} \left(\frac{1}{2}\Phi^0(x)^2 + \phi^0(x)^2 \right)dx - \frac{1}{6}\sum_{j=1}^{J} \int_0^t [\![\phi(s,d_j(s))]\!]^3 ds \\
&\quad - \int_0^t \left[\frac{U}{2}\Phi^2 + (\Phi+\psi)(\psi_x - \phi) + 2\left\{ \frac{U}{2}\phi^2 + \frac{1}{3}\phi^3 - \psi(\psi_x - \phi) \right\} \right]_{-M}^{M}(s)ds.
\end{aligned}$$

By (2.21) and the definition of Φ in (2.22), it holds that

$$|\Phi|_\infty(t) \le |\phi_0|_1 \tag{2.27}$$

for $t \in [0, T_0]$. The maximal principle (2.15) yields

$$|\phi|_\infty(t) \le |u|_\infty(t) + |U|_\infty \le |u^0|_\infty + |U|_\infty. \tag{2.28}$$

Owing to (2.20), (2.27), and (2.28), the integrand of the last term in (2.26) is uniformly bounded by a certain value determined by the initial data. By the entropy condition (2.13), we have from (2.26) that

$$\int_{-M}^{M} \left(\Phi^2 + \phi^2\right)(t,x)dx \le C\int_{\mathbb{R}} \left(\Phi^0(x)^2 + \phi^0(x)^2\right)dx + C\int_0^t \int_{-M}^{M} \phi^2(s,x)dxds + Ct,$$

where a positive constant C does not depend on M nor on t. Applying the Gronwall inequality to this and letting $M \to \infty$, we have $\Phi(t,\cdot), \phi(t,\cdot) \in L^2(\mathbb{R})$ and hence $\psi(t,\cdot), \psi_x(t,\cdot) \in L^2(\mathbb{R})$ from (2.20). ∎

3. Stability of subcritical shock waves. In the following two lemmata, we give a priori estimates to the local solution obtained in the previous section assuming the smallness of $|(\Phi^0, \phi^0)|_2$.

LEMMA 3.1. *Assume* (1.13)–(1.16). *If* $|(\Phi^0, \phi^0)|_2$ *is sufficiently small, it holds that*

$$|\Phi(t)|_2^2 + 2|\phi(t)|_2^2 \le |\Phi^0|_2^2 + 2|\phi^0|_2^2 \tag{3.1}$$

for an arbitrary $t \in [0, T_0]$.

Proof. Since Φ, ϕ, ψ and ψ_x are piecewise B^1 and belong to L^2, the last term of (2.26) vanishes as $M \to \infty$. Thus by letting $M \to \infty$ in (2.26) and using the Gronwall inequality, we have

$$|(\Phi,\phi)|_2(t) \le Ce^{Ct}|(\Phi^0,\phi^0)|_2.$$

Therefore, if $|(\Phi^0, \phi^0)|_2$ is sufficiently small, $|\Phi|_\infty(t) \le C|(\Phi,\phi)|_2(t) < 1 - |U'|_\infty$ holds for $t \in [0, \min\{T_0, 1\}]$. In that case, letting $M \to \infty$ in (2.26) again yields a better estimate of

$$\frac{1}{2}|\Phi(t)|_2^2 + |\phi(t)|_2^2 + \frac{1}{2}\int_0^t \int_{\mathbb{R}} |U'(x)|\Phi^2(s,x)dxds + \frac{1}{4}\int_0^t |(\phi,\psi,\psi_x)(s)|_2^2 ds \le \frac{1}{2}|\Phi^0|_2^2 + |\phi^0|_2^2 \tag{3.2}$$

for $t \in [0, \min\{T_0, 1\}]$ and, in particular, the smallness of $|(\Phi,\phi)|_2(t)$ is preserved. Repeating this argument if necessary, we complete the proof. ∎

LEMMA 3.2. *Assume the same conditions as in Theorem* 1.4. *If* $|(\Phi^0, \phi^0)|_2$ *is sufficiently small, then*

$$\inf_{x \in \Omega_t} u_x(t,x) \ge a_1 \tag{3.3}$$

and

$$[\![u^0]\!](d_j^0)e^{-(1+b_1)t} \le [\![u]\!](t, d_j(t)) \le [\![u^0]\!](d_j^0)e^{-(1+a_1)t} \tag{3.4}$$

hold for $t \in [0, T_0]$.

Proof. For arbitrary $t \in [0, T_0]$ and $x \in \Omega_t$, there exists a unique characteristic curve $\{(s, X(s)) | s \in [0,t]\}$ which reaches (t,x). By (2.14), along this trajectory we have

$$\frac{d}{dt}u_x(t, X(t)) = -u_x^2 - u_x + K * U' + K' * \phi. \tag{3.5}$$

First, we show that

$$\inf_{x\in\mathbb{R}} K * U'(x) = K * U'(0) = -Q(0) = -\delta_S^2/8. \tag{3.6}$$

The first equality is shown by noting (1.4) and (1.5) and that U is an odd function. The second equality is an immediate consequence of (1.3b). The last equality is obtained by integrating (1.3a) noting $s = 0$ and $U(0) = 0$. Substituting (3.6) into (3.5), we have

$$\begin{aligned}\frac{d}{dt}u_x(t, X(t)) &\geq -u_x^2 - u_x - \frac{1}{8}\delta_S^2 - |K' * \phi|_\infty(t)\\ &= -(u_x - a_-)(u_x - a_+) - |K' * \phi|_\infty(t),\end{aligned}$$

where $a_\pm := (-1 \pm \sqrt{1 - \delta_S^2/2})/2$. Note $a_+ = U'(0)$. By the Young inequality and Lemma 3.1, we see that

$$|K' * \phi|_\infty(t) \leq |K'|_2|\phi|_2(t) \leq |K'|_2|(\Phi^0, \phi^0)|_2 = |(\Phi^0, \phi^0)|_2/2 \tag{3.7}$$

provided $|(\Phi^0, \phi^0)|_2$ is sufficiently small. If in addition (1.17), that is, $\inf_{x\in\Omega_0} u_x^0(x) > a_-$ holds and $|(\Phi^0, \phi^0)|_2$ is further small if necessary, it holds that

$$\inf_{x\in\Omega_t} u_x(t, x) \geq \min\left\{\inf_{x\in\Omega_0} u_x^0(x), (-1/2 + a_+)/2\right\} = a_1$$

by the definition of a_1. Noting

$$-1 < a_1 \leq u_x(t, x) \leq b_1$$

for $t \in [0, T_0]$ and $x \in \Omega_t$, the estimate (3.4) is deduced in the same way as [18]. ∎

If $J = 0$ or $J = 1$, Lemmata 2.1 and 3.2 readily conclude the existence of a global solution. In the case $J \geq 2$, we need additional treatment. Now denote by $T_{0m}(m \in \mathbb{N})$ the local existence time of the m-times extended solution from the initial data following the method of Lemma 2.1. By its definition, the extension never stops in finite steps, but there is a possibility that $T_* := \sum_{m=1}^\infty T_{0m}$ is finite.

LEMMA 3.3. *Under the same conditions as in Theorem* 1.4, *assume that* $|(\Phi^0, \phi^0)|_2$ *is sufficiently small so that* (3.3) *holds. If* T_* *is finite, then*

$$L(t) := \min_{1\leq j\leq J-1}(d_{j+1} - d_j)(t) \to 0 \quad \text{as} \quad t \to T_* - 0.$$

Proof. We review (2) for the dependence of the existence time of a local solution when extended from a certain time $T(< T_*)$. Since the solution satisfies (2.15), (2.16), and (3.3) over $[0, T]$, we have a uniform bound for $\|u(t, \cdot)\|_{B^1}$. Recalling (2.1) and the definitions of C_1 and C_2 in (2.2), they are bounded above as

$$C_1 \leq CL(T)^{-1} + C, \quad C_2 \leq C$$

and hence

$$T_1 \geq c/(L(T)^{-1} + 1).$$

Lower bounds for T_2 and T_3 are given by (3.4) as

$$T_2 \geq ce^{-(1+b_1)T},$$
$$T_3 \geq cL(T).$$

If T_* is finite, $T_{0m} \to 0$ $(m \to \infty)$ whereas $T_2 \geq c > 0$. Therefore, $L(\sum_{k=1}^m T_{0k}) \to 0$ as $m \to \infty$. The proof is completed by noting (1.12) and (2.15). ◻

Proof of Theorem 1.4. When T_* is finite, we define the solution at $t = T_*$ by

$$d_j(T_*) := \lim_{t \to T_* - 0} d_j(t), \quad \Omega_{T_*} := \mathbb{R} \setminus \cup_{j=1}^J \{d_j(T_*)\},$$
$$u(T_*, x) := \lim_{t \to T_* - 0} u(t, x) \quad \text{for} \quad x \in \Omega_{T_*}.$$

This definition makes sense because of the uniform bounds for $\dot{d}_j(t) = \overline{u}(t, d_j(t))$ and u_t given by (2.15), (2.16), and (3.3). By the Ascoli–Arzelà theorem, $u(T_*, \cdot)$ is a piecewise B^1 function. The perturbation at $t = T_*$

$$\phi(T_*, x) := u(T_*, x) - U(x) \quad \text{for} \quad x \in \Omega_{T_*}$$

satisfies $\phi(T_*, \cdot) \in L^1$ due to the uniform bound (2.21). By the estimate (3.1), we see that both $\phi(T_*, \cdot)$ and the corresponding antiderivative $\Phi(T_*, \cdot)$ belong to L^2 and this estimate is valid also at $t = T_*$:

$$|\Phi(T_*)|_2^2 + 2|\phi(T_*)|_2^2 \leq |\Phi^0|_2^2 + 2|\phi^0|_2^2.$$

Therefore, we can construct the solution to (1.1) by regarding $u(T_*, \cdot)$ as the initial data without any further conditions until a possible collision of discontinuities at a later time. It is easy to see the extended solution indeed satisfies Definition 1.3 for an admissible solution following the argument in [4, p. 75]. By continuing this procedure finite times at most, we obtain the admissible solution globally in time. Also, by (2.15), (2.16), and (3.2)–(3.4) we have

$$\int_0^\infty |\phi(t)|_2^2 dt < \infty,$$
$$\int_0^\infty \left| \frac{d}{dt} |\phi(t)|_2^2 \right| dt \leq C \int_0^\infty |\phi(t)|_2^2 dt + C \int_0^\infty \sum_{j=1}^J [\![u]\!](t, d_j(t)) dt < \infty,$$

which imply $|\phi|_2(t) \to 0$ as $t \to \infty$. Noting the uniform bounds of u_x and the exponential decay of jump amplitude in (3.4), we have (1.18). In fact, if $\lim_{t\to\infty} \sup_{x \in \Omega_t} |u(t,x) - U(x)| = 0$ does not hold, there exists a positive constant ε, an increasing sequence of time $\{t_n\}_{n=1}^\infty$ with $t_n \to \infty$ as $n \to \infty$, and a sequence of points $\{x_n\}_{n=1}^\infty$ such that $|\phi(t_n, x_n)| > \varepsilon$ for all $n \in \mathbb{N}$. By (3.4) and $a_1 > -1$, there exists $T > 0$ such that

$$\sum_j [\![u]\!](t, d_j(t)) \leq \varepsilon/2 \quad \text{for} \quad t \geq T. \tag{3.8}$$

Let $\delta := \varepsilon/4(\max\{|a_1|, |b_1|\} + |U'|_\infty)$. Noting the definitions of ε, $\{t_n\}$, $\{x_n\}$, and δ, and using (2.16), (3.3), and (3.8), we have $|\phi(t_n, x)| \geq \varepsilon/4$ if $t_n > T$ and $|x - x_n| < \delta$ and hence

$$\int_{\mathbb{R}} \phi(t_n, x)^2 dx \geq 2\delta \left(\frac{\varepsilon}{4}\right)^2 > 0 \quad \text{for all} \quad t_n (> T),$$

which contradicts $|\phi|_2(t) \to 0$ as $t \to \infty$. The uniform convergence of q to Q is obtained using (2.20a) with the help of the Young inequality.

To prove (1.19), we recall (1.12):

$$\dot{d}_j(t) = U(d_j(t)) + \overline{\phi(t, d_j(t))}.$$

Since $|\phi|_\infty(t) \to 0$ as $t \to \infty$, for an arbitrary positive value X there exists $T(> 0)$ such that $|\phi|_\infty(t) \leq |U(X)|/2$ holds for $t \geq T$. Therefore, if $t \geq T$ and $|d_j(t)| \geq X$, then $\operatorname{sgn}(d_j(t))\dot{d}_j(t) \leq U(X)/2 < 0$. Hence for $t \geq T + 2\max\{|d_j(T)| - X, 0\}/|U(X)|$ it holds that $|d_j(t)| \leq X$, which means $d_j(t) \to 0$ as $t \to \infty$. ∎

Proof of Theorem 1.6.

(i) Using (1.12) and (3.3), the evolution of distances between adjacent discontinuities are estimated as

$$\begin{aligned}\frac{d}{dt}(d_{j+1}(t) - d_j(t)) &= \int_{d_j(t)+0}^{d_{j+1}(t)-0} u_x(t,x)dx \\ &\quad - \frac{1}{2}\big([\![u]\!](t, d_j(t)) + [\![u]\!](t, d_{j+1}(t))\big) \\ &\geq a_1(d_{j+1}(t) - d_j(t)) - \frac{1}{2}\big([\![u]\!](t, d_j(t)) + [\![u]\!](t, d_{j+1}(t))\big)\end{aligned}$$

over $[0, T_0]$. Integrate this inequality and use (3.4) to obtain

$$\begin{aligned}e^{-a_1 t}(d_{j+1}(t) - d_j(t)) &\geq (d_{j+1}^0 - d_j^0) \\ &\quad - \frac{1}{2}\int_0^t \big([\![u]\!](s, d_j(s)) + [\![u]\!](s, d_{j+1}(s))\big)e^{-a_1 s}ds \\ &\geq (d_{j+1}^0 - d_j^0) - \frac{[\![u^0]\!](d_j^0) + [\![u^0]\!](d_{j+1}^0)}{2(1+2a_1)}\big(1 - e^{-(1+2a_1)t}\big)\end{aligned} \tag{3.9}$$

for $t \in [0, T_0]$. If $a_1 > -1/2$ and (1.20) are satisfied, we have the desired estimate.

(ii) Here we see that T_* defined just prior to Lemma 3.3 is finite. By similar computations to deduce (3.9), using $u_x \leq b_1$ instead of $a_1 \leq u_x$, we have

$$e^{-b_1 t}(d_{j+1}(t) - d_j(t)) \leq (d_{j+1}^0 - d_j^0) - \frac{[\![u^0]\!](d_j^0) + [\![u^0]\!](d_{j+1}^0)}{2(1+2b_1)}\left(1 - e^{-(1+2b_1)t}\right) \tag{3.10}$$

for $t \in [0, T_*)$. Under (1.21), if T_* is not finite, the right-hand side of (3.10) gets negative within a finite time, which contradicts the positivity of the left-hand side for $t \in [0, T_*)$. Therefore, T_* is finite and Lemma 3.3 completes the proof. ∎

4. Stability problem of the critical shock wave.

4.1. Proofs of blowup results.

Proof of Theorem 1.7. Suppose that the conclusion is false and we have a B^1 solution globally in time. It is apparent that $u(t, \cdot)$ is an odd function for an arbitrary t for odd u^0. Then a characteristic curve initially within $x \geq 0$ remains always $x \geq 0$. By (3.5) and (3.6), we have

$$\frac{d}{dt}u_x(t, 0) = -(u_x(t, 0) + 1/2)^2 + K' * \phi(t, 0). \tag{4.1}$$

Now we show

$$\phi(t,x) \le 0 \quad \text{for arbitrary} \quad t \ge 0 \text{ and } x \ge 0 \tag{4.2}$$

and

$$K' * \phi\,(t,0) < 0 \quad \text{for arbitrary} \quad t \ge 0. \tag{4.3}$$

Consider an arbitrary characteristic curve $\{(t,X(t))\,|\,t \ge 0\}$ departing from $(0,X_0)$ with $X_0 \ge 0$. Substitute (2.20b) into (2.18a) and integrate the result along that characteristic curve to have

$$\begin{aligned} \phi(t,X(t)) = {} & \phi^0(X_0)\exp\left(-\int_0^t (1+U'(X(s)))ds\right) \\ & + \int_0^t (K*\phi)(\tau,X(\tau))\exp\left(-\int_\tau^t (1+U'(X(s)))ds\right)d\tau. \end{aligned} \tag{4.4}$$

Since $\phi(t,\cdot)$ is odd while $K(\cdot)$ is even and $K(a) \ge K(b)$ if $|a| \le |b|$, it holds for $x \ge 0$ that

$$\begin{aligned} K*\phi(t,x) = {} & \int_0^\infty \phi(t,y)(K(x-y)-K(x+y))dy \\ & \le \sup_{y\ge 0}\phi(t,y)\int_0^\infty (K(x-y)-K(x+y))dy = \sup_{y\ge 0}\phi(t,y)\,(1-e^{-x}). \end{aligned} \tag{4.5}$$

Noting $\phi^0(X_0) \le 0$, $U'(\cdot) \in [-1/2,0)$ and $\sup_{y\ge 0}\phi(t,y) \ge \phi(t,0) = 0$, we have from (4.4) that

$$\phi(t,X(t)) \le \int_0^t \sup_{y\ge 0}\phi(\tau,y)d\tau.$$

Taking the supremum among all characteristic curves departing from the right half line, we obtain

$$\sup_{y\ge 0}\phi(t,y) \le \int_0^t \sup_{y\ge 0}\phi(\tau,y)d\tau,$$

which readily yields (4.2) by the Gronwall inequality. Since both K' and $\phi(t,\cdot)$ are odd functions,

$$K' * \phi\,(t,0) = \int_{-\infty}^\infty K'(-x)\phi(t,x)dx = \int_0^\infty e^{-x}\phi(t,x)dx.$$

The conditions $\phi^0 \in B^1(\mathbb{R})$ and $\phi^0 \not\equiv 0$ imply $\phi(t,\cdot) \in B^1(\mathbb{R})$ and $\phi(t,\cdot) \not\equiv 0$. Thus (4.2) implies (4.3). Substituting (4.3) into (4.1) and noting $u_x(0,0) \le U'(0) = -1/2$, we conclude that

$$u_x(t,0) \to -\infty \quad \text{as} \quad t \to T_* - 0$$

for a certain finite value T_*, which contradicts the initial assumption. ∎

Proof of Theorem 1.8. Take an arbitrary constant $\alpha \in (-\delta_S/2, 0)$. See Figure 2 to facilitate the understanding of the subsequent arguments. For $B \in (-1/2, \alpha/U^{-1}(\alpha))$,

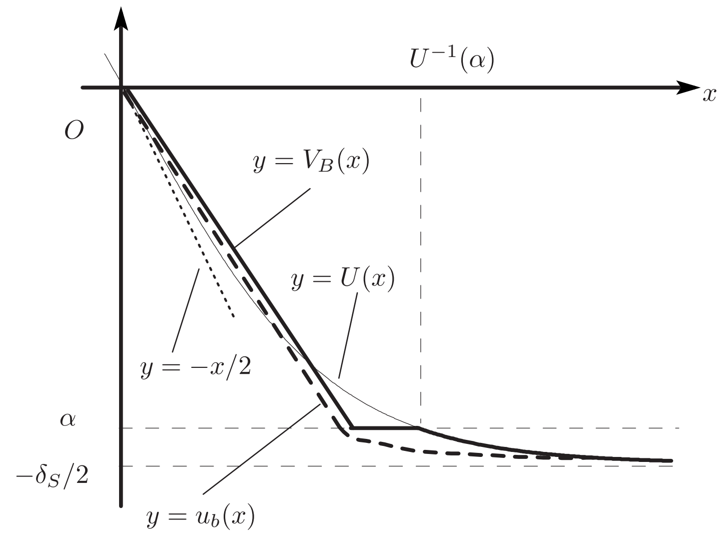

FIG. 2. *Profiles of $y = U(x)$(thin line), $y = V_B(x)$(thick line), $y = u_b(x)$(broken line), and $y = -x/2$(dotted line).*

define an odd function V_B in $B^0(\mathbb{R})$ by $V_B(x) = Bx$ for $x \in [0, \alpha/B]$, $V_B(x) = \alpha$ for $x \in [\alpha/B, U^{-1}(\alpha)]$, and $V_B(x) = U(x)$ for $x \geq U^{-1}(\alpha)$ and let $v_B(x) := V_B(x) - U(x)$. It is apparent that if B is sufficiently close to $U'(0) = -1/2$ and $|\alpha|$ is sufficiently small, then $\mu := -K' * v_B(0)/2 \in (0, (3\pi\delta_S/2)^{2/3})$. Fix arbitrary B and α so that μ satisfies this relation. For $b \in (-1/2, B]$, consider a nonincreasing odd function $u_b \in B^1(\mathbb{R})$ which satisfies $u_b(x) = bx$ for $x \in [0, \alpha/b]$, $u_b(x) \leq V_B(x)$ for $x \geq \alpha/b$, $u_b(x) \to -\delta_S/2$ as $x \to \infty$, and $\inf_{x\in\mathbb{R}} u_b'(x) = b$. Note that $K' * (u_b - U)(0) \leq -2\mu$ holds.

Let $u^0 = u_b$ with

$$b \in (-1/2, \sqrt{\mu}\tan(\sqrt{\mu}t_1/2) - 1/2), \text{ where } t_1 := 2\mu/3\delta_S. \tag{4.6}$$

Since $\sqrt{\mu}t_1/2 \in (0, \pi/2)$ follows from $\mu \in (0, (3\pi\delta_S/2)^{2/3})$, this definition makes sense. Suppose the conclusion is false and (1.1) and (1.2) has a global solution. We are going to show that $u_x(t, 0)$ blows up in a finite time.

First, we estimate the last term of (4.1). Using (2.18a), (2.20b), (4.5) and noting that $\phi(t, \cdot)$ is an odd function with $\inf_{x\geq 0} \phi(t, x) \leq 0$, we have by simple integration by parts that

$$\begin{aligned}
&\frac{d}{dt}\left(K' * \phi(t, 0)\right) \\
&\quad = \int_0^\infty e^{-x}\phi_t(t, x)dx \\
&\quad = \int_0^\infty e^{-x}\left\{(K * \phi)(t, x) - (1 + U(x))\phi(t, x)\right\}dx - \frac{1}{2}\int_0^\infty e^{-x}\phi^2(t, x)dx \\
&\quad \leq \int_0^\infty e^{-x}(1 - e^{-x})\sup_{y\geq 0}\phi(t, y)dx - \int_0^\infty e^{-x}(1 + U(x))\inf_{y\geq 0}\phi(t, y)dx \\
&\quad \leq \sup_{y\geq 0}\phi(t, y)/2 - \inf_{y\geq 0}\phi(t, y).
\end{aligned}$$

Due to (2.15) and the choice of nonincreasing $u_b = u^0$, we have

$$\frac{d}{dt}\left(K' * \phi(t,0)\right) \le \left(\sup_{y\in\mathbb{R}} u^0(y) + \delta_S/2\right)/2 - \left(\inf_{y\in\mathbb{R}} u^0(y) - \delta_S/2\right) = \frac{3}{2}\delta_S. \tag{4.7}$$

By definition, $K' * \phi^0(0) \le K' * v_B(0) = -2\mu$ and (4.7) deduces

$$K' * \phi(t,0) \le -\mu \quad \text{for} \quad t \le t_1 = 2\mu/3\delta_S.$$

Substituting this into (4.1), we see

$$u_x(t,0) \le \sqrt{\mu}\tan\left(\operatorname{Arctan}\left((u_x^0(0)+1/2)/\sqrt{\mu}\right) - \sqrt{\mu}t\right) - 1/2 \tag{4.8}$$

for $t \in [0,t_1]$. Since $t_2 := (\operatorname{Arctan}((u_x^0(0)+1/2)/\sqrt{\mu}) + \pi/2)/\sqrt{\mu}$ is the blowup time of the right-hand side of (4.8), the existence of C^1 global solution implies $t_1 < t_2$. Due to (4.6), (4.8) yields

$$u_x(t_1,0) < -2\gamma - 1/2, \tag{4.9}$$

where we define a positive constant γ by $\gamma = \sqrt{\mu}\tan(\sqrt{\mu}t_1/2)/2$. Noting $\sup_{x\ge 0}\phi(t,x) \ge \phi(t,0) = 0$, we have from (4.4) and (4.5) that

$$\sup_{x\ge0}\phi(t,x) \le \sup_{x\ge0}\phi^0(x) + \int_0^t \sup_{x\ge0}\phi(s,x)ds$$

and hence

$$\sup_{x\ge0}\phi(t,x) \le \sup_{x\ge0}\phi^0(x)\, e^t$$

with the aid of the Gronwall inequality. Let $t_3 := t_1 + (\log 3)/(2\gamma^2)$ and assume b is sufficiently close to but above $-1/2$ so that

$$\sup_{x\ge0}\phi^0(x)\, e^{t_3} \le \gamma^2$$

holds. Then for $t \le t_3$, we have $K' * \phi(t,0) \le \sup_{x\ge0}\phi(t,x) \le \sup_{x\ge0}\phi^0(x)\, e^t \le \gamma^2$. With $K' * \phi(t,0) \le \gamma^2$ for $t \in [t_1,t_3]$ and (4.9), (4.1) implies

$$u_x(t,0) \le \gamma\frac{e^{2\gamma^2(t-t_1)}+3}{e^{2\gamma^2(t-t_1)}-3} - \frac{1}{2} \quad \text{for} \quad t \ge t_1.$$

The right-hand side tends to $-\infty$ as $t \to t_3 - 0$, which contradicts the initial assumption.

Finally, assume further $\alpha \in (-\delta_S/4, 0)$ and $\operatorname{supp}\phi^0 \subset [-U^{-1}(2\alpha), U^{-1}(2\alpha)]$. Then the blowup initial data constructed above for an arbitrary $\alpha \in (-\delta_S/2, 0)$ satisfies $\phi^0 \in L^2 \cap L^1$, $\Phi^0 \in L^2$, and $|(\Phi^0,\phi^0)|_2 \to 0$ as $\alpha \to 0$. This is the desired initial data. □

4.2. Some convergence results toward the critical shock wave. In order to prove Theorem 1.9, we first see that a perturbed traveling wave in an "opposite direction" to the initial data considered in Theorem 1.7 converges uniformly to a traveling wave. By the analysis in section 3, we see that the key elements to construct a global solution in C^1 is to bound $|\phi|_2(t)$ and $|\phi_x|_\infty(t)$ uniformly in time. This is true for all shock strength δ_S.

THEOREM 4.1. *For $\delta_S \in (0, \sqrt{2}]$, let $U(x)$ be the associated traveling wave solution. Suppose an odd function $\phi^0 \in B^1 \cap L^1(\mathbb{R})$ satisfies $\phi^0(x) \geq 0$ for $x \geq 0$ and* (1.16)*, while $u^0(x) = U(x) + \phi^0(x)$ satisfies* (1.15) *and*

$$\inf_{x\in\mathbb{R}} u_x^0(x) \geq -1/2. \tag{4.10}$$

If $|\Phi^0|_2 + |\phi^0|_2$ and $\sup_{x\in\mathbb{R}} \phi_x^0(x)$ are sufficiently small, then the initial value problem (1.1) *and* (1.2) *has a unique global solution which satisfies*

$$\inf_{x\in\mathbb{R}} u_x(t,x) \geq -1/2 \quad \text{for an arbitrary} \quad t \geq 0. \tag{4.11}$$

Moreover, the solution converges uniformly to the traveling wave as in (1.18).

To prove Theorem 4.1, we give an estimate of $\sup_{x\in\mathbb{R}} \phi_x(t,x)$.

LEMMA 4.2. *Assume the same conditions as in Theorem* 4.1 *but not necessarily* (4.10)*. If $|\Phi^0|_2 + |\phi^0|_2$ is sufficiently small,*

$$\sup_{x\in\mathbb{R}} \phi_x(t,x) \leq \max\left\{\sup_{x\in\mathbb{R}} \phi_x^0(x), \sqrt{|(\Phi^0,\phi^0)|_2/2}\right\}$$

holds as long as the solution exists.

Proof. Let $\{(t,X(t)) \,|\, t \in [0,T]\}$ be an arbitrary characteristic curve with a certain $T \in (0,\infty]$. (This notation will frequently be used hereafter.) If $X(0) \neq 0$, then $X(t) \neq 0$ for all t and

$$\frac{d}{dt}\phi_x(t,X(t)) = -\phi_x(t,X(t))^2 - (2U'(X(t)) + 1)\,\phi_x(t,X(t)) + e(t,X(t))$$

holds, where

$$e(t,x) := K' * \phi(t,x) - \phi(t,x)U''(x).$$

In the case $e(t,X(t)) \leq 0$, $\frac{d}{dt}\phi_x(t,X(t)) \leq 0$ holds if $\phi_x(t,X(t)) \geq 0$ since $U' \geq -1/2$. In the case $e(t,X(t)) \geq 0$, if $\phi_x(t,X(t)) \geq \lambda_+$, where

$$\lambda_+ := e(t,X(t)) \,/\, \left(\sqrt{(U'(X(t)) + 1/2)^2 + e(t,X(t))} + (U'(X(t)) + 1/2)\right)$$

is the larger root of $-\lambda^2 - (2U'(X(t)) + 1)\lambda + e(t,X(t)) = 0$, then $\frac{d}{dt}\phi_x(t,X(t)) \leq 0$ holds. In the same way as proving (4.2), we obtain

$$\phi(t,x) \geq 0 \quad \text{for arbitrary} \quad t \geq 0 \text{ and } x \geq 0. \tag{4.12}$$

Using $U' \geq -1/2$, $U''(x) \geq 0$ for $x > 0$, (4.12) and (3.7), we have

$$\lambda_+ \leq \sqrt{e(t,X(t))} \leq \sqrt{K' * \phi(t,X(t))} \leq \sqrt{|(\Phi^0,\phi^0)|_2/2}.$$

In any case,

$$\frac{d}{dt}\phi_x(t,X(t)) \leq 0 \quad \text{if} \quad \phi_x(t,X(t)) \geq \sqrt{|(\Phi^0,\phi^0)|_2/2} \quad \text{and} \quad X(0) \neq 0.$$

Noting the continuity of $\phi_x(t,x)$ in x, we have the desired estimate. ∎

Proof of Theorem 4.1. The case with $\delta_S < \sqrt{2}$ is shown in Theorem 1.4 with fewer assumptions. Here we assume $\delta_S = \sqrt{2}$. Substituting $K * U' = -Q = U^2/2 - 1/4$ into (3.5), we have

$$\frac{d}{dt}u_x(t,X(t)) = -\left(u_x(t,X(t)) + 1/2\right)^2 + g(t,X(t)), \quad \text{where} \;\; g(t,x) := U(x)^2/2 + K' * \phi(t,x). \tag{4.13}$$

Hereafter we treat only the nontrivial case of $\phi^0 \not\equiv 0$. If

$$g(t,x) > 0 \tag{4.14}$$

is shown for arbitrary $t \geq 0$ and $x \in \mathbb{R}$, (4.10) guarantees (4.11) and hence the existence of a global solution. We define $R_0 \geq 0$ by $U(R_0)^2 = 2|(\Phi^0,\phi^0)|_2$ and show (4.14) for three states of $X(t)$: (i) $X(t) = 0$, (ii) $|X(t)| \geq R_0$, and (iii) $|X(t)| \in (0, R_0)$.

We first obtain from (4.12) and $\phi(t,\cdot) \not\equiv 0$ that

$$K' * \phi\,(t,0) > 0 \;\; \text{for an arbitrary} \;\; t \geq 0. \tag{4.15}$$

Thus the case with $X(t) = 0$ is shown. In the case of $|X(t)| \geq R_0$, (3.7) assures $g(t,X(t)) \geq U(R_0)^2/2 - |(\Phi^0,\phi^0)|_2/2 = |(\Phi^0,\phi^0)|_2/2 > 0$. Finally, we consider the case $|X(t)| \in (0,R_0)$. Applying the Taylor expansion, we rewrite $g(t,x)$ into

$$g(t,x) = K' * \phi(t,0) + \left(U'(\theta_1 X(t))^2 + K' * \phi\,(t,\theta_2 X(t)) - \phi_x(t,\theta_2 X(t))\right) X(t)^2/2 \tag{4.16}$$

using certain values $\theta_1, \theta_2 \in (0,1)$. Here $\partial_x(K' * \phi)(t,0) = 0$ and $\partial_x^2(K' * \phi) = K' * \phi - \phi_x$ are used noting that $K*$ is an inverse operator to $1 - \partial_x^2$. For $|X(t)| \leq R_0$, it holds that $U'(\theta_1 X(t))^2 \geq U'(R_0)^2$. Noting that R_0 is an increasing function of $|(\Phi^0,\phi^0)|_2$, if $|(\Phi^0,\phi^0)|_2$ is sufficiently small, (3.7) implies $|K' * \phi\,(t,\cdot)|_\infty \leq U'(R_0)^2/2$. By Lemma 4.2, if $|(\Phi^0,\phi^0)|_2$ and $\sup_{x\in\mathbb{R}} \phi_x^0(x)$ are sufficiently small, then $\sup_{x\in\mathbb{R}} \phi_x(t,x) \leq U'(R_0)^2/2$ and

$$U'(\theta_1 X(t))^2 + K' * \phi\,(t,\theta_2 X(t)) - \phi_x(t,\theta_2 X(t)) \geq 0. \tag{4.17}$$

Combining (4.15), (4.16), and (4.17), we have (4.14) and hence (4.11). The uniform convergence towards the traveling wave is shown as in the proof of Theorem 1.4. ∎

Theorem 1.9 is shown based on Theorems 1.4 and 4.1.

Proof of Theorem 1.9. For an arbitrary $\delta_S \in (0,\sqrt{2}]$, let $U(x)$ be the associated traveling waves solution with $U(0) = 0$ and $u^0 \in B^1(\mathbb{R})$ be an odd function which satisfies $\phi^0(x) = u^0(x) - U(x) \geq 0$ for $x \geq 0$. Assume that ϕ^0 has a compact support included in $\{x \in \mathbb{R} \,|\, |x| \geq S_0 + T_0 V_0\}$ with $\inf_{x\in\mathbb{R}} u_x^0(x) = -1 + \varepsilon$, where constants S_0, T_0, and V_0 are defined by $S_0 := |U^{-1}(\sqrt{\max\{(\delta_S/2)^2 - \varepsilon(1-\varepsilon)/2, 0\}}\,)|$, $T_0 := 2/\varepsilon^2$, and $V_0 := \max\{|\inf_{x\in\mathbb{R}} u^0(x)|, |\sup_{x\in\mathbb{R}} u^0(x)|\}$. Note that (1.15), (1.16), and $\inf_{x\geq 0} \phi(t,x) \geq 0$ for all t are satisfied by these requirements.

By substituting $K * U' = -Q = U^2/2 - \delta_S^2/8$ into (3.5), for any characteristic curve $\{(t,X(t)) \,|\, t \in [0,T]\}$ it holds that

$$\frac{d}{dt}u_x(t,X(t)) = -\left(u_x(t,X(t)) + 1/2\right)^2 + U(X(t))^2/2 + \frac{1}{8}(2 - \delta_S^2) + K' * \phi(t,X(t)). \tag{4.18}$$

If $|(\Phi^0, \phi^0)|_2$ and $\sup_{x\in\mathbb{R}} \phi_x^0(x)$ are sufficiently small, the same arguments as in the proofs of Lemma 3.2 (if $\delta_S < \sqrt{2}$) or Theorem 4.1 (if $\delta_S = \sqrt{2}$) deduce

$$\frac{d}{dt}u_x(t, X(t)) > -\left(u_x(t, X(t)) + 1/2\right)^2.$$

Therefore, if $u_x(t_0, X(t_0)) \geq -1/2$ holds for a certain $t_0 \geq 0$ and a certain characteristic curve, then along the same characteristic curve, $u_x(t, X(t)) \geq -1/2$ holds for $t \geq t_0$ as long as the solution exists.

Assume that $|(\Phi^0, \phi^0)|_2$ is sufficiently smaller than $\varepsilon(1-\varepsilon)/2$ so that Lemma 3.1 and hence (3.7) hold. If

$$u_x(t, X(t)) \in [-1+\varepsilon/2, -\varepsilon/2] \quad \text{and} \quad (\delta_S/2)^2 - U(X(t))^2 \leq \varepsilon(1-\varepsilon)/2 \tag{4.19}$$

are satisfied, then

(4.20)
$$\begin{aligned}\frac{d}{dt}u_x(t, X(t)) &\geq -\left(-1/2+\varepsilon/2\right)^2 + U(X(t))^2/2 + \frac{1}{8}(2-\delta_S^2) - |K' * \phi(t, X(t))|_\infty \\ &\geq (1-\varepsilon/2)\varepsilon/2 - \left((\delta_S/2)^2 - U(X(t))^2\right)/2 - |(\Phi^0, \phi^0)|_2/2 \geq \varepsilon^2/4.\end{aligned}$$

Consider an arbitrary point x_0 with $u_x^0(x_0) \in [-1+\varepsilon, -1/2)$ and the characteristic curve $\{(t, X(t)) \,|\, t \in [0, T]\}$ departing from x_0. Such x_0 belongs to $\mathrm{supp}(u^0 - U)$. With the aid of (2.15), if $t \leq T_0$, we have $|X(t)| \geq S_0 + T_0V_0 - t\sup_{t\geq 0, x\in\mathbb{R}} |u(t,x)| \geq S_0$ and $|U(X(t))|^2 \geq \max\{(\delta_S/2)^2 - \varepsilon(1-\varepsilon)/2, 0\}$. Therefore, if $t_1 \leq T_0$ and $u_x(t_1, X(t_1)) < -1/2$, then $u_x^0(X(0)) \in [-1+\varepsilon, -1/2)$ and (4.19) hold for $t \in [0, t_1]$ and hence (4.20). Due to this and the definition of T_0, there exists $t_2 (\leq T_0)$ such that $u_x(t_2, X(t_2)) = -1/2$. Therefore, $\inf_{x\in\mathbb{R}} u_x(T_0, x) \geq -1/2$ and the application of Theorems 1.4 or 4.1 at time T_0 yields the conclusion. $\square$

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