Chapter 18

Entropy Stable Schemes

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Chapter Outline

ABSTRACT
We review the topic of entropy stability of discrete schemes, finite-difference and finite-volume schemes, for the approximate solution of nonlinear systems of conservation laws. The question of entropy stability plays an important role in both, the theory and computation of such systems, which is reflected by the extensive literature on this topic. Here we focus on a several key ingredients in the study of entropy stable schemes. Our main theme is the investigation of entropy stability using a *comparison principle*. Thus for example, the entropy stability of scalar *monotone* schemes follows from a comparison with constant solutions, and the more general E-schemes are stable

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by a comparison with Godunov solvers. For system of conservation laws, entropy stability is investigated by comparing the amount of their *numerical viscosity* with that of certain *entropy conservative* fluxes. These ingredients are explored in the context of first- and second-order schemes and lend themselves to *higher-order methods* and multidimensional schemes on *unstructured grids*.

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1 ENTROPIC SYSTEMS OF CONSERVATION LAWS

We are concerned with discrete approximation to hyperbolic systems of conservation laws. These laws take the form,

$$\frac{\partial}{\partial t}\mathbf{u}+\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\mathbf{f}^{(j)}(\mathbf{u})=0, \quad \mathbf{x}=(x_1,\ldots,x_d)\in\Omega, \qquad t\in\mathbb{R}_+, \tag{1}$$

governing the flow of n conservative variables,[a] $\mathbf{u}(x, t) = (u_1(x, t), \ldots, u_n(x, t))^\top$, by their fluxes, $\mathbf{f}(\mathbf{u})=\left(\mathbf{f}^{(1)},\ldots,\mathbf{f}^{(d)}\right)$, where $\mathbf{f}^{(j)}(\mathbf{u}):\mathbb{R}^n\mapsto\mathbb{R}^n$. We consider the Cauchy problem, where a solution of (1) is sought subject to prescribed initial data $\mathbf{u}_0(x)$, in either the whole space, $\Omega=\mathbb{R}^d$, or the periodic torus, $\Omega=\mathbb{T}^d$; in either case, there are no contributions from the boundaries. The system is hyperbolic if the eigenvalues of the $n \times n$ symbol, $\sum_j \xi_j A_j(\mathbf{u}),\ A_j(\mathbf{u}):=\partial_{\mathbf{u}}\mathbf{f}^{(j)}(\mathbf{u})$, are real for all real $\xi = (\xi_1, \ldots, \xi_d)$.

The progress in the development of mathematical theory for nonlinear systems of conservation laws was summarized over the years in a series of monographs and books, starting with the classical work by Courant and Hilbert (1962) and followed by Lax (1973), Smoller (1983), Whitham (1999), Serre (2000), Bressan (2000), Dafermos (2016), as well as a series of biannual conferences devoted to the theory, numerics and applications of hyperbolic problems (Hyp. series, 1984–2016).

The study of such systems was motivated, to a large extent, by the canonical example of the compressible Euler equations.

Example 1 (Euler equations). The compressible Euler equations given by

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \mathbf{m}\\ E\end{bmatrix}+\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\begin{bmatrix}m_j\\ v_j\mathbf{m}+p\\ v_j(E+p)\end{bmatrix}=0, \tag{2}$$

express the conservative flow of density ρ, the d-dimensional momentum $\mathbf{m} := \rho\mathbf{v}$, and (total) energy E, in terms of the fluxes $\mathbf{f}^{(j)}(\mathbf{u}) = (\rho v_j, \rho v_j\mathbf{v}+p,$

[a]Here and later, scalars are distinguished from vectors which are denoted by **bold** letters.

$v_j(E+p))^\top$, where the closure for the pressure is determined by the γ-law, $p := (\gamma - 1)(E - \rho|\mathbf{v}|^2/2)$.

Euler equations (2) admit yet another conservation law which is expressed in terms of the *specific entropy* $S := \ln(p\rho^{-\gamma})$,

$$\frac{\partial}{\partial t}(-\rho S) + \sum_{j=1}^{d}\frac{\partial}{\partial x_j}(-\rho v_j S) = 0. \tag{3}$$

The last equality, which follows by *formal* manipulations of (2), asserts the conservation of the entropy $\eta(\mathbf{u}) = -\rho S$ in terms of the entropy flux $F(\mathbf{u}) = -\rho \mathbf{v} S$. This motivates the notion of *entropy pairs* for general system of conservation laws.

1.1 Entropy Pairs

An *entropy pair* associated with (1) consists of a convex entropy $\eta : \mathbb{R}^n \mapsto \mathbb{R}$ and the corresponding entropy flux $\mathbf{F} = \left(F^{(1)}, \ldots, F^{(d)}\right) : \mathbb{R}^n \mapsto \mathbb{R}^d$, such that the following compatibility relations hold[b]

$$\eta'(\mathbf{u})A_j(\mathbf{u}) = F^{(j)\prime}(\mathbf{u}), \quad A_j(\mathbf{u}) = \frac{\partial}{\partial \mathbf{u}}\mathbf{f}^{(j)}(\mathbf{u}), \;\; j = 1, \ldots, d. \tag{4}$$

The existence of such compatible entropy pair allows us to proceed with the following *formal* manipulation

$$\begin{aligned}\eta(\mathbf{u})_t + \nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{u}) &= \langle \eta'(\mathbf{u}), \mathbf{u}_t\rangle + \sum_{j=1}^{d}\left\langle F^{(j)\prime}(\mathbf{u}), \mathbf{u}_{x_j}\right\rangle \\ &= \left\langle \eta'(\mathbf{u}), \mathbf{u}_t + \sum_{j=1}^{d} A_j(\mathbf{u})\mathbf{u}_{x_j}\right\rangle = \langle \eta'(\mathbf{u}), \mathbf{u}_t + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u})\rangle = 0.\end{aligned} \tag{5}$$

Thus, if $\mathbf{u}$ is a classical C^1-solution of the conservation law (1), then the pair $(\eta(\mathbf{u}), F(\mathbf{u}))$ forms a conservative extension of it, in complete analogy to the conservation of physical entropy in Euler equations (3). The convexity of $\eta = \eta(\cdot)$ signifies that $\eta(\mathbf{u})$ is a nontrivial extension, beyond the obvious conserved linear combinations $\mathbf{c} \cdot \mathbf{u}$. Thus for example, the judicious minus sign in (3) is chosen to make the corresponding Euler's entropy, $\eta(\mathbf{u}) = -\rho S$, a convex entropy function of the conservative variables ρ, $\mathbf{m}$ and E.

Nonlinear conservation laws may admit one or more entropy pairs, or none at all. This depends on whether there exists a Hessian $\eta''(\mathbf{u})$ which symmetrizes the Jacobians $A_j(\mathbf{u}) = \partial_{\mathbf{u}}\mathbf{f}^{(j)}(\mathbf{u})$. Observe that systems which do admit an entropic extension are necessarily (symmetric) hyperbolic: since terms on the right of the identity (which follows by differentiation of (4)),

[b] We let prime denotes the gradient w.r.t. to specified variable, $X'(\mathbf{u}) := (X_{u_1}, \ldots, X_{u_n})$.

$\eta'' A_j \equiv F_j'' - \eta' A_j'$, are symmetric, the A_j's are *symmetrizable* and hence have a complete set of real eigen-system. Scalar equations have all convex functions as admissible entropy functions. One-dimensional system in n unknowns admit entropy functions for $n = 2$, but the overdetermined symmerazibility condition, $\eta'' A_j = A_j^\top \eta''$, may fail for $n \times n$ systems with $n > 2$ unknowns. We focus our discussion on *entropic systems of conservation laws*—those that are endowed with at least one entropy pair. Most "physically relevant" systems, Euler equations, the shallow-water equations, MHD equations, etc., are entropic.

1.2 Entropy Inequality

The generic phenomena associated with these nonlinear equations are the finite-time breakdown of differentiability of their solutions. Thereafter, one must admit *weak solutions*, where (1) are interpreted in distribution sense (Dafermos, 2016). Among the possibly many weak solutions, physically relevant solutions are postulated as those realized by *vanishing viscosity* limits, $\mathbf{u} = \lim_{\epsilon \downarrow 0} \mathbf{u}^\epsilon$, where

$$\frac{\partial}{\partial t}\mathbf{u}^\epsilon + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\mathbf{f}^{(j)}(\mathbf{u}^\epsilon) = \epsilon \Delta \mathbf{u}^\epsilon. \tag{6}$$

In this context of weak solutions, one cannot proceed with the formal manipulations (5) which led to the entropy equality $\eta(\mathbf{u})_t + \nabla_\mathbf{x} \cdot \mathbf{F}(\mathbf{u}) = 0$. Instead, arguing along the lines of (5) while using the convexity of $\eta(\cdot)$, we end up with

$$\begin{aligned}\eta(\mathbf{u}^\epsilon)_t + \nabla_\mathbf{x} \cdot \mathbf{F}(\mathbf{u}^\epsilon) &= \left\langle \eta'(\mathbf{u}^\epsilon), \mathbf{u}_t^\epsilon + \nabla_\mathbf{x} \cdot \mathbf{f}(\mathbf{u}^\epsilon) \right\rangle \\ &= -\epsilon \langle \eta'(\mathbf{u}^\epsilon), \Delta \mathbf{u}^\epsilon \rangle \equiv \epsilon \Delta \eta(\mathbf{u}^\epsilon) - \epsilon \langle \nabla_\mathbf{x} \mathbf{u}^\epsilon, \eta''(\mathbf{u}^\epsilon) \nabla_\mathbf{x} \mathbf{u}^\epsilon \rangle \leq \epsilon \Delta \eta(\mathbf{u}^\epsilon).\end{aligned}$$

It follows that boundedly a.e. limits of vanishing viscosity solutions satisfy the *entropy inequality*,

$$\eta(\mathbf{u})_t + \nabla_\mathbf{x} \cdot \mathbf{F}(\mathbf{u}) \leq 0. \tag{7}$$

A weak solution of (1) is *entropic* if it satisfies the *entropy inequality* (7) *for all* admissible entropy pairs $(\eta, \mathbf{F})$ associated with (1). This notion of entropy solution is the cornerstone for the theory of hyperbolic systems of nonlinear conservation laws. We mention here the pioneering contributions (Godunov, 1961; Kruzkhov, 1970, §7; Friedrichs and Lax, 1971; Lax, 1957, 1971).

1.3 The One-Dimensional Setup

The entropy inequality involves an entropy flux whose components, $\mathbf{F} = (F^{(1)}, \ldots, F^{(d)})^\top$, are aligned with the Cartesian coordinates and sought to satisfy the compatibility requirement (4), one component at the time. We can therefore reduce the question of entropic solution to the one-dimensional case, where the conservative variables $\mathbf{u} = (u_1, \ldots, u_n)^\top$ are balanced by the flux $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \ldots, f_n(\mathbf{u}))^\top$,

$$\frac{\partial}{\partial t}\mathbf{u}(x,t) + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}(x,t)) = 0, \quad (x,t) \in \Omega \times \mathbb{R}_+. \tag{8}$$

The system is augmented with (one or more) entropy inequalities

$$\frac{\partial}{\partial t}\eta(\mathbf{u}(x,t)) + \frac{\partial}{\partial x}F(\mathbf{u}(x,t)) \leq 0, \tag{9}$$

which should hold for all admissible entropy pairs (η, F), satisfying the compatibility condition (4), $\eta' A = F'$, realizing the boundedly a.e. limits of vanishing viscosity limit (Bianchini and Bressan, 2005). Again, the prototype example is the one-dimensional Euler equations, where the density, momentum, $m := \rho v$, and energy, $\mathbf{u} = (\rho, m, E)^\top$ are balanced by the flux $\mathbf{f}(\mathbf{u}) = (\rho v, \rho v^2 + p, v(E+p))^\top$. One seeks entropic solutions which satisfy, in addition, the entropy inequality $(-\rho S)_t + (-\rho v S)_x \leq 0$.

2 DISCRETE APPROXIMATIONS AND ENTROPY STABILITY

Weak solutions of (8) can be observed in terms of their (sliding) averages, $\bar{\mathbf{u}}(x,t) := \frac{1}{\Delta x}\int_{I_x} \mathbf{u}(y,t)\mathrm{d}y$ across the cell $I_x := \left[x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2}\right]$. Integrating (8) over the *control volume* $\Omega_\Delta := I_x \times [t, t + \Delta t]$ we find

$$\frac{\bar{\mathbf{u}}(x,t+\Delta t) - \bar{\mathbf{u}}(x,t)}{\Delta t} = -\frac{1}{\Delta x}\left[\tilde{\mathbf{f}}\left(x + \frac{\Delta x}{2}\right) - \tilde{\mathbf{f}}\left(x - \frac{\Delta x}{2}\right)\right]. \tag{10}$$

This reflects the balance between the difference of spatial averages on the left and the *temporal* averages of fluxes on the right, $\tilde{\mathbf{f}}\left(x + \frac{\Delta x}{2}\right) := \int_{\tau=t}^{t+\Delta t} \mathbf{f}\left(\mathbf{u}\left(x + \frac{\Delta x}{2}, \tau\right)\right)\mathrm{d}\tau$.

We are interested in computation of approximate entropy solutions of (8) and (9). To this end we now *fix* the mesh ratio between a small time-step and the size of spatial cells, $\lambda := \frac{\Delta t}{\Delta x}$, and we consider the corresponding class of *conservative schemes* of the form

$$\mathbf{u}_\nu(t+\Delta t) = \mathbf{u}_\nu(t) - \frac{\Delta t}{\Delta x}\left(\mathbf{f}_{\nu+\frac{1}{2}} - \mathbf{f}_{\nu-\frac{1}{2}}\right). \tag{11a}$$

Here, $\mathbf{u}_\nu(t)$ denotes the discrete solution, viewed as an approximate *cell average* $\mathbf{u}_\nu(t) \approx \bar{\mathbf{u}}(\mathbf{x}_\nu, t)$ centred at (x_ν, t). At the heart of matter are the *numerical* *fluxes* depending on $2p$ neighbouring gridvalues,[c] $\mathbf{f}_{\nu+\frac{1}{2}} := \mathbf{f}(\mathbf{u}_{\nu-p+1}(t), \ldots, \mathbf{u}_{\nu+p}(t))$, which approximate the differential flux, $\mathbf{f}_{\nu+\frac{1}{2}} \approx \overline{\mathbf{f}}(x_{\nu+\frac{1}{2}}, t)$, and in particular, are consistent with the differential flux in the sense that

$$\mathbf{f}(\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u}) \equiv \mathbf{f}(\mathbf{u}). \tag{11b}$$

The framework of conservative difference schemes (11) was initiated in the seminal paper of Lax and Wendroff (1960).

2.1 Examples

We mention four canonical examples.

Use forward differencing in time and centred differencing in space to discretize (8). The resulting so-called forward Euler scheme reads, $\mathbf{u}_\nu(t+\Delta t) = \mathbf{u}_\nu(t) - \frac{\Delta t}{2\Delta x}(\mathbf{f}(\mathbf{u}_{\nu+1}(t)) - \mathbf{f}(\mathbf{u}_{\nu-1}(t)))$, which is associated with the numerical flux $\mathbf{f}^{\rm FE}_{\nu+\frac{1}{2}} = \frac{1}{2}(\mathbf{f}(\mathbf{u}_\nu) + \mathbf{f}(\mathbf{u}_{\nu+1}))$. It is a prototype example for an *unstable* scheme due to amplification of high-modes and lack of numerical dissipation to tame their unbounded growth.

The *Lax–Friedrichs scheme* is the canonical example for a robust numerical solver

$$\mathbf{u}_\nu(t+\Delta t) = \frac{\mathbf{u}_{\nu+1}(t) + \mathbf{u}_{\nu-1}(t)}{2} - \frac{\Delta t}{2\Delta x}(\mathbf{f}(\mathbf{u}_{\nu+1}(t)) - \mathbf{f}(\mathbf{u}_{\nu-1}(t))). \tag{12}$$

It is associated with numerical flux[d] $\mathbf{f}^{\rm LxF}_{\nu+\frac{1}{2}} = \frac{1}{2}(\mathbf{f}(\mathbf{u}_\nu) + \mathbf{f}(\mathbf{u}_{\nu+1})) - \frac{1}{2\lambda}\Delta \mathbf{u}_{\nu+\frac{1}{2}}$.

The *Lax–Wendroff scheme* is the prototypical example of a second-order accurate finite-difference scheme

$$\begin{aligned} \mathbf{u}_\nu(t+\Delta t) = \ & \mathbf{u}_\nu(t) - \frac{\Delta t}{2\Delta x}(\mathbf{f}(\mathbf{u}_{\nu+1}(t)) - \mathbf{f}(\mathbf{u}_{\nu-1}(t))) \\ & + \frac{(\Delta t)^2}{2(\Delta x)^2}\left(A_{\nu+\frac{1}{2}}\Delta \mathbf{f}_{\nu+\frac{1}{2}} - A_{\nu-\frac{1}{2}}\,\Delta \mathbf{f}_{\nu-\frac{1}{2}}\right). \end{aligned} \tag{13}$$

Here, $A_{\nu+\frac{1}{2}}$ is the mid-value of the Jacobian such that $\Delta \mathbf{f}_{\nu+\frac{1}{2}} = A_{\nu+\frac{1}{2}}\Delta \mathbf{u}_{\nu+\frac{1}{2}}$. The corresponding flux is found to be $\mathbf{f}^{\rm LxW}_{\nu+\frac{1}{2}} = \frac{1}{2}(\mathbf{f}(\mathbf{u}_\nu) + \mathbf{f}(\mathbf{u}_{\nu+1})) - \frac{\lambda}{2}A^2_{\nu+\frac{1}{2}}\Delta \mathbf{u}_{\nu+\frac{1}{2}}$.

Godunov's scheme (Godunov, 1959) is the forerunner for the class of finite-volume schemes. It evolves a piecewise-constant approximate solution

[c]Remark that the numerical flux involves a stencil of $2p$ neighbouring grid values centred at half-indexed gridpoints, $\mathbf{f}(\cdot, \cdot, \ldots, \cdot) \rightsquigarrow \mathbf{f}_{\nu\pm\frac{1}{2}}$, and as such, could be clearly distinguished from the (same notation of) the differential flux tagged at integer indexed gridpoints, $\mathbf{f}(\cdot) \rightsquigarrow \mathbf{f}_\nu$.

[d]For a given gridfunction $\{X_\nu\}$ we let $\Delta X_{\nu+\frac{1}{2}}$ denote the forward difference, $\Delta X_{\nu+\frac{1}{2}} := X_{\nu+1} - X_\nu$, centred at $x_{\nu+\frac{1}{2}}$. Thus for example, $\Delta \mathbf{f}_{\nu+\frac{1}{2}} = \mathbf{f}(\mathbf{u}_{\nu+1}(t)) - \mathbf{f}(\mathbf{u}_\nu(t))$ and $\Delta \mathbf{u}_{\nu+\frac{1}{2}} = \mathbf{u}_{\nu+1}(t) - \mathbf{u}_\nu(t)$.

$\mathbf{u}_\Delta(x,t) := \sum_\nu \mathbf{u}_\nu(t)\boldsymbol{I}_{I_{x_\nu}}(x)$, using the *exact* entropic solution operator. The "pushed-forward" solution, $\{\mathbf{u}_\Delta(x, \tau), \tau > t\}$ is then realized at $\tau = t + \Delta t$ in terms of its cell-averages, $\mathbf{u}_\nu(t+\Delta t) = \frac{1}{\Delta x}\int_{I_{x_\nu}} \mathbf{u}_\Delta(y, t+\Delta t)dy$: appealing to (10) we find that these cell averages, $\mathbf{u}_\nu(t + \Delta t)$, satisfy

$$\mathbf{u}_\nu(t+\Delta t) = \mathbf{u}_\nu(t) - \frac{\Delta t}{\Delta x}\left(\mathbf{f}\left(\mathbf{u}_R\left(x_{\nu+\frac{1}{2}}, t+\frac{\Delta t}{2}\right)\right) - \mathbf{f}\left(\mathbf{u}_R\left(x_{\nu-\frac{1}{2}}, t+\frac{\Delta t}{2}\right)\right)\right). \tag{14}$$

Here, $\mathbf{u}_R\left(x_{\nu+\frac{1}{2}}, t+\frac{\Delta t}{2}\right)$ is the centred value of the entropic solution for the Riemann fan which resolves the discontinuous jump from $\mathbf{u}_\ell = \mathbf{u}_{\nu-\frac{1}{2}}$ to $\mathbf{u}_r = \mathbf{u}_{\nu+\frac{1}{2}}$ at $(x_{\nu+\frac{1}{2}}, t)$.

2.2 Entropy Stability

Let (η, F) be an entropy pair associated with the system (8). The scheme (11) is *entropy stable* w.r.t. such a pair, if it satisfies a discrete entropy inequality analogous to the entropy inequality $\eta(\mathbf{u})_t + F(\mathbf{u})_x \leq 0$, namely, if

$$\eta(\mathbf{u}_\nu(t+\Delta t)) \leq \eta(\mathbf{u}_\nu(t)) - \frac{\Delta t}{\Delta x}\left(F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}\right). \tag{15}$$

Here, $F_{\nu+\frac{1}{2}} := F\left(\mathbf{u}_{\nu-p+1}(t), \ldots, \mathbf{u}_{\nu+p}(t)\right)$ is a *numerical entropy flux* which is consistent with the differential one, $F(\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u}) = F(\mathbf{u})$.

The development of numerical methods for approximate solution of nonlinear conservation laws paralleled the development of the analytical theory. It was driven, to a large extent, by the need for scientific computation of stable, high-resolution simulations which in many cases, superseded the analytical theories at the time. We mention the pioneering work of von Neumann (Lax, 2014; von Neumann and Richtmyer, 1950). The progress in the development of numerical methods for nonlinear systems of conservation laws was summarized over the years in a series of monographs and books, starting with the classical work by Richtmyer and Morton (1967) and followed by LeVeque (1992), Godlewski and Raviart (1996), Cockburn et al. (1997), LeVeque (2002) and Gustafsson et al. (2013).

3 ENTROPY STABLE SCHEMES FOR SCALAR CONSERVATION LAWS

3.1 Monotone Schemes

A main feature of *scalar* conservation laws is *monotonicity*. Let $u_1(\cdot, t)$ and $u_2(\cdot, t)$ be two entropy solutions of the scalar law $u_t + f(u)_x = 0$ subject to two different initial data, u_{10} and u_{20}, and assume that u_{20} *dominates* u_{10}, denoted $u_{20} \succeq u_{10}$, in the sense that $u_{20}(x) \geq u_{10}(x) \quad \forall x$. Then $u_2(\cdot, t) \succeq u_1(\cdot, t)$. Namely, a relative ordering among entropy solutions propagates in time. This follows at once from the corresponding ordering of the viscosity solutions, $u_{20} \succeq u_{10} \rightsquigarrow u_2^\epsilon(\cdot,t) \succeq u_1^\epsilon(\cdot,t)$. We turn to the discrete case,

$$u_\nu(t+\Delta t) = u_\nu(t) - \frac{\Delta t}{\Delta x}\Big(f_{\nu+\frac{1}{2}} - f_{\nu-\frac{1}{2}}\Big), \quad f_{\nu+\frac{1}{2}} = f(u_{\nu-p+1}(t), \ldots, u_{\nu+p}(t)). \tag{16}$$

The scheme is *monotone* if $u_\nu(t+\Delta t)$ is an increasing function of its $2p+1$ arguments, $(u_{\nu-p}(t), \ldots, u_{\nu+p}(t))$ on the right of (16). Let $\mathbf{u}(t) := (\ldots, u_{\nu-1}(t), u_\nu(t), u_{\nu+1}(t), \ldots)$ and $\boldsymbol{v}(t)$ be two different discrete states at time level t. Monotone schemes propagate their order, namely, if $\mathbf{u}(t) \succeq \boldsymbol{v}(t)$ in the sense that $u_\nu(t) \geq v_\nu(t) \ \forall \nu$, then $\mathbf{u}(t+\Delta t) \succeq \boldsymbol{v}(t+\Delta t)$.

The entropy stability of monotone schemes, originally due to Harten et al. (1976) and Sanders (1983), follows from a comparison with the constant solution,[e] $\mathbf{c} := (\ldots, c, c, c, \ldots)$. We recall here the elegant argument of Crandall and Majda (1980). Consider the discrete grid function $(\mathbf{u}(t) \vee \mathbf{c})_\nu := \max\{u_\nu(t), c\}$: since $\mathbf{u}(t) \vee \mathbf{c}$ dominates both — $\mathbf{u}(t)$ and $\mathbf{c}$, monotonicity implies

$$(u_\nu(t+\Delta t) \vee c) \leq (u_\nu(t) \vee c) - \frac{\Delta t}{\Delta x}\Big(f_{\nu+\frac{1}{2}}(\mathbf{u}(t) \vee \mathbf{c}) - f_{\nu-\frac{1}{2}}(\mathbf{u}(t) \vee \mathbf{c})\Big),$$

where we abbreviate $f_{\nu+\frac{1}{2}}(\mathbf{u} \vee \mathbf{c}) := f(u_{\nu-p+1} \vee c, \ldots, u_{\nu+p} \vee c)$. Similarly, since $(\mathbf{u} \wedge \mathbf{c})_\nu = \min\{u_\nu, c\}$ is dominated by both $\mathbf{u}(t)$ and $\mathbf{c}$, it follows that

$$(u_\nu(t+\Delta t) \wedge c) \geq (u_\nu(t) \wedge c) - \frac{\Delta t}{\Delta x}\Big(f_{\nu+\frac{1}{2}}(\mathbf{u}(t) \wedge \mathbf{c}) - f_{\nu-\frac{1}{2}}(\mathbf{u}(t) \wedge \mathbf{c})\Big),$$

Taking the difference of the last two inequalities yields

$$|u_\nu(t+\Delta t) - c| \leq |u_\nu(t) - c| - \frac{\Delta t}{\Delta x}\Big(F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}\Big),$$

where $F_{\nu+\frac{1}{2}} := f_{\nu+\frac{1}{2}}(\mathbf{u}(t) \vee \mathbf{c}) - f_{\nu+\frac{1}{2}}(\mathbf{u}(t) \wedge \mathbf{c})$ is a numerical flux consistent with the entropy flux $F(u;c) = f(u \vee c) - f(u \wedge c)$. We conclude that monotone schemes are entropy stable with respect to the class of Kruzkov entropy pairs (Kruzkhov, 1970) $\eta(u;c) = |u-c|, \ F(u;c)) = \text{sign}(u-c)(f(u) - f(c))$.

Example 2 (3-point schemes). Consider the class of scalar schemes based on *3-point stencils*,

$$u_\nu(t+\Delta t) = u_\nu(t) - \frac{\Delta t}{\Delta x}(f(u_\nu(t), u_{\nu+1}(t)) - f(u_{\nu-1}(t), u_\nu(t))).$$

The scheme has monotone dependence on $u_{\nu\pm1}(t)$ if and only if its two-point flux, $f(u_\ell, u_r)$, is increasing in u_ℓ and respectively decreasing in u_r, abbreviated as $f(\uparrow, \downarrow)$. The monotone dependence on $u_\nu(t)$ follows from a CFL condition

[e]Observe that $\mathbf{c}$ is a steady solution of (16) for an arbitrary c.

$$\lambda\left(\partial_{u_\ell} f_{\nu+\frac{1}{2}}(u_\nu,\cdot) - \partial_{u_r} f_{\nu-\frac{1}{2}}(\cdot,u_\nu)\right) \le 1.$$

Thus, the LxF scheme is monotone but LxW scheme is not. The scalar Godunov scheme is monotone because it combines the exact entropic evolution operator together with projection to cell averages, $u(\cdot,t+\Delta t)\mapsto \sum_\nu \bar{u}_\nu(t)\boldsymbol{1}_{I_{x_\nu}}(x)$, which are both monotone. Indeed, the numerical flux of scalar Godunov schemes is given by Osher (1984, lemma 1.1)

$$f^{\rm G}(u_\nu,u_{\nu+1}) = \begin{cases} \min\limits_{u_\nu\le u\le u_{\nu+1}} f(u), & \text{if} \quad u_\nu < u_{\nu+1}, \\ \max\limits_{u_{\nu+1}\le u\le u_\nu} f(u), & \text{if} \quad u_\nu > u_{\nu+1}. \end{cases}$$

which is readily verified to be of monotone type $f^{\rm G}(\uparrow,\downarrow)$.

3.2 E-Schemes

In the particular case of scalar conservation laws, all convex η's are admissible entropy functions: the compatibility condition (4), $\eta' f' = F'$, merely recovers the corresponding entropy flux as $F(u) = \int^u \eta'(v) f'(v)\mathrm{d}v$. A discrete scheme is an *E-scheme* (Osher, 1984; Tadmor, 1984; Osher, 1985) if it is entropy stable w.r.t. *all convex* entropies. Godunov and Lax–Friedrichs schemes are primary examples: they are entropy stable w.r.t. all entropy pairs associated with an underlying conservation law, and in the particular case of scalar laws—w.r.t. all convex entropies. Furthermore, in Example 7 we show that Godunov scheme has the distinct feature of having the *least* numerical viscosity among those scalar schemes which are entropy stability w.r.t. *all* convex entropies. The characterization of the scalar E-class is accomplished by a comparison with Godunov scheme.

3.3 Numerical Viscosity I

To carry out this comparison (Tadmor, 1984), consider the class of discrete schemes written in the *viscosity form*

$$\begin{aligned} u_\nu(t+\Delta t) = \ & u_\nu(t) - \frac{\Delta t}{2\Delta x}(f(u_{\nu+1}(t)) - f(u_{\nu-1}(t))) \\ & + \frac{\Delta t}{2\Delta x}\left(q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}} - q_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}\right). \end{aligned} \tag{17}$$

The role of $\{q_{\nu+\frac{1}{2}}\}$ as the *numerical viscosity coefficients* is revealed once we view (17) as an approximation to the *modified equation*, $u_t + f(u)_x = \dfrac{\Delta x}{2}(q u_x)_x$, with vanishing viscosity amplitude of order $\sim \dfrac{\Delta x}{2} q(\cdot)$.

When compared with (16), we observe that these are conservative schemes with numerical flux[f]

$$f_{\nu+\frac{1}{2}} = \frac{1}{2}(f(u_{\nu+1} + f(u_\nu)) - \frac{1}{2}q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}. \tag{18}$$

Conversely, every 3-point scheme admits a viscosity form (17) with a numerical viscosity coefficient dictated by (18).[g] Thus, for example, $q^{\mathrm{LxF}}_{\nu+\frac{1}{2}} \equiv \frac{1}{\lambda}$ and

$$\begin{aligned} q^{\mathrm{G}}_{\nu+\frac{1}{2}} &= \max_{u \in C_{\nu+\frac{1}{2}}} \frac{f(u_{\nu+1}) + f(u_\nu) - 2f(u)}{u_{\nu+1} - u_\nu}, \\ C_{\nu+\frac{1}{2}} &:= [\min\{u_\nu, u_{\nu+1}\}, \max\{u_\nu, u_{\nu+1}\}]. \end{aligned} \tag{19}$$

The class of E-schemes consists of those schemes which contain *more* numerical viscosity than Godunov's, so that their numerical viscosity coefficient, $q^{\mathrm{E}}_{\nu+\frac{1}{2}}$, satisfies, (Tadmor, 1984; Makridakis and Perthame, 2003)

$$\lambda q^{\mathrm{G}}_{\nu+\frac{1}{2}} \leq \lambda q^{\mathrm{E}}_{\nu+\frac{1}{2}} \leq 1.$$

Indeed, such a scheme satisfies the discrete entropy inequality

$$\eta(u_\nu(t+\Delta t)) \leq \eta(u_\nu(t)) - \frac{\Delta t}{\Delta x}\left(F^{\mathrm{E}}_{\nu+\frac{1}{2}} - F^{\mathrm{E}}_{\nu-\frac{1}{2}}\right),$$

for an arbitrary convex entropy $\eta(\cdot)$. The numerical entropy flux F^{E} is given by a convex combination of the corresponding fluxes, $F^{\mathrm{G}}_{\nu+\frac{1}{2}}$ and $F^{\mathrm{LxF}}_{\nu+\frac{1}{2}}$.

Example 3. (Engquist–Osher scheme (Engquist and Osher, 1980)). The Engquist–Osher scheme is an example for an E-scheme. Its numerical viscosity coefficient, $q^{\mathrm{EO}} := \frac{1}{\Delta u_{\nu+\frac{1}{2}}}\int_{u_\nu}^{u_{\nu+1}} |f'(u)|\mathrm{d}u$, satisfies $q^{\mathrm{EO}}_{\nu+\frac{1}{2}} \geq q^{\mathrm{G}}_{\nu+\frac{1}{2}}$, under the CFL condition $\lambda q^{\mathrm{EO}}_{\nu+\frac{1}{2}} \leq 1$.

[f]In certain references, the numerical viscosity coefficient is rescaled *with* the mesh-ration λ, so that (17) reads $u_\nu(t+\Delta t) = u_\nu(t) - \frac{\Delta t}{2\Delta x}(f(u_{\nu+1}(t)) - f(u_{\nu-1}(t)) + \frac{1}{2}\left(q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}} - q_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}\right)$, with a numerical flux $f_{\nu+\frac{1}{2}} = \frac{1}{2}(f(u_{\nu+1} + f(u_\nu)) - \frac{1}{2\lambda}q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}$.

[g]Indeed, every *essentially* 3-point scheme in the sense that its flux satisfies $f(u_{\nu-p+1}, \ldots, u, u, \ldots, u_{\nu+p}) = f(u)$, admits the viscosity form (17).

4 SEMIDISCRETE SCHEMES FOR SYSTEMS OF CONSERVATION LAWS

We focus our attention on the *semidiscrete* limit, $\Delta t \downarrow 0$, where (11) recasts into the form (so-called method of lines)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x}\left(\mathbf{f}_{\nu+\frac{1}{2}} - \mathbf{f}_{\nu-\frac{1}{2}}\right), \quad \mathbf{f}_{\nu+\frac{1}{2}} = \mathbf{f}\left(\mathbf{u}_{\nu-p+1}(t), \ldots \mathbf{u}_{\nu+p}(t)\right). \tag{20}$$

We now fix an entropy pair, (η, F), and seek the corresponding entropy stability, where

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(\mathbf{u}_\nu(t)) \le -\frac{1}{\Delta x}\left(F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}\right) \tag{21}$$

holds for a consistent numerical entropy flux $F_{\nu+\frac{1}{2}}$. In the particular case that entropy *equality* holds in (21), we say that the scheme (20) is *entropy conservative*.

To address the question on entropy stability w.r.t. to this pair, we seek special schemes which do not dissipate this entropy. These are the *entropy conservative* schemes constructed in Tadmor (1987). The study of entropy stability then proceeds using two main ingredients: (i) the use of the *entropy variables* which enables us to compare numerical viscosity *matrix* coefficients by the natural ordering of symmetric matrices; and (ii) comparison with the appropriate *entropy conservative* schemes. We discuss these two ingredients.

4.1 Entropy Variables (Godunov, 1961; Mock, 1980; see also Godunov and Peshkov, 2008)

Define the *entropy variables* $\boldsymbol{v} \equiv \boldsymbol{v}(\mathbf{u}) := \eta'(\mathbf{u})$. Thanks to the convexity of $\eta(\mathbf{u})$, the mapping $\mathbf{u} \to \boldsymbol{v}$ is one-to-one and hence we can make the (*local*) change of variables $\mathbf{u}_\nu = \mathbf{u}(\boldsymbol{v}_\nu)$. The scheme (11) then recasts into an equivalent form expressed in terms of the discrete entropy variables $\boldsymbol{v}_\nu = \boldsymbol{v}_\nu(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(\boldsymbol{v}_\nu(t)) = -\frac{1}{\Delta x}\left(\mathbf{f}_{\nu+\frac{1}{2}} - \mathbf{f}_{\nu-\frac{1}{2}}\right), \tag{22}$$

with a numerical flux[h] $\mathbf{f}_{\nu+\frac{1}{2}} = \mathbf{f}(\boldsymbol{v}_{\nu-p+1}, \ldots, \boldsymbol{v}_{\nu+p}) := \mathbf{f}(\mathbf{u}(\boldsymbol{v}_{\nu-p+1}), \ldots, \mathbf{u}(\boldsymbol{v}_{\nu+p}))$, consistent with the differential flux, $\mathbf{f}(\boldsymbol{v}, \boldsymbol{v}, \ldots, \boldsymbol{v}) = \mathbf{f}(\mathbf{u}(\boldsymbol{v}))$.

[h]We shall often abuse the notation using the same $\mathbf{f}(\cdot)$ as a vector function of the conservative variables $\mathbf{f}(\mathbf{u})$ and of the entropy variables, $\mathbf{f}(\mathbf{u}(\boldsymbol{v})) \rightsquigarrow \mathbf{f}(\boldsymbol{v})$, whenever their dependence is clear from context and there is no ambiguity.

4.2 Entropy Conservative Fluxes

We seek entropy conservative fluxes, denoted $\mathbf{f}^*_{\nu+\frac{1}{2}}$, such that

$$\frac{d}{dt}\mathbf{u}_\nu(t)+\frac{1}{\Delta x}\left(\mathbf{f}^*_{\nu+\frac{1}{2}}-\mathbf{f}^*_{\nu-\frac{1}{2}}\right)=0 \overset{?}{\rightsquigarrow} \frac{d}{dt}\eta(\mathbf{u}_\nu(t))+\frac{1}{\Delta x}\left(F_{\nu+\frac{1}{2}}-F_{\nu-\frac{1}{2}}\right)=0. \quad (23)$$

Premultiply both sides by $\eta'(\mathbf{u})$: we conclude that $\mathbf{f}^*_{\nu-\frac{1}{2}}$ is an *entropy conservative numerical flux* if preserves the structure of 'perfect differences' in the sense that $\left\langle \eta'(\mathbf{u}_\nu),\ \mathbf{f}^*_{\nu+\frac{1}{2}}-\mathbf{f}^*_{\nu-\frac{1}{2}}\right\rangle = F_{\nu+\frac{1}{2}}-F_{\nu-\frac{1}{2}}$. Expressed in terms of the entropy variables, $\boldsymbol{v}_\nu=\eta'(\mathbf{u}_\nu)$, the requirement that $\left\langle \boldsymbol{v}_\nu, \mathbf{f}^*_{\nu+\frac{1}{2}}-\mathbf{f}^*_{\nu-\frac{1}{2}}\right\rangle$ is a 'perfect difference' holds iff $\left\langle \boldsymbol{v}_{\nu+1}-\boldsymbol{v}_\nu, \mathbf{f}^*_{\nu+\frac{1}{2}}\right\rangle$ is a perfect difference. Specifically, the following identity holds,

$$\begin{aligned}&\frac{d}{dt}\eta(\mathbf{u}_\nu(t))+\frac{1}{\Delta x}\left(F_{\nu+\frac{1}{2}}-F_{\nu-\frac{1}{2}}\right)\\ &\quad\equiv\frac{1}{2\Delta x}\left[\left\langle\Delta\boldsymbol{v}_{\nu+\frac{1}{2}},\mathbf{f}_{\nu+\frac{1}{2}}\right\rangle-\Delta\psi_{\nu+\frac{1}{2}}\right]+\frac{1}{2\Delta x}\left[\left\langle\Delta\boldsymbol{v}_{\nu-\frac{1}{2}},\mathbf{f}_{\nu-\frac{1}{2}}\right\rangle-\Delta\psi_{\nu-\frac{1}{2}}\right].\end{aligned} \quad (24)$$

Here $F_{\nu+\frac{1}{2}}$ is a numerical entropy flux expressed in terms of the corresponding *entropy flux potential*,

$$\psi(\boldsymbol{v}):=\langle\boldsymbol{v},\mathbf{f}(\boldsymbol{v})\rangle-F(\mathbf{u}(\boldsymbol{v})). \quad (25)$$

This brings us to the following (Tadmor, 1987, §3).

(i) *[Entropy conservative scheme]*. The difference scheme (22) is entropy conservative so that (23) holds, if its numerical flux, denoted $\mathbf{f}_{\nu+\frac{1}{2}}=\mathbf{f}^*_{\nu+\frac{1}{2}}$, satisfies

$$\left\langle\boldsymbol{v}_{\nu+1}-\boldsymbol{v}_\nu,\mathbf{f}^*_{\nu+\frac{1}{2}}\right\rangle=\psi_{\nu+1}-\psi_\nu,\quad \psi_\nu=\langle\boldsymbol{v}_\nu,\mathbf{f}(\boldsymbol{v}_\nu)\rangle-F(\mathbf{u}(\boldsymbol{v}_\nu)) \quad (26)$$

(ii) *[Entropy stable schemes]*. Consider a numeral flux $\mathbf{f}_{\nu+\frac{1}{2}}$ of the form

$$\mathbf{f}_{\nu+\frac{1}{2}}=\mathbf{f}^*_{\nu+\frac{1}{2}}-\frac{1}{2}D_{\nu+\frac{1}{2}}(\boldsymbol{v}_{\nu+1}-\boldsymbol{v}_\nu),\quad D_{\nu+\frac{1}{2}}\geq 0; \quad (27)$$

Here, $\mathbf{f}^*_{\nu+\frac{1}{2}}$ is any entropy conservative flux satisfying (26) and $D_{\nu+\frac{1}{2}}$ is *any* positive definite symmetric matrix. Then the resulting scheme (20) is entropy stable,

$$\begin{aligned}&\frac{d}{dt}\eta(\mathbf{u}_\nu(t))+\frac{1}{\Delta x}\left(F^*_{\nu+\frac{1}{2}}-F^*_{\nu-\frac{1}{2}}\right)\\ &\quad=-\frac{1}{4\Delta x}\left\langle\Delta\boldsymbol{v}_{\nu-\frac{1}{2}},\ D_{\nu-\frac{1}{2}}\Delta\boldsymbol{v}_{\nu-\frac{1}{2}}\right\rangle-\frac{1}{4\Delta x}\left\langle\Delta\boldsymbol{v}_{\nu+\frac{1}{2}},\ D_{\nu+\frac{1}{2}}\Delta\boldsymbol{v}_{\nu+\frac{1}{2}}\right\rangle\leq 0.\end{aligned} \quad (28)$$

Remark that a general framework for *explicit* construction of entropy conservative fluxes (26) is outlined in Section 4.5. Together with (27), they provide an explicit recipe for constructing entropy stable schemes. In particular, the fluxes (27) satisfy the entropy stability condition in Osher (1984, lemma 3.1).

4.3 How Much Numerical Viscosity

The entropy inequality $\eta(\mathbf{u})_t + F(\mathbf{u})_x \leq 0$ is imposed as a stability condition which excludes nonphysically relevant shock discontinuities. In particular, the entropy decay follows $\int \eta(\mathbf{u}(x,t_2))\mathrm{d}\mathbf{x} \leq \int \eta(\mathbf{u}(x,t_1))\mathrm{d}\mathbf{x}, \ \ t_2 > t_1$. The question is to quantify the inequality, namely—how much entropy decay will suffice? "physically relevant" entropy decay could be dictated by various mechanisms. We mention the most important two:

(i) *Physical diffusion*. The canonical example of the conservative Euler equations vs. the entropy decay dictated by Navier–Stokes equations. However, in practical simulations one does not often fully resolve the small scales governed by physical diffusion, and an "artificial" numerical viscosity is being used.

(ii) *Numerical viscosity*. According to (28), one can add *any* amount of numerical viscosity to enforce entropy stability. The goal is to add a judicious amount of vanishing viscosity so that in the resulting scheme admits additional desirable and often competing properties of high-resolution and nonoscillatory behaviour. A recent discussion along these lines with the arbitrarily high-order nonoscillatory ENO schemes can be found in Fjordholm et al. (2012, 2015).

4.4 Scalar Entropy Stability Revisited

We discuss the question of entropy stability for semidiscrete scalar schemes $\frac{\mathrm{d}}{\mathrm{d}t}u_\nu(t) = -\frac{1}{\Delta x}\left(f_{\nu+\frac{1}{2}} - f_{\nu-\frac{1}{2}}\right)$, which is expressed in its equivalent viscosity form (17),

$$\frac{\mathrm{d}}{\mathrm{d}t}u_\nu(t) = -\frac{1}{2\Delta x}(f(u_{\nu+1}) - f(u_{\nu-1})) + \frac{1}{2\Delta x}\left(q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}} - q_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}\right), \quad (29)$$

To simplify matters we now *fix the quadratic entropy* $\eta(u) = \frac{1}{2}u^2$, where the entropy variables coincide with the conservative variables, $v = u$. The corresponding entropy conservative flux is now uniquely determined as $f^*_{\nu+\frac{1}{2}} = \frac{\psi(u_{\nu+1}) - \psi(u_\nu)}{u_{\nu+1} - u_\nu}$, which can be expressed as

$$f^*_{\nu+\frac{1}{2}} := \frac{\psi(u_{\nu+1}) - \psi(u_\nu)}{u_{\nu+1} - u_\nu} \equiv \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \psi'\left(u_{\nu+\frac{1}{2}}(\xi)\right)\mathrm{d}\xi$$
$$= \frac{1}{2}(f(u_{\nu+1}) + f(u_\nu)) - \frac{1}{2}q^*_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}.$$

Recall (18): we recognize $q^*_{\nu+\frac{1}{2}}$ as the entropy conservative numerical viscosity coefficient, which is given by

$$q^*_{\nu+\frac{1}{2}} := \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi f'\left(u_{\nu+\frac{1}{2}}(\xi)\right)\mathrm{d}\xi, \quad u_{\nu+\frac{1}{2}}(\xi) := \frac{1}{2}(u_\nu + u_{\nu+1}) + \xi\Delta u_{\nu+\frac{1}{2}}.$$

The resulting entropy conservative scheme then takes the *viscosity form*

$$\frac{\mathrm{d}}{\mathrm{d}t}u_\nu(t) = -\frac{1}{2\Delta x}(f(u_{\nu+1}) - f(u_{\nu-1})) + \frac{1}{2\Delta x}\left(q^*_{\nu+\frac{1}{2}}\,\Delta u_{\nu+\frac{1}{2}} - q^*_{\nu-\frac{1}{2}}\,\Delta u_{\nu-\frac{1}{2}}\right). \quad (30)$$

The statement of entropy stability, (27) and (28), can be rephrased by stating that the conservative scheme (17) is entropy stable if it contains *more* viscosity than the entropy conservative scheme (30), in the sense that $q_{\nu+\frac{1}{2}} \geq q^*_{\nu+\frac{1}{2}}$. Indeed, the numerical flux associated with (29) can be expressed as

$$f_{\nu+\frac{1}{2}} = \frac{1}{2}(f(u_{\nu+1}) + f(u_\nu)) - \frac{1}{2}q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}} \equiv f^*_{\nu+\frac{1}{2}} + \frac{1}{2}\left(q_{\nu+\frac{1}{2}} - q^*_{\nu+\frac{1}{2}}\right)\Delta u_{\nu+\frac{1}{2}},$$

and entropy stability follows from (27) with $D_{\nu+\frac{1}{2}} = q_{\nu+\frac{1}{2}} - q^*_{\nu+\frac{1}{2}} \geq 0$.

The corollary above enables to verify the entropy stability of first- and second-order accurate schemes. A host of examples can be found in Tadmor (2003). We mention a couple of them.

Example 4 (Burgers' equation). Consider the inviscid Burgers' equation, $u_t + \left(\frac{1}{2}u^2\right)_x = 0$, augmented with the quadratic entropy inequality, $\left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3\right)_x = 0$. The entropy variable $\boldsymbol{v}(u) = u$ and entropy potential $\psi(v) := vf - F = \frac{1}{6}u^3$ yield the entropy conservative flux which is the "$\frac{1}{3}$"-rule

$$\frac{\mathrm{d}}{\mathrm{d}t}u_\nu(t) = -\frac{2}{3}\left(\frac{u^2_{\nu+1} - u^2_{\nu-1}}{4\Delta x}\right) - \frac{1}{3}\left(u_\nu\frac{u_{\nu+1} - u_{\nu-1}}{2\Delta x}\right) \rightsquigarrow \sum u^2_\nu(t)\Delta x = \sum u^2_\nu(0)\Delta x.$$

Example 5 (Lax–Wendroff viscosity (Lax and Wendroff, 1960)). Consider the case of a convex flux $f(u)$ and fix the quadratic entropy $\eta(u) = \frac{1}{2}u^2$. To see how much viscosity is required to guarantee the quadratic entropy stability,[i] we use the fact that the f' is increasing, leading to the upper bound

[i]We note in passing that quadratic entropy stability is sufficient to single out the unique physically relevant solution in the case of convex flux, e.g., Chen (2000).

$$q^*_{v+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi f'\left(u_{v+\frac{1}{2}}(\xi)\right)\mathrm{d}\xi \le \frac{1}{4}\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} f''\left(u_{v+\frac{1}{2}}(\xi)\right)\mathrm{d}\xi$$
$$= \frac{1}{4}(f'(u_{v+1}) - f'(u_v))_+.$$

The resulting viscosity coefficient on the right is the second-order Lax–Wendroff viscosity proposed in Lax and Wendroff (1960) with numerical viscosity coefficient $q^{LxW}_{v+\frac{1}{2}} = \frac{1}{4}(f'(u_{v+1}) - f'(u_v))_+$. It follows that this version of LxW scheme is entropy stable, $\frac{1}{2}\frac{d}{dt}u_v^2(t) + \frac{1}{\Delta x}\left(F_{v+\frac{1}{2}} - F_{v-\frac{1}{2}}\right) \le 0$.

4.5 Numerical Viscosity II

We extend the previous discussion to an arbitrary convex entropy. Let $=\eta'(u)$ denote the corresponding entropy variables. The starting point is the corresponding conservative entropy flux (26)

$$f^*_{v+\frac{1}{2}} := \frac{\psi(u_{v+1}) - \psi(u_v)}{v_{v+1} - v_v} \equiv \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \psi'\left(u(v_{v+\frac{1}{2}}(\xi))\right)\mathrm{d}\xi = \frac{1}{2}(f(u_{v+1}) + f(u_v)) - \frac{1}{2}p^*_{v+\frac{1}{2}}\Delta v_{v+\frac{1}{2}},$$

which yields the entropy conservative schemes in its viscosity form

$$\frac{\mathrm{d}}{\mathrm{d}t}u_v(t) = -\frac{1}{2\Delta x}(f(u_{v+1}) - f(u_{v-1})) + \frac{1}{2\Delta x}(p^*_{v+\frac{1}{2}}\Delta_{vv+\frac{1}{2}} - p^*_{v-\frac{1}{2}}\Delta_{vv-\frac{1}{2}}).$$

Observe that the viscosity term on the right is expressed in terms of the jump in entropy variables, $\Delta_{vv+\frac{1}{2}}$}. (Of course, in the case of quadratic entropy $v = u$ hence $p_{v+\frac{1}{2}} = q_{v+\frac{1}{2}}$ and $p^*_{v+\frac{1}{2}} = q^*_{v+\frac{1}{2}}$ recovering (29)). This motivates the general viscosity form

$$\frac{\mathrm{d}}{\mathrm{d}t}u_v(t) = -\frac{1}{2\Delta x}(f(u_{v+1}) - f(u_{v-1})) + \frac{1}{2\Delta x}\left(p_{v+\frac{1}{2}}\Delta v_{v+\frac{1}{2}} - p_{v-\frac{1}{2}}\Delta v_{v-\frac{1}{2}}\right). \quad (31)$$

corresponding to the vanishing viscosity $u_t + f(u)_x = \frac{\Delta x}{2}(pv_x)_x$. The η-entropy stability follows if and only if $p_{v+\frac{1}{2}} \ge p^*_{v+\frac{1}{2}}$. We conclude with a couple of examples.

Example 6 (Entropy conservative Toda flow). Consider the equation $u_t + (e^u)_x = 0$ augmented with exponential entropy pair, $(e^u)_t + (e^{2u})_x = 0$. The entropy variable associated with $\eta(u) = e^u$ are $\boldsymbol{v}(u) = e^u$, the entropy potential is $\psi(v) := vf - F = \frac{1}{2}v^2$, and we end up with the entropy conservative flux:

$$f^*_{v+\frac{1}{2}} = \frac{\psi(v_{v+1}) - \psi(v_v)}{v_{v+1} - v_v} = \frac{\frac{1}{2}v^2_{v+1} - \frac{1}{2}v^2_v}{v_{v+1} - v_v} = \frac{1}{2}(v_v + v_{v+1}) = \frac{1}{2}(e^{u_v} + e^{u_{v+1}}).$$

This leads to the *dispersive* centred scheme, interesting for its own sake, e.g., Lax (1986) and Deift and McLaughlin (1998)

$$\frac{\mathrm{d}}{\mathrm{d}t}u_\nu(t) + \frac{e^{u_{\nu+1}(t)} - e^{u_{\nu-1}(t)}}{2\Delta x} = 0,$$

which conserve the exponential entropy $\eta(u_\nu(t)) = e^{u_\nu(t)}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_\nu e^{u_\nu(t)}\Delta x &= -\sum_\nu \frac{e^{u_\nu + u_{\nu+1}} - e^{u_\nu + u_{\nu-1}}}{2\Delta x}\Delta x \\ &= 0 \rightsquigarrow \sum_\nu \eta(u_\nu(t))\Delta x = \sum_\nu \eta(u_\nu(0))\Delta x.\end{aligned}$$

Example 7 (On the optimality of Godunov flux (Tadmor, 2003, Example 4.4)). We normalize the viscous term on the right of (31) in terms of the conservative variables

$$\frac{1}{2\Delta x}\left(\left(p_{\nu+\frac{1}{2}}\frac{\Delta v_{\nu+\frac{1}{2}}}{\Delta u_{\nu+\frac{1}{2}}}\right)\Delta u_{\nu+\frac{1}{2}} - \left(p_{\nu-\frac{1}{2}}\frac{\Delta v_{\nu-\frac{1}{2}}}{\Delta u_{\nu-\frac{1}{2}}}\right)\Delta u_{\nu-\frac{1}{2}}\right),$$

It follows that in order to maintain entropy stability w.r.t. all η's, we need to *maximize* the corresponding entropy viscous factors $p^*_{\nu+\frac{1}{2}}\left(\Delta v_{\nu+\frac{1}{2}}/\Delta u_{\nu+\frac{1}{2}}\right)$,

$$\sup_v \frac{f(u_\nu) + f(u_{\nu+1}) - 2f^*_{\nu+\frac{1}{2}}}{\Delta u_{\nu+\frac{1}{2}}}, \quad f^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} f\left(u\left(v_{\nu+\frac{1}{2}}(\xi)\right)\right)\mathrm{d}\xi,$$

where the supremum is taken over *all increasing* $v = v(u)$. This is precisely the Godunov's viscosity coefficient (19). Thus, the scalar schemes which are entropy stable with respect to all convex entropies are precisely those that contain at least as much numerical viscosity as the Godunov scheme does.

4.6 Entropy Conservative Fluxes—Systems of Conservation Laws

Unlike the scalar case, there is more than one way to meet the requirement of entropy conservative flux, (26), for systems of conservation laws. In particular, one can set (Tadmor, 1987),

$$\mathbf{f}^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}\left(\mathbf{u}(\boldsymbol{v}_{\nu+\frac{1}{2}}(\xi))\right)\mathrm{d}\xi, \tag{32}$$

integrated along the *straight-path* $\boldsymbol{v}_{\nu+\frac{1}{2}}(\xi) := \frac{1}{2}(\boldsymbol{v}_\nu + \boldsymbol{v}_{\nu+1}) + \xi\Delta\boldsymbol{v}_{\nu+\frac{1}{2}}$. Observe that when viewed as a function of the entropy variables, the $\boldsymbol{v}$-dependent flux $\mathbf{f}(\boldsymbol{v}) \equiv \mathbf{f}(\mathbf{u}(\boldsymbol{v}))$ becomes a gradient, $\mathbf{f}(\boldsymbol{v}) = \nabla_{\boldsymbol{v}}\,\psi(\boldsymbol{v})$ of the entropy potential, $\psi(\boldsymbol{v}) := \langle \boldsymbol{v}, \mathbf{f}(\boldsymbol{v})\rangle - F(\mathbf{u}(\boldsymbol{v}))$. Hence, the value of $\mathbf{f}^*$ is in fact, independent of the path of integration. In particular, a more accessible recipe, amenable for explicit evaluation of such fluxes is given by integration along a *piecewise-path* in phase-space, connecting the two neighbouring values $\boldsymbol{v}_\nu$ and $\boldsymbol{v}_{\nu+1}$. To this end, we begin at $\boldsymbol{v}^1 := \boldsymbol{v}_\nu$, and follow the intermediate steps $\boldsymbol{v}^{j+1} = \boldsymbol{v}^j + \left\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta\boldsymbol{v}_{\nu+\frac{1}{2}} \right\rangle \mathbf{r}^j_{\nu+\frac{1}{2}}$ for $j = 1, 2, \ldots$, ending at $\boldsymbol{v}^{n+1} = \boldsymbol{v}_{\nu+1}$. Here, $\{\mathbf{r}^j\}_{j=1}^n$ be an *arbitrary* set of n linearly independent n-directions, and let $\{\boldsymbol{\ell}^j\}_{j=1}^n$ denote the corresponding orthogonal set, $\langle \boldsymbol{\ell}^j, \mathbf{r}^k \rangle = \delta_{jk}$. (since the mapping $\mathbf{u} \mapsto \boldsymbol{v}$ is one-to-one, the path is mirrored in the usual phase space of conservative variables, starting with $\mathbf{u}^1_{\nu+\frac{1}{2}} = \mathbf{u}_\nu$ and ending with $\mathbf{u}^{n+1}_{\nu+\frac{1}{2}} = \mathbf{u}_{\nu+1}$). The entropy conservative flux $\mathbf{f}^*_{\nu+\frac{1}{2}}$ is then given by the *explicit* formula (Tadmor, 2003, theorem 6.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x}\left(\mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}}\right), \quad \mathbf{f}^*_{\nu+\frac{1}{2}} := \sum_{j=1}^{n} \frac{\psi(\mathbf{v}^{j+1}) - \psi(\boldsymbol{v}^j)}{\left\langle \boldsymbol{\ell}^j, \Delta\boldsymbol{v}_{\nu+\frac{1}{2}} \right\rangle} \boldsymbol{\ell}^j, \tag{33}$$

We demonstrate the above approach in the context of entropic Euler equations, with entropy pair $(\eta, F) = (-\rho S, -\rho v S)$.

Example 8 (Entropy conservative Euler flux (Tadmor and Zhong, 2006)). The entropy function $\eta(\mathbf{u}) = -\rho S$ induces the entropy variables, $\boldsymbol{v} = \eta'(\mathbf{u}) = \left(-{}^{E}\!/_{e} - S + \gamma + 1, {}^{q}\!/_{\theta}, -{}^{1}\!/_{\theta}\right)^\top$ expressed in terms of the internal energy $e := E - \frac{1}{2}\rho v^2 = C_\upsilon \rho\theta$. The corresponding entropy flux potential amounts to $\psi(\boldsymbol{v}) = \langle \boldsymbol{v}\ , \mathbf{f}\rangle - F(\mathbf{u}) = (\gamma - 1)m$ and the entropy conservative Euler flux is then given by $\mathbf{f}^*_{\nu+\frac{1}{2}} = (\gamma - 1)\sum_{j=1}^{3} \frac{m^{j+1} - m^j}{\langle \boldsymbol{\ell}^j\ , \Delta\boldsymbol{v}_{\nu+\frac{1}{2}}\rangle} \boldsymbol{\ell}^j$.

Example 9 (An affordable recipe for entropy conservative flux). An "affordable" entropy conservative flux for Euler equations was derived by Ismail and Roe in 2009 by clever manipulation of the algebraic relations (26). Expressed in terms of the normalized vector $\mathbf{z} := \sqrt{\frac{\rho}{p}}(1, v, p)^\top$, the entropy conservative flux, $\mathbf{f}^*_{\nu+\frac{1}{2}} := (f^1, f^2, f^3)^\top$, is given by the explicit recipe, in terms of the averages $\bar{z}_{\nu+\frac{1}{2}} := \frac{1}{2}(z_\nu + z_{\nu+1})$ and $z^{ln}_{\nu+\frac{1}{2}} := \frac{\Delta z_{\nu+\frac{1}{2}}}{\Delta \log(z)_{\nu+\frac{1}{2}}}$,

$$f^1_{\nu+\frac{1}{2}} = (\bar{z}_2)_{\nu+\frac{1}{2}} (z_3)^{ln}_{\nu+\frac{1}{2}}, \qquad f^2_{\nu+\frac{1}{2}} = \frac{(\bar{z}_3)_{\nu+\frac{1}{2}}}{(\bar{z}_1)_{\nu+\frac{1}{2}}} + \frac{(\bar{z}_2)_{\nu+\frac{1}{2}}}{(\bar{z}_1)_{\nu+\frac{1}{2}}} f^1_{\nu+\frac{1}{2}},$$

and

$$f^3_{\nu+\frac{1}{2}} = \frac{1}{2(\bar{z}_1)_{\nu+\frac{1}{2}}}\left(\frac{\gamma+1}{\gamma-1}\frac{1}{(z_1)^{ln}_{\nu+\frac{1}{2}}}f^1_{\nu+\frac{1}{2}} + (\bar{z}_2)_{\nu+\frac{1}{2}}f^2_{\nu+\frac{1}{2}}\right).$$

5 FULLY DISCRETE SCHEMES FOR SYSTEMS OF CONSERVATION LAWS

Godunov scheme (14) is based on "pushing-forward" an exact entropic solution, $\mathbf{u}_\Delta(\cdot,\tau)$ for $\tau > t$, subject to piecewise-constant data, $\mathbf{u}_\Delta(x,t) = \sum_\nu \mathbf{u}_\nu(t)\boldsymbol{1}_{I_{x_\nu}}(x)$. As such, $\mathbf{u}_\Delta(x,\cdot)$ satisfies the entropy inequality $\partial_\tau\eta(\mathbf{u}_\Delta) + \partial_x F(\mathbf{u}_\Delta) \le 0$. Integrated across the control volume $I_{x_\nu} \times [t, t+\Delta t]$ we find the balance between spatial and temporal averages,

$$\overline{\eta(\mathbf{u}_\Delta)}(x_\nu, t+\Delta t) \le \overline{\eta(\mathbf{u}_\Delta)}(x_\nu, t) - \frac{\Delta t}{\Delta x}\left(\widetilde{F}(\mathbf{u}_R)\left(x_{\nu+\frac{1}{2}}\right) - \widetilde{F}(\mathbf{u}_R)\left(x_{\nu-\frac{1}{2}}\right)\right).$$

But $\overline{\eta(\mathbf{u}_\Delta)}(x_\nu, t) = \eta(\mathbf{u}_\nu(t))$, and by Jensen's inequality,

$$\eta(\mathbf{u}_\nu(t+\Delta t)) = \eta(\overline{\mathbf{u}_\Delta})(x_{\nu+\frac{1}{2}}, t+\Delta t) \le \overline{\eta(\mathbf{u}_\Delta)}(x_\nu, t+\Delta t),$$

and we conclude that Godunov scheme is entropic for all admissible pairs,

$$\eta(\mathbf{u}_\nu(t+\Delta t)) \le \eta(\mathbf{u}_\nu(t)) - \frac{\Delta t}{\Delta x}\left(F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}\right),$$
$$\boxed{F_{\nu\pm\frac{1}{2}}} = \frac{1}{\Delta t}\int_{\tau=t}^{t+\Delta t} F\left(\mathbf{u}_R(x_{\nu\pm\frac{1}{2}},\tau)\right)\mathrm{d}\tau.$$

Lax–Friedrich scheme, (12), can be interpreted as a Godunov scheme, where a piecewise-constant solution, $\mathbf{u}_\Delta(t) = \sum_\nu\left(\mathbf{u}_{\nu-1}(t)\boldsymbol{1}_{I_{x_{\nu-\frac{1}{2}}}}(x) + \mathbf{u}_{\nu+1}(t)\boldsymbol{1}_{I_{x_{\nu+\frac{1}{2}}}}(x)\right)$ is being "pushed-forward by the exact entropy solution operator, and then realized at $t+\Delta t$ by its averages over the *staggered grid*, $\mathbf{u}_\nu(t+\Delta t) = \frac{1}{\Delta x}\int_{I_{x_\nu}} \mathbf{u}_\Delta(x, t+\Delta t)\mathrm{d}x$. Arguing along the above lines for Godunov scheme, we find that LxF is an E-scheme (it is entropy stable w.r.t. all admissible entropy pairs η, F),

$$\eta(\mathbf{u}_\nu(t+\Delta t)) \le \eta(\mathbf{u}_\nu(t)) - \frac{\Delta t}{2\Delta x}(F(\mathbf{u}_{\nu+1}(t)) - F(\mathbf{u}_{\nu-1}(t))), \tag{34}$$

under the CFL condition[j] $\lambda\rho(A(\mathbf{u})) \le \frac{1}{2}$.

[j]We let $\rho(M) := \max_k|\lambda_k(M)|$ denote the *spectral radius* of a matrix M.

5.1 Numerical Viscosity III

Godunov and LxF are the prototype for the class of (essentially) 3-point schemes, which take the viscosity form

$$\mathbf{u}_\nu(t+\Delta t) = \mathbf{u}_\nu(t) - \frac{\Delta t}{2\Delta x}(\mathbf{f}(\mathbf{u}_{\nu+1}(t) - \mathbf{f}(\mathbf{u}_{\nu-1}(t))) + \frac{\Delta t}{2\Delta x}\left(P_{\nu+\frac{1}{2}}\Delta \boldsymbol{v}_{\nu+\frac{1}{2}} - P_{\nu-\frac{1}{2}}\Delta \boldsymbol{v}_{\nu-\frac{1}{2}}\right). \tag{35}$$

In the case of systems of conservation law, $P_{\nu+\frac{1}{2}}$ are $n\times n$ *matrix* numerical viscosity coefficients. The viscosity form for the entropy conservative schemes (32) is given in Tadmor (2003, §5) in terms of the *symmetric* Jacobian $\boldsymbol{B}(\boldsymbol{v}) = \partial_{\boldsymbol{v}}\mathbf{f}(\mathbf{u}(\boldsymbol{v}))$,

$$\mathbf{f}^*_{\nu+\frac{1}{2}} = \frac{1}{2}(\mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_\nu)) - \frac{1}{2}P^*_{\nu+\frac{1}{2}}\Delta \boldsymbol{v}_{\nu+\frac{1}{2}}, \qquad P^*_{\nu+\frac{1}{2}} := \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi B\left(\mathbf{u}(\boldsymbol{v}_{\nu+\frac{1}{2}}(\xi))\right)\mathrm{d}\xi.$$

The key point is the expression of the viscosity on the right of (35) in terms of the entropy variables which yields symmetric matrices, and which turn are amenable to a *comparison*: (35) is entropy stable if $P_{\nu+\frac{1}{2}} \geq P^*_{\nu+\frac{1}{2}}$ in the usual sense of ordering among symmetric matrices. For example, Lax–Friedrichs viscosity in (12), expressed in terms of the usual conservative variables is given by $Q^{\mathrm{LxF}}_{\nu+\frac{1}{2}} = \frac{1}{\lambda}I_{n\times n}$. Translated into the entropy variables, $\Delta \boldsymbol{v}_{\nu+\frac{1}{2}} = H_{\nu+\frac{1}{2}}\Delta \mathbf{u}_{\nu+\frac{1}{2}}$, we find,[k]

$$P^{LxF}_{\nu+\frac{1}{2}} = \frac{1}{\lambda}\left[\frac{\Delta \mathbf{u}_{\nu+\frac{1}{2}}}{\Delta \boldsymbol{v}_{\nu+\frac{1}{2}}}\right] := \frac{1}{\lambda}H^{-1}_{\nu+\frac{1}{2}}, \qquad H_{\nu+\frac{1}{2}} := \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \eta''(\mathbf{u}(\boldsymbol{v}_{\nu+\frac{1}{2}}(\xi))\mathrm{d}\xi.$$

It dominates $P^*_{\nu+\frac{1}{2}}$ and hence LxF scheme entropy stable w.r.t. all admissible entropy function associated with (8).

We demonstrate the derivation of entropy stability for the more general class of schemes (35) by a comparison with the entropy conservative flux. We have

$$\eta(\mathbf{u}_\nu(t+\Delta t)) = \eta(\mathbf{u}_\nu(t)) - \frac{\Delta t}{\Delta x}\left(F^*_{\nu+\frac{1}{2}}(t) - F^*_{\nu-\frac{1}{2}}(t)\right) - \mathcal{E}^{(x)}_\nu + \mathcal{E}^{(t)}_\nu.$$

here, $F^*_{\nu+\frac{1}{2}}$ is the entropy conservative flux, $\mathcal{E}^{(x)}_\nu$ is the amount of spatial entropy dissipation quoted in (28),

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[k]We use abbreviated notation for $\left[\frac{\Delta \boldsymbol{v}_{\nu+\frac{1}{2}}}{\Delta \mathbf{u}_{\nu+\frac{1}{2}}}\right]$ for any matrix such that $\Delta \boldsymbol{v}_{\nu+\frac{1}{2}} = \left[\frac{\Delta \boldsymbol{v}_{\nu+\frac{1}{2}}}{\Delta \mathbf{u}_{\nu+\frac{1}{2}}}\right]\Delta \mathbf{u}_{\nu+\frac{1}{2}}$. $H_{\nu+\frac{1}{2}}$ is such a matrix realized by integration along the usual straight path in phase space.

$$-\mathcal{E}_\nu^{(x)} = -\frac{\lambda}{4}\left\langle \Delta \boldsymbol{v}_{\nu-\frac{1}{2}}, D_{\nu-\frac{1}{2}} \Delta \boldsymbol{v}_{\nu-\frac{1}{2}} \right\rangle - \frac{\lambda}{4}\left\langle \Delta \boldsymbol{v}_{\nu+\frac{1}{2}}, D_{\nu+\frac{1}{2}} \Delta \boldsymbol{v}_{\nu+\frac{1}{2}} \right\rangle, \quad D_{\nu+\frac{1}{2}} := P_{\nu+\frac{1}{2}} - P^*_{\nu+\frac{1}{2}},$$

and $\mathcal{E}_\nu^{(t)} = \frac{1}{2}|\mathbf{u}_\nu(t+\Delta t) - \mathbf{u}_\nu(t)|^2$ is the entropy *production* due to the forward time-differencing. Thus, entropy stability is guaranteed if the former dominates the latter, and to this end, one needs to employ large enough numerical viscosity, $D_{\nu+\frac{1}{2}}$. How much is "enough"? observe that all the matrices involved are symmetric, and one is led to the matrix inequality

$$Q_{\nu+\frac{1}{2}} \geq |A_{\nu+\frac{1}{2}}| + 2|Q^*_{\nu+\frac{1}{2}}|, \qquad Q^*_{\nu+\frac{1}{2}} := P^*_{\nu+\frac{1}{2}} \left[\frac{\Delta \boldsymbol{v}_{\nu+\frac{1}{2}}}{\Delta \mathbf{u}_{\nu+\frac{1}{2}}}\right] = P^*_{\nu+\frac{1}{2}} H_{\nu+\frac{1}{2}}, \tag{36}$$

under CFL condition $\lambda|Q_{\nu+\frac{1}{2}}| \leq \frac{1}{4}$.

An alternative approach of securing entropy stability is achieved by adding a minimal amount of numerical viscosity *correction* of Khalfallah and Lerat (1989): starting with a given viscosity matrix $Q_{\nu+\frac{1}{2}}$, we use the *scalar* correction wherever $P_{\nu+\frac{1}{2}}$ has a smaller entropy dissipation then required by $P^*_{\nu+\frac{1}{2}}$, quantified by how negative $\left(D_{\nu+\frac{1}{2}}\right)_-$ is,

$$Q^c_{\nu+\frac{1}{2}} := Q_{\nu+\frac{1}{2}} + \beta^c_{\mu+\frac{1}{2}} I_{n\times n}, \qquad \beta^c := \frac{\left|\left\langle \Delta \boldsymbol{v}_{\nu+\frac{1}{2}}, \left(D_{\nu+\frac{1}{2}}\right)_- \Delta \mathbf{u}_{\nu+\frac{1}{2}} \right\rangle\right|}{\left\langle \Delta \boldsymbol{v}_{\nu+\frac{1}{2}}, \Delta \mathbf{u}_{\nu+\frac{1}{2}} \right\rangle}.$$

One can readily verify that $\langle \Delta \boldsymbol{v}, Q^c_{\nu+\frac{1}{2}} \Delta \mathbf{u}\rangle \geq \langle \Delta \boldsymbol{v}, P^*_{\nu+\frac{1}{2}} \Delta \boldsymbol{v}\rangle$.

Roe scheme (Roe, 1981) is the canonical example for an "upwind scheme": one sets[1] $Q^{\mathrm{Roe}}_{\nu+\frac{1}{2}} = |A_{\nu+\frac{1}{2}}|$ where $A_{\nu+\frac{1}{2}}$ is an averaged Jacobian such that $\Delta \mathbf{f}_{\nu+\frac{1}{2}} = A_{\nu+\frac{1}{2}} \Delta \mathbf{u}_{\nu+\frac{1}{2}}$. It has the attractive feature of keeping sharp resolution of shock discontinuities, whether they are physical or not, and it therefore fails to be entropy stable across steady rarefactions. The entropy stability condition (36) shows that one needs to add a minimal amount of numerical viscosity of order $|Q^*_{\nu+\frac{1}{2}}| \sim |\Delta \mathbf{u}_{\nu+\frac{1}{2}}|$ to enforce entropy stability.

[1]The function value of a diagonalizable matrix $M = T\Lambda T^{-1}$, is set as $h(M) = T\begin{pmatrix} h(\lambda_1) & \dots & \\ & \ddots & \\ & & h(\lambda_n)\end{pmatrix}T^{-1}$. In particular, a mid-value Jacobian of entropic system $A_{\nu+\frac{1}{2}}$ is symmetrizable, hence diagonalizable (Barth, 1999) and $|A_{\nu+\frac{1}{2}}| := T\begin{pmatrix} |\lambda_1| & \dots & \\ & \ddots & \\ & & |\lambda_n|\end{pmatrix}T^{-1}$.

5.2 A Homotopy Method

The entropy stability of LxF scheme was derived in Lax (1971) using a homotopy method, *independent* of the existence of entropic solution for Riemann problem. The general case (Tadmor, 2003, §8) implies entropy stability of (35) provided $Q_{\nu+\frac{1}{2}}$ is "large enough", $Q_{\nu+\frac{1}{2}} \geq \max_{\xi}|A\left(\mathbf{u}(\boldsymbol{v}_{\nu+\frac{1}{2}}(\xi))\right)|$, under the CFL condition $\lambda Q_{\nu+\frac{1}{2}} \leq \frac{1}{2}$. Observe that in this version of the so-called local Lax–Friedrichs scheme (Rusanov, 1961), the viscosity coefficient matrix domains *all* intermediate states rather than the one state offered by the Roe matrix $\boldsymbol{A}_{\nu+\frac{1}{2}} = \boldsymbol{B}_{\nu+\frac{1}{2}}\boldsymbol{H}_{\nu+\frac{1}{2}}$, $B_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} B(\mathbf{u}(\boldsymbol{v}_{\nu+\frac{1}{2}}(\xi)))d\xi$.

6 HIGHER-ORDER METHODS

The entropy conservative fluxes (32) and (33) are second-order accurate, leading to second-order entropy stable semidiscrete schemes. Extension to arbitrarily high-order entropy stable schemes was introduced in Fjordholm et al. (2012). The question of entropy stability for *fully-discrete* schemes is more intricate. Observe that the results in Section 5 compares with the first-order Roe numerical viscosity. A rigorous entropy stability analysis for fully-discrete *second-order* schemes can be found in Majda and Osher (1978, 1979) for modified Lax–Wendroff scheme, in Nessyahu and Tadmor (1990), Popov and Trifonov (2006) and Kurganov (2016) for Nessyahu–Tadmor scheme, in Osher and Tadmor (1988), Bouchut et al. (1996) and Coquel and LeFloch (1995) for the MUSCL scheme, in LeFloch and Rohde (2000), Chalons and LeFloch (2001), LeFloch et al. (2002) for high-order extensions based on caparison with entropy conservative fluxes, in Fjordholm et al. (2012) for the class of ENO-based schemes developed in Harten et al. (1987), Shu and Osher (1989), and in Jiang and Shu (1994), Qiu and Zhang (2016) and the references therein for DG method.

In practical applications, one proceeds by discretization of the entropy stable semidiscrete schemes using Runge–Kutta (RK) time integrators (Gottlieb and Ketcheso, 2016). The first- and second-order RK solvers are responsible for entropy *production*, and require entropy dissipation to compete with entropy production, making the overall fully-discrete scheme entropy stable. In contrast, the generic cases of third- and higher-order RK time integrators *retain* the entropy stability of the underlying semidiscrete scheme. The *linear* stability of third- and higher-order RK methods is well known for diagonalizable systems and was shown for general linear operators (Tadmor, 2002). This question of *nonlinear* entropy stability was demonstrated in Fjordholm et al. (2009, §4.2.5), but the rigorous entropy stability analysis for high-order RK solvers is, to our knowledge, completely open.

7 MULTIDIMENSIONAL SYSTEMS OF CONSERVATION LAWS

7.1 Cartesian Grids

When multidimensional conservation laws are discretized over grids which are *aligned with the Cartesian coordinates*, the question of entropy stability can be addressed along these coordinates, one dimension at the time. Thus, our one-dimensional setup applies.

Example 10 (Well balanced shallow-water equations). We consider the 2D shallow water equations, e.g., Xing (2017)

$$\frac{\partial}{\partial t}\mathbf{u}+\frac{\partial}{\partial x_1}\mathbf{f}^{(1)}(\mathbf{u})+\frac{\partial}{\partial x_2}\mathbf{f}^{(2)}(\mathbf{u})=-gh\nabla b(\mathbf{x}),\ \mathbf{u}:=[h,h\mathbf{v}]^\top,\ \mathbf{x}=(x_1,x_2)\in\Omega\subset\mathbb{R}^2,$$

which govern the motion of shallow-water with height h above bottom topography $b(\mathbf{x})$, and velocity field, $\mathbf{v}=(v_1,\ v_2)^\top$, driven by the convective fluxes, $\mathbf{f}^{(j)}=\left(hv_j,\ hv_1v_j+\frac{1}{2}gh^2\delta_{1j},\ hv_2v_j+\frac{1}{2}gh^2\delta_{2j}\right)^\top$,

$$h_t+(hv_1)_{x_1}+(hv_2)_{x_2}=0$$

$$(hv_1)_t+\left(hv_1^2+\frac{1}{2}gh^2\right)_{x_1}+(hv_1v_2)_{x_2}=-ghb_{x_1}$$

$$(hv_2)_t+(hv_2v_1)_{x_1}+\left(hv_2^2+\frac{1}{2}gh^2\right)_{x_2}=-ghb_{x_2}.$$

The entropy function is the total energy, $E(\mathbf{u})=\frac{1}{2}(gh(h+b)+h|\mathbf{v}|^2)$ with energy variables, $\boldsymbol{v}=(gh-\frac{1}{2}|\mathbf{v}|^2,v_1,v_2)^\top$. Observe that the shallow-water fluxes are quadratic in $z:=(h,\sqrt{h}v_1,\sqrt{h}v_2)^\top$. This enables a straightforward "affordable" algebraic approach for satisfying the energy conservative compatibility relation (26), $\left\langle \boldsymbol{v}_{\nu+1,\mu}-\boldsymbol{v}_{\nu,\mu}\,,\,\mathbf{f}^{(1)*}_{\nu+\frac{1}{2},\mu}\right\rangle=\psi(\boldsymbol{v}_{\nu+1,\mu})-\psi(\boldsymbol{v}_{\nu,\mu})$. Here we use the usual indexing of two-dimensional grid-functions attached to grid points $\mathbf{x}_{\nu,\mu}:=\left(x_{1\nu},x_{2\mu}\right)$. Using the average values, $\bar{z}_{\nu+\frac{1}{2}}:={}^1/_2(z_\nu+z_{\nu+1})$, one finds the x_1-entropy conservative flux (Fjordholm et al., 2011)

$$\mathbf{f}^{(1)*}_{\nu+\frac{1}{2},\mu}=\begin{bmatrix}\bar{h}_{\nu+\frac{1}{2},\mu}(\overline{v_1})_{\nu+\frac{1}{2},\mu}\\ \bar{h}_{\nu+\frac{1}{2},\mu}(\overline{v_1})^2_{\nu+\frac{1}{2},\mu}+\frac{g}{2}\left(\overline{h^2}\right)_{\nu+\frac{1}{2},\mu}+gh\overline{(b_{x_1})}\bar{h}_{\nu+\frac{1}{2},\mu}(\overline{v_1})_{\nu+\frac{1}{2},\mu}(\overline{v_2})_{\nu+\frac{1}{2},\mu}\end{bmatrix}. \tag{37a}$$

Similar expression applies for the conservative flux $\mathbf{f}^{(2)*}_{\nu,\mu+\frac{1}{2}}$ in the x_2-direction. We end up with the energy conservative scheme

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_{\nu,\mu}(t) = -\frac{1}{\Delta x_1}\left(\mathbf{f}^{(1)*}_{\nu+\frac{1}{2},\mu} - \mathbf{f}^{(1)*}_{\nu-\frac{1}{2},\mu}\right) - \frac{1}{\Delta x_2}\left(\mathbf{f}^{(2)*}_{\nu,\mu+\frac{1}{2}} - \mathbf{f}^{(2)*}_{\nu,\mu-\frac{1}{2}}\right). \tag{37b}$$

These schemes recover the precise energy balance, $E(\mathbf{u})_t + F^{(1)}(\mathbf{u})_{x_1} + F^{(2)}(\mathbf{u})_{x_2} = 0$, in terms of the energy fluxes $F^{(j)}(\mathbf{u}) = \frac{1}{2}\left(hv_j|\mathbf{v}|^2 + gh(h+b)\right)$.

7.2 Unstructured Grids

We consider a computational domain which is *partitioned* to a set of nonoverlaping cells, $\Omega_h = \bigcup_i C_i$. Let $\mathbf{n}_{ij} = \int_{\partial C_i \cap \partial C_j} \mathbf{n}\,\mathrm{d}\sigma$ be the unit normal on the nonempty interface pointing out of the control volume C_i. Note that when we sum the normals over all neighbouring cells $\mathcal{N}_i$ then $\sum_{j\in\mathcal{N}_i} \mathbf{n}_{ij} = 0$. The semidiscrete finite volume scheme of (1) reads,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu = -\frac{1}{|C_\nu|}\sum_{\mu\in\mathcal{N}_\nu} \mathbf{f}_{\nu\mu},$$

where the 2-point numerical flux, $\mathbf{f}_{\nu\mu} = \mathbf{f}\left(\mathbf{u}_\nu(t), \mathbf{u}_\mu(t), \mathbf{n}_{\nu\mu}\right)$, is assumed to consistent, $\mathbf{f}(\mathbf{u},\ \mathbf{u},\ \mathbf{n}) = \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}$. The scheme is conservative in the sense that $\sum_\nu |C_\nu|\,\mathbf{u}_\nu(t)$ is conserved in time, since $\mathbf{f}_{\nu\mu}(\mathbf{n}) = -\mathbf{f}_{\mu\nu}(-\mathbf{n})$.

The corresponding question of entropy stability for such schemes, satisfying the cell entropy inequality, $\frac{\mathrm{d}}{\mathrm{d}t}\eta(\mathbf{u}_\nu(t)) \leq -\frac{1}{|C_\nu|}\sum_{\mu\in\mathcal{N}_\nu} \mathbf{F}_{\nu\mu}$, was investigated by extension of the tools outlined above. In particular, the question of *scalar* entropy stability was studied in a long series of papers and we mention here (Barth, 1999; Eymard et al., 2000; Kroner et al., 1995; Sonar, 2016) and the references therein.

To design or investigate entropy stable fluxes for systems of conservation laws, one may proceed by comparing their numerical viscosities with entropy conservative schemes. A numerical flux $\mathbf{f}^*_{\nu\mu} = \mathbf{f}^*(\mathbf{u}_\nu, \mathbf{u}_\mu, \mathbf{n}_{\nu\mu})$ is *entropy conservative* if its components, projected along the normal directions, $\mathbf{f}_{\nu\mu} = \mathbf{f}^{(1)}_{\nu\mu}\mathbf{n}^{(1)}_{\nu\mu} + \mathbf{f}^{(2)}_{\nu\mu}\mathbf{n}^{(2)}_{ij}$, satisfy the compatibility relations,

$$\left\langle \boldsymbol{v}_\mu - \boldsymbol{v}_\nu, \mathbf{f}^{*(j)}_{\nu\mu} \right\rangle = \psi^{(j)}_\nu - \psi^{(j)}_\mu,$$

where $\psi^{(j)}(\mathbf{u})$ is the usual entropy potential $\psi^{(j)} := \left\langle \boldsymbol{v}, \mathbf{f}^{(j)} \right\rangle - F^{(j)},\ \ j = 1,2$. Entropy stable fluxes then take the form $\mathbf{f}_{\nu\mu} = \mathbf{f}^*_{\nu\mu} - \frac{1}{2}D_{\nu\mu}(\boldsymbol{v}_\nu - \boldsymbol{v}_\mu)$ for positive-definite entropy dissipation matrices D's. The study of entropy stable schemes on two-dimensional unstructured grids by comparing numerical viscosities along these lines was carried out in Ray et al. (2016).

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