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THE OPTIMAL CONVERGENCE RATE OF MONOTONE FINITE DIFFERENCE METHODS FOR HYPERBOLIC CONSERVATION LAWS*

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Abstract. We are interested in the rate of convergence in L^1 of the approximate solution of a conservation law generated by a monotone finite difference scheme. Kuznetsov has proved that this rate is 1/2 [*USSR Comput. Math. Math. Phys.*, 16 (1976), pp. 105–119 and *Topics Numer. Anal.* III, in Proc. Roy. Irish Acad. Conf., Dublin, 1976, pp. 183–197], and recently Teng and Zhang have proved this estimate to be sharp for a linear flux [*SIAM J. Numer. Anal.*, 34 (1997), pp. 959–978]. We prove, by constructing appropriate initial data for the Cauchy problem, that Kuznetsov's estimates are sharp for a nonlinear flux as well.

Key words. conservation laws, error estimates, monotone finite difference schemes, accuracy

AMS subject classifications. 35L65, 65M15

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1. Introduction.

There are many ways of looking at the accuracy of numerical methods for solving partial differential equations. One way is to take as a measure of accuracy the order of the error term obtained by substituting the numerical solution in the differential equation; this is usually called the order of the method [LeV]. In what follows, we shall refer to it as the formal order of accuracy of the numerical method. Another way is to measure the error of approximation $\|u - u_{\Delta x}\|$ of the exact solution u by the numerical solution $u_{\Delta x}$ in a suitable norm. The parameter Δx gives the scale of approximation and converges to zero as the scale becomes finer. We shall call the order of this error the convergence rate of the numerical solution generated by the given numerical method, or simply the convergence rate of the method.

For linear partial differential equations with constant coefficients, the convergence rate is the same as the formal order of accuracy, provided the solution is smooth enough [BTW]. This justifies the use of either one as a measure of the accuracy of a numerical method.

For conservation laws, this is no longer true. Formally, high-order methods have low convergence rates. For many numerical methods, this is explained in part by the relation between high-order convergence rates and high-order regularity of the solution. If a numerical method has a high order of convergence, the solution will have a high order of smoothness. But for conservation laws, due to the formation of shock discontinuities, there are a priori limits on the smoothness of solutions. A discussion of regularity spaces for conservation laws and of the implications for approximation of solutions can be found in the papers of DeVore and Lucier [DeVLu1, DeVLu2].

Moreover, there are discrepancies between the expected accuracy, given by the formal accuracy of a method, and the actual convergence rates in cases where the smoothness of the solution is not an obstacle. The aim of this paper is to show that this discrepancy is not due to our inability to find better error estimates.

In what follows, we restrict our attention to monotone finite difference schemes for scalar conservation laws. The fundamental results on monotone schemes were

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obtained by Kuznetsov [Ku2], Kuznetsov and Volosin [KuVl], and Crandall and Majda [CrMj]. Their papers contain proofs of convergence of monotone schemes to the entropy solution and estimates of the convergence rate in L^1.

Monotone difference schemes are quite natural, since the solution operator of the conservation law is monotone. The problem with monotone schemes, as shown by Harten, Hyman, and Lax [HHL], is that they are at most first-order accurate, and as a consequence, their rate of convergence in L^1 is restricted to $O(\Delta x)$, where Δx represents the meshsize of the uniform spatial grid on which the exact solution is approximated. On the other hand, the general proofs for convergence rates of Kruzkhov [Kr] and Kuznetsov [Ku1, Ku2] give only $O(\sqrt{\Delta x})$. Recently Nessyahu, Tadmor, and Tassa [Td], [NsTd], [NTT] used a different method from the one employed by Kuznetsov to prove a convergence rate of $O(\Delta x)$ in $W^{-1,1}$. This higher convergence rate in $W^{-1,1}$ translates into the same $O(\sqrt{\Delta x})$ for the convergence rate in L^1. In some special cases, like Riemann problems, it was shown by Bakhvalov [Bk] and Harabetian [Hb] that faster convergence rates, like $O(\Delta x\,|\log \Delta x|)$, can be obtained for a rarefaction wave; Teng and Zhang [TnZh] proved the optimal convergence rate of $O(\Delta x)$ for solutions corresponding to initial data that are piecewise constant with a finite number of pieces. One common feature of these results is that they make use of the nonlinearity of the flux, while it has been known (for example [Lu1, Lu2]) that in the case of a linear flux (that is, for the linear advection equation) and an upwind scheme, the rate of convergence for solutions of Riemann problems (traveling waves) is precisely $O(\sqrt{\Delta x})$. Tang and Teng [TgTn] proved that the same is true for all monotone finite difference schemes applied to a linear conservation law.

All this raises the question: are better estimates possible for the case of a nonlinear flux and general initial data? As we show in this paper, the answer is no; even for a nonlinear flux, we can construct initial data for which the convergence rate in L^1 of the corresponding numerical solution to the exact solution is no better than $O(\sqrt{\Delta x})$.

Our construction consists of an initial condition that is piecewise constant with infinitely many pieces. We obtain the error estimate by using linearized equations associated with the conservation law and the error estimates already available for linear equations.

2. Monotone finite difference schemes. In this section we review the basic notations, definitions, and theorems concerning monotone finite difference schemes.

In order to construct an approximate solution to the conservation law

$$\text{(CL)} \qquad \partial_t u(x,t) + \operatorname{div}_x f(u(x,t)) = 0, \quad u(x,0) = u_0(x)\,,$$

we replace derivatives by finite differences and the solution u by a piecewise constant function on a uniform grid. For simplicity of notation, we restrict ourselves to the one-dimensional case.

Let $I_j := [\,j, j+1)$ for any $j \in \mathbb{Z}$, and let $(\delta_a u)(x) := u(ax)$. The piecewise constant numerical approximation to the solution is given by

$$u^n_{\Delta x} := u_{\Delta x}(\,\cdot\,, n\Delta t) := \sum_{j\in\mathbb{Z}} u^n_j \,\delta_{1/\Delta x}\, \chi_{I_j}\ . \tag{2.1}$$

Here, Δx is the meshsize, Δt is the time step, and n represents the number of steps in time performed. For the initial condition $u^0_{\Delta x}$ we take the orthogonal projection of the initial datum u_0 onto the space of piecewise constant functions on the given grid

$$u^0_{\Delta x} := P_{\Delta x}\, u_0 := \sum_{j\in\mathbb{Z}} u^0_j \delta_{1/\Delta x}\chi_{I_j}\ , \quad u^0_j := \frac{1}{\Delta x}\int_{j\Delta x}^{(j+1)\Delta x} u_0(x)\, dx\ . \tag{2.2}$$

By an *explicit numerical scheme*, we mean a transformation $(u^n_j)_{j\in\mathbb{Z}} \mapsto (u^{n+1}_j)_{j\in\mathbb{Z}}$ of the form

$$u^{n+1}_j = u^n_j - \frac{\Delta t}{\Delta x}\left[h(f)(u^n_{j-p+1},\dots,u^n_{j+q}) - h(f)(u^n_{j-p},\dots,u^n_{j+q-1})\right]\ . \tag{2.3}$$

The quotient $\lambda := \Delta t/\Delta x$ is constant, meaning that the grids we consider are uniform in space and time. The numerical flux $h(f)$ is a function of $p+q$ variables, and the method is called a $(p+q+1)$-point scheme. We note that the numerical scheme can be rewritten as

$$\begin{aligned} u^{n+1}_{\Delta x} &= \delta_{1/\Delta x}\, S\, \delta_{\Delta x}\, u^n_{\Delta x} \\ &= \delta_{1/\Delta x}\, S^n\, \delta_{\Delta x}\, u^0_{\Delta x}\ , \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} Su(x) := u(x) - \lambda\,[h(f)\,(u(x-p+1),\dots,u(x+q)) \\ -h(f)\,(u(x-p),\dots,u(x+q-1))]\ . \end{aligned} \tag{2.5}$$

A three-point scheme can be written as

$$u^{n+1}_j = u^n_j - \lambda\left[h(f)(u^n_j,u^n_{j+1}) - h(f)(u^n_{j-1},u^n_j)\right]\ . \tag{2.6}$$

In this particular case, the operator S becomes

$$Su(x) := u(x) - \lambda\,[h(f)(u(x),u(x+1)) - h(f)(u(x-1),u(x))]\ . \tag{2.7}$$

Using the piecewise constant functions $u^n_{\Delta x}$ that approximate the exact solution at $n\Delta t$, $n = 0,1,\dots$, we can construct various approximations to u in the whole half-space. For example, we can choose piecewise constant functions in time

$$u_{\Delta x}(\,\cdot\,,t) = u_{\Delta x}(\,\cdot\,,n\Delta t), \quad n\Delta t \le t < (n+1)\Delta t\ , \tag{2.8}$$

or we can take piecewise linear functions in time

$$\begin{aligned} u_{\Delta x}(\,\cdot\,,(1-\alpha)n\Delta t + \alpha(n+1)\Delta t) \\ = (1-\alpha)u_{\Delta x}(\cdot,n\Delta t) + \alpha u_{\Delta x}(\cdot,(n+1)\Delta t), \quad 0\le\alpha\le 1\,. \end{aligned} \tag{2.9}$$

We are interested in the convergence of the approximations $u_{\Delta x}$ to the entropy solution of the conservation law. We say that $u_{\Delta x} \to u$ in $L^\infty([0,T],L^1(\mathbb{R}))$ when $\Delta x \to 0$ if

$$\sup_{0\le t\le T} \|u_{\Delta x}(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})} \longrightarrow 0 \ \text{ for }\ \Delta x \longrightarrow 0\ . \tag{2.10}$$

We note that, since $\lambda = \Delta t/\Delta x$ is constant, $\Delta x \to 0$ means $\Delta t \to 0$ as well.

We say that the numerical flux $h(f)$ is *consistent* with the differential flux f if $h(f)(c,\dots,c) = f(c)$.

The numerical scheme (2.3) is said to be *conservative* since the operator S defined by (2.5) preserves integrals, that is

$$\int_{\mathbb{R}} Su(x)\,dx = \int_{\mathbb{R}} u(x)\,dx\ , \tag{2.11}$$

whenever u, Su are integrable. Indeed, the integrals of the last two terms on the right in (2.5) are equal, since each can be obtained from the other by a translation in the variable x.

The numerical scheme (2.3) is called *monotone* if the operator S is monotone, that is, if $u \geq v$ implies $Su \geq Sv$.

By a theorem of Lax and Wendroff [LaW], it is known that *if* a consistent scheme of the form (2.3) converges, in the sense that $u_{\Delta x}$ converges boundedly almost everywhere to a function u, then the limit u is a weak solution of (CL). The problem is that it might not converge, or it is possible that it converges to the wrong solution [HHL]. However, if the operator S is monotone, the scheme converges, and it converges to the entropy solution. Kuznetsov [Ku2] proved the following theorem.

THEOREM 2.1. *The approximate solution generated by a monotone, conservative, and consistent numerical scheme* (2.3) *with a Lipschitz continuous numerical flux converges in $L^\infty([0,T], L^1(\mathbb{R}))$ to the entropy solution of* (CL) *for any initial condition in $L^1 \cap L^\infty(\mathbb{R})$.*

In addition, if the initial condition has bounded variation,

$$\|u_{\Delta x}(\cdot,t) - u(\cdot,t)\| \leq C_T\sqrt{\Delta x} \tag{2.12}$$

for all $0 \leq t \leq T$.

Next, we give some examples of monotone schemes.

(1) The modified Lax–Friedrichs scheme

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{\lambda}{2}\left[f(u_{j+1}^n) - f(u_{j-1}^n)\right] + \frac{Q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \\ h^{LF}(f)(u,v) &= \frac{f(v)+f(u)}{2} - \frac{Q}{2\lambda}(v-u)\ , \end{aligned} \tag{2.13}$$

where $0 < Q \leq 1$. Here, *modified* refers to the coefficient Q, which in the Lax–Friedrichs scheme is equal to 1. If Q is chosen greater than 1, the scheme cannot be monotone. Sufficient conditions for monotonicity are that $Q\lambda \leq 1$ and $Q \geq \lambda \max|f'(u)|$.

(2) The Engquist–Osher scheme

$$\begin{aligned} u_j^{n+1} &= u_j^n - \lambda\left[f^+(u_j^n) - f^+(u_{j-1}^n) + f^-(u_{j+1}^n) - f^-(u_j^n)\right], \\ h^{EO}(f)(u,v) &= f^-(v) + f^+(u)\ , \end{aligned} \tag{2.14}$$

where $f^+ = \int f' \vee 0$ and $f^- = \int f' \wedge 0$. A sufficient condition for monotonicity in this case is $\lambda \max|f'(u)| \leq 1$.

(3) The Godunov scheme is defined as follows: let $P_{\Delta x}$ be the orthogonal projector onto the space of piecewise constant functions as defined by (2.2) and let $E(t)$ denote the exact solution operator of (CL); then, we define

$$u_{\Delta x}^{n+1} := P_{\Delta x}E(\Delta t)u_{\Delta x}^n\ . \tag{2.15}$$

Since both $P_{\Delta x}$ and $E(\Delta t)$ are monotone operators, it follows that S, the numerical evolution operator, is also monotone. In this case, S is given by the formula $S =$

$\delta_{\Delta x} P_{\Delta x} E(\Delta t) \delta_{1/\Delta x}$. Under the additional assumption $\lambda \max|f'(u)| \leq 1/2$, Osher [Os] showed that for a general, possibly nonconvex flux f, the Godunov scheme can be written as

$$u_j^{n+1} = u_j^n - \lambda \left[f(u^*(u_j^n, u_{j+1}^n)) - f(u^*(u_{j-1}^n, u_j^n)) \right] ,$$
$$h^G(f)(u,v) = f(u^*(u,v)) \ ,$$

where

$$f(u^*(u,v)) = \begin{cases} \min\limits_{w \in [u,v]} f(w) & \text{if } u < v, \\ \max\limits_{w \in [v,u]} f(w) & \text{if } u \geq v \ . \end{cases} \tag{2.16}$$

The strengthened Courant–Friedrichs–Levy (CFL) condition $\lambda \max|f'(u)| \leq 1/2$ is equivalent to the requirement that the Riemann problems we solve at consecutive gridpoints in Godunov's scheme do not interact.

In our examples, the conditions we give as sufficient for monotonicity also enforce the CFL condition.

We note that, for Burgers's equation ($f(u) = \frac{1}{2}u^2$) and positive initial data, or a linear advection equation with positive transport velocity, (2) and (3) coincide and become an upwind scheme:

$$u_j^{n+1} = u_j^n - \lambda \left[f(u_j^n) - f(u_{j-1}^n) \right] . \tag{2.17}$$

We now present sharp estimates for the L^1 error of approximation, $\|u(\,\cdot\,, N\Delta t) - u_{\Delta x}(\,\cdot\,, N\Delta t)\|_{L^1(\mathbb{R})}$, in the case $f(u) = cu$, that is, when the equation is linear. Detailed proofs can be found in the lecture notes of Lucier [Lu2] and in the paper of Tang and Teng [TgTn].

We apply a monotone finite difference scheme to the linear advection equation

$$\partial_t v(x,t) + c\,\partial_x v(x,t) = 0, \quad v(x,0) = u_0(x), \tag{LCL}$$

where

$$u_0(x) = \begin{cases} a \text{ if } x < 0, \\ b \text{ if } x \geq 0, \end{cases} \tag{2.18}$$

and $c > 0$. The exact solution is

$$w(x,T) = \begin{cases} a \text{ if } x < cT, \\ b \text{ if } x \geq cT \ . \end{cases} \tag{2.19}$$

We assume that the monotone $(p+q+1)$-point scheme we chose has the form

$$v_j^{n+1} = \sum_{s=-p}^{q} a_s v_{j+s}^n \tag{2.20}$$

when applied to the linear advection equation. For such a scheme applied to the Riemann problem (2.18), we have the following estimate of Tang and Teng [TgTn]:

$$\begin{aligned} \|v(\cdot,T) - v_{\Delta x}(\cdot,T)\|_{L^1(\mathbb{R})} &\geq c(p,q)|b-a| \frac{T}{\lambda} \sum_{s \neq s_0} \sqrt{a_s}\ N^{-1/2} \\ &= c(p,q)|b-a| \sqrt{\frac{T}{\lambda}} \sum_{s \neq s_0} \sqrt{a_s}\ \sqrt{\Delta x} \ , \end{aligned} \tag{2.21}$$

where $T = N\Delta t$, $c(p,q)$ is a universal constant depending only on p and q, and s_0 is determined by $a_{s_0} := \max_s a_s$. We note that this estimate holds for both $a < b$ and for $b < a$.

3. Comparison of the linear and the nonlinear schemes. Our next goal is to compare the numerical solutions provided by the *same* numerical scheme applied to the nonlinear problem

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \quad u(x,0) = u_0(x) \tag{CL}$$

and to the linear transport equation

$$\partial_t v(x,t) + c\,\partial_x v(x,t) = 0, \quad v(x,0) = u_0(x) \tag{LCL}$$

in the case of a Riemann problem. That is, given the nonlinear conservation law (CL) and some numerical method for its solution, we shall apply the same numerical method to a linear problem (LCL) and compare the solutions of the two. We shall show that, if the variation of the initial datum u_0 is small, the error of approximation for (CL) is close to the error of approximation for (LCL). This is due to the fact that, for small discontinuities, the nonlinear equation exhibits behavior very similar to that of the linear equation.

For the remainder of this section, we take

$$u_0(x) = \begin{cases} a \text{ if } x < 0\ , \\ b \text{ if } x \geq 0\ . \end{cases} \tag{3.1}$$

As before, our estimates hold for both $a < b$ and $b < a$. We fix $T > 0$ and we allow N, the number of time steps performed, to vary. That means that we let $\Delta x = \Delta t/\lambda = T/\lambda N$. Thus, our parameter is N, and we want to express everything in terms of N, rather than Δx or Δt. For this purpose, we need to make a small modification in the way we write the numerical scheme. Let $d := T/\lambda$ and let $l_c(u) := cu$; then, we claim, for $0 \leq n \leq N$,

$$u_{\Delta x}(\,\cdot\,, n\Delta t) = \delta_N\, S_1^n\, u_0 \quad \text{ and } \quad v_{\Delta x}(\,\cdot\,, n\Delta t) = \delta_N\, S_2^n\, u_0\ , \tag{3.2}$$

where

$$\begin{aligned} S_1 u(x) &:= u(x) - \frac{T}{d}[h(f)(u(x-(p-1)d), \dots, u(x+qd)) \\ &\qquad\qquad - h(f)(u(x-pd), \dots, u(x+(q-1)d))]\,, \\ S_2 u(x) &:= u(x) - \frac{T}{d}[h(l_c)(u(x-(p-1)d), \dots, u(x+qd)) \\ &\qquad\qquad - h(l_c)(u(x-pd), \dots, u(x+(q-1)d))]\,. \end{aligned} \tag{3.3}$$

We remark that $u^0_{\Delta x} = u_0$ due to the particular form of u_0. We have only rewritten (2.4)–(2.5) since

$$\begin{aligned} u_{\Delta x}(\,\cdot\,, n\Delta t) &= \delta_{1/\Delta x}\, S^n\, \delta_{\Delta x}\, u^0_{\Delta x} \\ &= \delta_N\, \delta_{1/N\Delta x}\, S^n\, \delta_{N\Delta x}\, \delta_{1/N}\, u^0_{\Delta x} \\ &= \delta_N\, \left(\delta_{1/N\Delta x}\, S\, \delta_{N\Delta x}\right)^n\, \delta_{1/N}\, u^0_{\Delta x} \\ &= \delta_N\, S_1^n\, u^0_{\Delta x}\ , \end{aligned} \tag{3.4}$$

with the observation that S is the one given by (2.5) and the initial datum u_0 is dilation invariant. The same analysis gives S_2.

Thus, $u_{\Delta x}(\,\cdot\,, N\Delta t)$ and $v_{\Delta x}(\,\cdot\,, N\Delta t)$ represent the numerical solutions for $t = T$ corresponding to the same numerical method applied to (CL) and (LCL), respectively. We recall that $h(f)$ is the numerical flux obtained by applying our numerical method to (CL) and that $h(l_c)$ is the corresponding numerical flux for (LCL).

We now make some supplementary assumptions on the numerical method under consideration. We shall need the following two conditions for f and the numerical method:

(C_1) $h(l_c)$ is a linear function, $h(l_c)(x_1, \ldots, x_{p+q}) = \sum_{i=1}^{p+q} b_i x_i$ for $l_c(u) = cu$, and

(C_2) $\sup_i |\partial_i h(f)(\xi) - b_i| \le C(m, M, f)|b - a|$ uniformly for $\xi \in [a, b]^{p+q}$ (or $[b, a]^{p+q}$), $m \le a, b \le M$, and $c = f'(\zeta)$ for some $\zeta \in [a, b]$ (or $[b, a]$); the b_i are determined by c through condition (C_1).

To understand the nature of these conditions, suppose we have a numerical method for solving a nonlinear conservation law (CL) with initial condition (3.1). We want to show that the numerical behavior of this method is closely related to a numerical scheme for the linear equation (LCL) with the same initial condition, provided we choose $c = f'(\zeta)$, with ζ between a and b and a, b close to each other.

Essentially, (C_1) and (C_2) express the fact that the nonlinearity of the numerical scheme for the nonlinear conservation law is due only to the nonlinearity of the differential flux f, and not to the numerical method being based on nonlinear approximation.

For example, it is easy to see that these conditions are satisfied by the modified Lax–Friedrichs scheme, by the Engquist–Osher scheme, and by Godunov's scheme in the simple case when it reduces to an upwind scheme.

We want to estimate $E_N := \|u_{\Delta x}(\,\cdot\,, T) - v_{\Delta x}(\,\cdot\,, T)\|_{L^1(\mathbb{R})} = \|u_{\Delta x}(\,\cdot\,, N\Delta t) - v_{\Delta x}(\,\cdot\,, N\Delta t)\|_{L^1(\mathbb{R})}$. To do this, we use an idea of Harabetian [Hb], which is essentially induction on N. Let $\Delta t' := T/(N+1)$ and let $\Delta x'$ be the corresponding x-grid scaling parameter. Then

$$
\begin{aligned}
u_{\Delta x'}(\,\cdot\,, T) &- v_{\Delta x'}(\,\cdot\,, T)\\
&= u_{\Delta x'}(\,\cdot\,, (N+1)\Delta t') - v_{\Delta x'}(\,\cdot\,, (N+1)\Delta t')\\
&= \delta_{N+1}\, S_1^{N+1}\, u_0 - \delta_{N+1}\, S_2^{N+1}\, u_0\\
&= \delta_{1+1/N}\, \delta_N\, S_1\, \delta_{1/N}\, \delta_N\, S_1^N\, u_0 - \delta_{1+1/N}\, \delta_N\, S_1\, \delta_{1/N}\, \delta_N\, S_2^N\, u_0\\
&+ \delta_{1+1/N}\, \delta_N\, S_1\, \delta_{1/N}\, \delta_N\, S_2^N\, u_0 - \delta_{1+1/N}\, \delta_N\, S_2\, \delta_{1/N}\, \delta_N\, S_2^N\, u_0\ .
\end{aligned}
\tag{3.5}
$$

Applying the triangle inequality to (3.5), we obtain

$$
\begin{aligned}
E_{N+1} &\le \left\|\delta_{1+1/N}\left[(\delta_N S_1 \delta_{1/N})\delta_N S_1^N u_0 - (\delta_N S_1 \delta_{1/N})\delta_N S_2^N u_0\right]\right\|_{L^1(\mathbb{R})}\\
&+ \ \left\|\delta_{1+1/N}\left[(\delta_N S_1 \delta_{1/N})\delta_N S_2^N u_0 - (\delta_N S_2 \delta_{1/N})\delta_N S_2^N u_0\right]\right\|_{L^1(\mathbb{R})}\\
&\le \frac{N}{N+1} E_N + \frac{N}{N+1}\left\|\delta_N\, S_1 \delta_{1/N}\, g - \delta_N\, S_2\, \delta_{1/N}\, g\right\|_{L^1(\mathbb{R})}\ ,
\end{aligned}
\tag{3.6}
$$

where $g = \delta_N\, S_2^N\, u_0$. To estimate the first term, we also used the fact that $\delta_N\, S_1\, \delta_{1/N}$ is a contraction in L^1. This follows from the fact that the numerical method is monotone and conservative and from the Crandall–Tartar lemma [CrTt]. Also, the method is total variation diminishing and thus preserves monotone profiles, so g is monotone and $|g|_{BV} = |b - a|$. Now, we estimate the second term in the inequality.

By using the mean value theorem, we have

$$
\begin{aligned}
&\delta_N\, S_1\, \delta_{1/N}\, g - \delta_N\, S_2\, \delta_{1/N}\, g \\
&= \frac{1}{d}\left[h(f)(g(x-(p-1)d/N),\dots,g(x+qd/N))\right. \\
&\left.-h(f)(g(x-pd/N),\dots,g(x+(q-1)d/N))\right] \\
&-\frac{1}{d}\left[h(l_c)(g(x-(p-1)d/N),\dots,g(x+qd/N))\right. \\
&\left.-h(l_c)(g(x-pd/N),\dots,g(x+(q-1)d/N))\right] \\
&= \frac{1}{d}\left(h(f)-h(l_c)\right)(\xi_1-\xi_2) \\
&= \frac{1}{d}\,\triangledown\left(h(f)-h(l_c)\right)(\xi)\cdot(\xi_1-\xi_2)\ ,
\end{aligned} \tag{3.7}
$$

where ξ is a point on the line segment determined by $\xi_1 = (g(x-(p-1)d/N), \dots, g(x+qd/N))$, and $\xi_2 = (g(x-pd/N),\dots,g(x+(q-1)d/N))$. We also note that ξ_1,ξ_2,ξ are all in $[a,b]^{p+q}$, and using (C_2) in (3.7), we have

$$
\begin{aligned}
&\|\delta_N\, S_1\, \delta_{1/N}\, g - \delta_N\, S_2\, \delta_{1/N}\, g\|_{L^1(\mathbb{R})} \\
&\le \frac{1}{d}C(m,M,f)|b-a|\frac{d}{N}\|\xi_1-\xi_2\|_{L^1(\mathbb{R})} \\
&\le \frac{p+q}{d}C(m,M,f)|b-a|\frac{d}{N}|g|_{BV} = C\frac{(b-a)^2}{N}\ ,
\end{aligned} \tag{3.8}
$$

where the constant C depends only on p,q,λ,m,M, and f. Hence, by using (3.8) in (3.6), we have

$$
E_{N+1} \le \frac{N}{N+1}E_N + \frac{C}{N+1}(b-a)^2\ . \tag{3.9}
$$

By repeatedly applying (3.9) for $N-1, N-2, \dots,$ we conclude with the following estimate for the error:

$$
E_N \le C(b-a)^2\ , \tag{3.10}
$$

where the constant C depends only on p,q,λ,m,M, and f.

Next, we estimate how close the exact solutions of the two problems are. Let $u(x,t)$ be the exact solution of (CL) with initial datum (3.1). To keep the description of the exact solution simple, we assume f to be strictly convex on $[m,M]$ and we have, when $a<b$ (the rarefaction case),

$$
u(x,T) = \begin{cases} a & \text{if } x < f'(a)T, \\ (f')^{-1}(x/T) & \text{if } f'(a)T \le x \le f'(b)T, \\ b & \text{if } f'(b)T < x \end{cases} \tag{3.11}
$$

for any $m \le a < b \le M$. If $a > b$ (the shock case), we have

$$
u(x,T) = \begin{cases} a & \text{if} \quad x < T(f(a)-f(b))/(a-b), \\ b & \text{if} \quad x > T(f(a)-f(b))/(a-b) \end{cases} \tag{3.12}
$$

for any $m \le b < a \le M$.

Let $v(x,t)$ be the exact solution of (LCL) with initial datum (3.1). Then

$$v(x,T) = \begin{cases} a & \text{if } x < cT, \\ b & \text{if } x > cT. \end{cases} \tag{3.13}$$

Taking $c = f'(\zeta)$ for some ζ in $[a,b]$ (or $[b,a]$), we obtain

$$\begin{aligned} \|u(\,\cdot\,,T) - v(\,\cdot\,,T)\|_{L^1(\mathbb{R})} &\leq (b-a)^2 T \sup_{a\leq\xi\leq b} |f''(\xi)| \\ &\leq C(m,M,f,T)(b-a)^2\ , \end{aligned} \tag{3.14}$$

uniformly for $m \leq a, b \leq M$.

Returning to the general case of a nonlinear flux f, we estimate from below the error for a single Riemann problem. Combining (3.10) and (3.14), we have

$$\begin{aligned} &\|u(\,\cdot\,,T) - u_{\Delta x}(\,\cdot\,,T)\|_{L^1(\mathbb{R})} \\ &\geq \|v(\,\cdot\,,T) - v_{\Delta x}(\,\cdot\,,T)\|_{L^1(\mathbb{R})} \\ &- \|u(\,\cdot\,,T) - v(\,\cdot\,,T)\|_{L^1(\mathbb{R})} - \|u_{\Delta x}(\,\cdot\,,T) - v_{\Delta x}(\,\cdot\,,T)\|_{L^1(\mathbb{R})} \\ &\geq \|v(\,\cdot\,,T) - v_{\Delta x}(\,\cdot\,,T)\|_{L^1(\mathbb{R})} - C(b-a)^2\ . \end{aligned} \tag{3.15}$$

The functions v and $v^N_{\Delta x}$ are solutions of an auxiliary linear transport equation with transport velocity $c = f'(\zeta)$ for the same ζ. The point is that for small $|b-a|$, the error is as in the linear case.

To summarize, we have proved the following theorem.

THEOREM 3.1. *We assume that for a given monotone, conservative, consistent finite difference method* (2.4), (2.5)*, there exist* m, M *such that* (C_1), (C_2) *hold. Furthermore, we assume* f *to be strictly convex on* $[m,M]$*. Let* u *be the exact solution of* (CL) *with initial condition* (3.1)*, and let* $u_{\Delta x}$ *be the corresponding approximate solution generated by the numerical method. Then,*

$$\begin{aligned} &\|u(\cdot,T) - u_{\Delta x}(\cdot,T)\|_{L^1(\mathbb{R})} \\ &\qquad \geq \|v(\cdot,T) - v_{\Delta x}(\cdot,T)\|_{L^1(\mathbb{R})} - C(m,M,f,T)(b-a)^2\ , \end{aligned} \tag{3.16}$$

where v *is the exact solution of* (LCL) *with initial datum* (3.1) *and* $v_{\Delta x}$ *is the corresponding approximate solution generated by the numerical method. The estimate is uniform with respect to the transport velocity in the linear problem,* $c = f'(\zeta)$ *for* ζ *in* $[a,b]$ *(or* $[b,a]$*) and uniform with respect to* a, b *such that* $m \leq a, b \leq M$.

4. The main result. We return now to the error analysis for the nonlinear conservation law

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \quad u(x,0) = u_0(x)\ . \tag{CL}$$

Throughout this section, we assume that we have fixed a monotone finite difference method that, together with the flux f, satisfies the hypotheses of Theorem 3.1. We define

$$u_0(x) := \sum_{i\geq 0} \sum_{1\leq j\leq k_i} c_i\ \chi_{I_{ij}}(x)\,, \tag{4.1}$$

where $c_i \geq 0, c_0 \geq c_1 \geq c_2 \geq \cdots$, k_i are nonnegative integers and the intervals I_{ij} are pairwise disjoint. In addition, the intervals I_{ij} are situated on the real axis in a sequence $I_{01}, \ldots, I_{0k_0}, \ldots, I_{11}, \ldots, I_{1k_1}, \ldots$.

The CFL number is determined by $\|u_0\|_{L^\infty(\mathbb{R})} = c_0$ and f. For simplicity, we set $T = 1$ and we take $\Delta t = 1/N$ and $\Delta x = 1/\lambda N = d/N$. For technical reasons, we will assume N to be an integer power of 2. We carry the numerical computation for N steps on the uniform grid, with Δt and Δx as defined above.

First, we choose the intervals I_{ij} such that the following two conditions are satisfied uniformly for N:

$$P_{\Delta x} u_0 = u_0 \quad \text{(see (2.2))}, \tag{4.2}$$

$$\|u(\cdot,1) - u_{\Delta x}(\cdot,1)\|_{L^1(\mathbb{R})} = \sum_{i,j} \|u^{ij}(\cdot,1) - u^{ij}_{\Delta x}(\cdot,1)\|_{L^1(\mathbb{R})} \ , \tag{4.3}$$

where u^{ij} and $u^{ij}_{\Delta x}$ are the exact solutions and the numerical approximations corresponding to the initial data $c_i\ \chi_{I_{ij}}(x)$. To obtain (4.2) and (4.3), we fix the endpoints of each interval to be points from $d\mathbb{Z}$. Then, we take the length of each interval to be $(p+q+1)d$, and we set the distance between any two adjacent intervals to be at least $(p+q+1)d$. This guarantees that after N steps in the computation, the disturbances propagating from each interval endpoint will not interact with each other, neither in the exact solution nor in the numerical solution.

Indeed, for the numerical solution we see that after one iteration, the value at one gridpoint can influence at most q cells to the left and p cells to the right, for a total of $(p+q+1)$ influenced cells at the next time level. Since we are iterating the scheme N times, and the size of one cell is $\Delta x = d/N$, we obtain a domain of influence of length $(p+q+1)d$. For the exact solution, the assertion is also true, since the CFL condition says that the domain of numerical propagation should include the domain of maximum propagation of the exact solution.

Now, we need to define the constants c_i and k_i. We want $u_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R})$. Calculating the L^1-norm and the BV-norm of u_0,

$$\|u_0\|_{L^1(\mathbb{R})} = \sum_{i\geq 0} k_i c_i (p+q+1)d = (p+q+1)d \sum_{i\geq 0} k_i c_i, \tag{4.4}$$

$$\|u_0\|_{BV(\mathbb{R})} = \sum_{i\geq 0} k_i\, 2\, c_i = 2\sum_{i\geq 0} k_i c_i \ , \tag{4.5}$$

we conclude that $u_0 \in L^1 \cap BV(\mathbb{R})$ if and only if $\sum_{i\geq 0} k_i c_i < +\infty$.

We recall that, under the assumption (C_1), the numerical method applied to a linear problem gives a linear numerical scheme

$$v_j^{n+1} = \sum_{s=-p}^{q} a_s v_{j+s}^n \ . \tag{4.6}$$

Returning to the error estimates, we combine (3.16) with (2.21) to obtain

$$\begin{aligned} \|u^{ij} - u^{ij}_{\Delta x}\|_{L^1(\mathbb{R})} &\geq 2c(p,q)\, c_i \sum_{s\neq s_0} \sqrt{a_s^i}\, d\, N^{-1/2} - 2C(0,c_0,f,1)\, c_i^2 \\ &= 2c_i \left(d\, c(p,q) \sum_{s\neq s_0} \sqrt{a_s^i}\, N^{-1/2} - C(0,c_0,f,T)\, c_i \right) \ . \end{aligned} \tag{4.7}$$

The coefficients a_s^i are those determined by (4.6). The differential flux of the auxiliary linear problem is $l_c(u) = cu$ with $c = f'(\zeta)\, , 0 < \zeta \leq c_i$.

We need a uniform estimate from below for $\sum_{s\neq s_0}\sqrt{a_s^i}$. We shall make the following assumption concerning the flux f and the numerical method we have chosen: there exist $M>0$ and $\beta>0$, such that for all $0<\zeta<M$,

$$\sum_{s\neq s_0}\sqrt{a_s}\geq\beta>0 \tag{4.8}$$

uniformly, where a_s are the coefficients from (4.6) and $l_c(u)=cu$ with $c=f'(\zeta)$. Here are the coefficients a_s in two particular instances.

(1) For the modified Lax–Friedrichs scheme:

$$a_{-1}=(Q+\lambda c)/2\,,\quad a_0=1-Q\,,\quad a_1=(Q-\lambda c)/2\ . \tag{4.9}$$

(2) For the upwind scheme (which in the linear case under consideration coincides with both the Engquist–Osher and Godunov schemes):

$$a_{-1}=\lambda c\ ,a_0=1-\lambda c\ ,\ \text{if } c>0\ ,a_0=1+\lambda c\ ,a_1=-\lambda c\ ,\ \text{if } c<0\ . \tag{4.10}$$

In the first case, $\sum_{s\neq s_0}\sqrt{a_s}\geq\beta>0$ is satisfied always. In the second case, if $f'(0)=0$, it is no longer satisfied since $\lambda c\to 0$ when $c=f'(\zeta)$ and $\zeta\to 0$. In case $f'(0)\neq 0$, there are no problems with the upwind scheme either.

Returning to the error estimate, we have under the assumption (4.8)

$$\sum_{j=1}^{k_i}\|u^{ij}(\cdot,1)-u^{ij}_{\Delta x}(\cdot,1)\|_{L^1(\mathbb{R})}\geq Ak_ic_i(N^{-1/2}-Bc_i) \tag{4.11}$$

for all i, with $A,B>0$ independent of i and N. From (4.3) and (4.11) we get

$$\begin{aligned}\|u(\cdot,1)-u_{\Delta x}(\cdot,1)\|_{L^1(\mathbb{R})}&\geq\sum_{j=1}^{k_i}\|u^{ij}(\cdot,1)-u^{ij}_{\Delta x}(\cdot,1)\|_{L^1(\mathbb{R})}\\&\geq Ak_ic_i(N^{-1/2}-Bc_i)\end{aligned} \tag{4.12}$$

for all i and N. Now we fix $\alpha>1$, and we let $c_i=2^{-i}$, $k_i=[2^i i^{-\alpha}]+1$. With this choice of c_i and k_i, we have

$$\begin{aligned}\sum_{i\geq 0}k_ic_i&=\sum_{i\geq 0}2^{-i}([2^i i^{-\alpha}]+1)\\&\leq 2+\sum_{i\geq 1}i^{-\alpha}<+\infty\ .\end{aligned} \tag{4.13}$$

Thus, $u_0\in L^1\cap L^\infty\cap BV(\mathbb{R})$. Now, given N an integer power of 2, $\Delta t=1/N$, $\Delta x=d/N$, let $i(N)=\log_2 N$. From (4.12) with $i=i(N)$ we obtain

$$\begin{aligned}\|u(\cdot,1)-u_{\Delta x}(\cdot,1)\|_{L^1(\mathbb{R})}&\geq Ak_{i(N)}c_{i(N)}(N^{-1/2}-Bc_{i(N)})\\&=A([2^{i(N)}i(N)^{-\alpha}]+1)2^{-i(N)}(N^{-1/2}-B2^{-i(N)})\\&\geq AN(\log_2 N)^{-\alpha}N^{-1}(N^{-1/2}-BN^{-1})\\&=A(\log_2 N)^{-\alpha}N^{-1/2}(1-BN^{-1/2})\\&\geq C(\log_2 N)^{-\alpha}N^{-1/2}\end{aligned} \tag{4.14}$$

for $N \geq N_0$ such that $1 - BN^{-1/2} > 1/2$, which is the same as requiring that $\Delta x < \varepsilon_0$ for some $\varepsilon_0 > 0$. In other words, we proved that

$$\|u(\cdot,1) - u_{\Delta x}(\cdot,1)\|_{L^1(\mathbb{R})} \geq C(\alpha)|\log_2 \Delta x|^{-\alpha}\sqrt{\Delta x} \tag{4.15}$$

for a sequence $\Delta x = d/N < \varepsilon_0$ for any fixed $\alpha > 1$.

Thus, we have proved the following theorem.

THEOREM 4.1. *We assume that for a given monotone numerical method and a given differential flux, the hypotheses of Theorem* 3.1 *are satisfied. Furthermore, we assume that* (4.8) *holds. Then*

$$\begin{aligned} &\sup\{s > 0 : \text{ for all } T > 0 \text{ there exists } C_T > 0 \text{ such that} \\ &\text{for all } u_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}), \|u(\cdot,T) - u_{\Delta x}(\cdot,T)\|_{L^1(\mathbb{R})} \leq C_T(\Delta x)^s\} = 1/2 \ , \end{aligned} \tag{4.16}$$

where u and $u_{\Delta x}$ are the exact and numerical solutions to (CL) *for a uniform grid with meshsize Δx.*

Remark. As we noted earlier, in the case of the modified Lax–Friedrichs scheme, all the hypotheses of the theorem hold if f is strictly convex. However, in the case of an upwind scheme (to which Godunov's scheme would reduce if $f' > 0$), (4.8) might fail to hold if $f'(0) = 0$. Specifically, our example doesn't work for Godunov's method applied to Burgers's equation. In spite of this, if we drop the requirement $u_0 \in L^1(\mathbb{R})$, we obtain an analogous result replacing u_0 defined by (4.1) and the paragraph following by $\tilde{u}_0 = K + u_0$, with K some positive constant. Then $\tilde{u}_0 \in L^\infty \cap BV(\mathbb{R})$ and (4.16) becomes

$$\begin{aligned} &\sup\{s > 0 : \text{ for all } T > 0 \text{ there exists } C_T > 0 \text{ such that} \\ &\text{for all } u_0 \in L^\infty \cap BV(\mathbb{R}), \|u(\cdot,T) - u_{\Delta x}(\,\cdot\,,T)\|_{L^1(\mathbb{R})} \leq C_T(\Delta x)^s\} = 1/2 \ . \end{aligned} \tag{4.17}$$

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