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CRITICAL THRESHOLDS IN MULTI-DIMENSIONAL EULER-POISSON EQUATIONS WITH RADIAL SYMMETRY*

DONGMING WEI†, EITAN TADMOR‡, AND HANTAEK BAE§

To Dave Levermore on his 60th birthday with friendship and appreciation

Abstract. We study the global regularity of multi-dimensional repulsive Euler-Poisson equations in the radial setup. We show that the question of global regularity vs. finite breakdown of smooth solutions depends on whether the initial configuration crosses an initial *critical threshold* in configuration space. Specifically, there exists a global-in-time smooth solution if and only if the initial configuration of density ρ_0, radial velocity R_0, and electrical charge e_0 satisfies $R_0' \geq F(\rho_0, e_0, R_0)$ for a certain threshold F. Similarly, we characterize the critical threshold for global smooth solutions subject to two-dimensional radially symmetric data with swirl. We also discuss a possible framework for global regularity analysis beyond the radial case, which indicates that the main difficulty lies with bounding the *spectral gap*, $\lambda_2(\nabla \mathbf{u}) - \lambda_1(\nabla \mathbf{u})$.

Key words. Repulsive Euler-Poisson equations, radial symmetry, axisymmetric solutions with or without swirl, critical threshold phenomena.

AMS subject classifications. 35L65, 35D05.

1. Introduction

We are concerned here with the Euler-Poisson equations, where the density $n(\cdot,t): \mathbb{R}^d \mapsto \mathbb{R}$ and velocity field, $\mathbf{u}(\cdot,t): \mathbb{R}^d \mapsto \mathbb{R}^d$ are governed by the system of equations

$$n_t + \nabla \cdot (n\mathbf{u}) = 0, \tag{1.1a}$$

$$(n\mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(n) = kn\nabla\phi, \qquad \Delta\phi = n - b(x). \tag{1.1b}$$

This system represents the usual statements of the conservation of mass and Newton's second law subject to isentropic pressure term $p(n) = An^\gamma$ with amplitude $A > 0$, and an electrical charge $\nabla\phi$ induced by the density n with background mass which we set to zero ($b(x) \equiv 0$). The parameter k is a scaled physical constant signifying the property of the underlying force; the force is *repulsive* if $k > 0$, and *attractive* if $k < 0$. This system describes dynamic behaviors of many important physical flows, from small scale models of charged transport [20, 12], expansion of cold ions [9], and collisional plasma [10], to large scale models of cosmological waves [1, 2]. For smooth solutions away from vacuum, (1.1) can be reduced to

$$n_t + \nabla \cdot (n\mathbf{u}) = 0, \tag{1.2a}$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{A\gamma}{\gamma - 1}\nabla\left(n^{\gamma-1}\right) = k\nabla\Delta^{-1} n. \tag{1.2b}$$

Let us list some known results regarding (1.1). For the local existence in the small H^s neighborhood of a steady state, see [7, 17, 19]. Global existence due to

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†Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706 (dwei@math.wisc.edu). http://www.math.wisc.edu/~dwei/.

‡Department of Mathematics, Center of Scientific Computation and Mathematical Modeling (CSCAMM), and Institute for Physical Science and Technology (IPST), University of Maryland, College Park, MD 20742 (tadmor@cscamm.umd.edu). http://www.cscamm.umd.edu/~tadmor.

§Center of Scientific Computation and Mathematical Modeling (CSCAMM), University of Maryland, College Park, MD 20742 (hbae@cscamm.umd.edu).

damping relaxation with nonzero background can be found in [16, 23, 24]. For the model without damping relaxation, global existence in the neighborhood of a steady state was obtained in [8]. On the other hand, finite time blowup results for attractive forces were obtained in [18], and for repulsive forces in [21, 3, 4].

Beyond the two scenarios of global existence of smooth solutions and finite-time breakdown, a third scenario of *conditional regularity* was promoted in [5, 6, 14, 15], where it was shown that there exists a "large" set of $\mathcal{O}(1)$ initial configurations which lead to global smooth solutions, and the complementary "large" set of $\mathcal{O}(1)$ initial configurations which yield finite-time breakdown. That is, global regularity versus finite-time breakdown is separated by a non-trivial *critical threshold* in the configuration space. The critical threshold in one-dimensional models of (1.1) were studied in [5, 6]. The critical thresholds in higher dimensional models were analyzed in [14, 15] via spectral dynamics. In particular, in [15] the authors studied the so-called restricted Euler-Poisson model in two spatial dimensions, and showed the existence of the critical threshold in terms of the initial density, initial divergence, and the initial spectral gap.

The goal of this paper is to answer the question of global regularity versus finite-time breakdown of *radial* solutions to the multi-dimensional repulsive Euler-Poisson equations (1.2). Most of our discussion is devoted to the *pressureless* case, $p(n)\equiv 0$, where we distinguish between two different types of radial solutions: an axisymmetric flow without swirl discussed in Section 2, and the two-dimensional axisymmetric flow with swirl discussed in Section 3. Finally, we briefly comment on critical thresholds for radial solutions of the full system (1.2) with pressure in Section 4, and on the difficulties of addressing the question of global regularity of Euler-Poisson equations beyond the radial case in Section 5.

2. Radial solutions of Euler-Poisson equations without swirl

We consider the d-dimensional pressureless Euler-Poisson equations (1.1):

$$n_t+\nabla\cdot(n\mathbf{u})=0, \qquad \mathbf{u}(\cdot,t):\mathbb{R}^d\mapsto\mathbb{R}^d, \tag{2.1a}$$

$$(n\mathbf{u})_t+\nabla\cdot(n\mathbf{u}\otimes\mathbf{u})=\kappa n\nabla\phi, \qquad \Delta\phi=n, \tag{2.1b}$$

subject to spherically symmetric initial data

$$\rho_0(\mathbf{x})=\rho_0(r), \quad \mathbf{u}_0(\mathbf{x})=R_0(r)\frac{\mathbf{x}}{r}, \quad r=|\mathbf{x}|.$$

Then, a radial solution of (2.1) of the form $\rho:=r^{d-1}n$, and $\mathbf{u}(\mathbf{x},t)=u(r,t)\frac{\mathbf{x}}{r}$ is sought, where (ρ,u) solves the corresponding system

$$\rho_t+(\rho u)_r=0, \tag{2.2a}$$

$$u_t+uu_r=\kappa\phi_r, \qquad (r^{d-1}\phi_r)_r=\rho, \tag{2.2b}$$

subject to initial conditions $\rho(r,0)=\rho_0(r)$ and $u(r,0)=R_0(r)$. Let $e:=r^{d-1}\phi_r$ be the radial electric field so that $e_r=\rho$, and the charge e satisfies a transport equation $e_t+ue_r=0$. Therefore, e remains constant along particle paths: $\{r(\alpha,t): \dfrac{dr}{dt}= u(r,t), \quad r(\alpha,0)=\alpha\}$,

$$\frac{d}{dt}e(r(\cdot,t))=0.$$

Along these particle paths we also have

$$\frac{du}{dt} = \frac{\kappa e}{r^{d-1}} = \frac{\kappa e_0}{r^{d-1}},$$

and we end up with the second-order equation

$$\frac{d^2 r}{dt^2} = \frac{\kappa e_0}{r^{d-1}}, \qquad r(\alpha,0) = \alpha, \qquad \frac{dr}{dt}(\alpha,0) = R_0(\alpha). \tag{2.3}$$

We distinguish between two cases. If $\kappa < 0$, then the solution of (2.3) always breaks down at a finite time. We turn to the repulsive case, $\kappa > 0$, which will occupy the rest of this section. According to [6, Corollary 5.2], a smooth solution of the repulsive Euler-Poisson equations (2.1) blows up at a finite time, $t = t_c$, if and only if there exist an $\alpha \in \mathbb{R}$ such that $(\partial r/\partial\alpha)(\alpha, t_c) = 0$. Thus, the solution to the Euler-Poisson equations (2.1) remains smooth as long as

$$\frac{\partial r}{\partial \alpha}(\alpha,t) > 0, \quad \forall \alpha > 0. \tag{2.4}$$

To verify (2.4), we multiply (2.3) by $\dfrac{dr}{dt}$, obtaining

$$\frac{1}{2}\frac{d}{dt}\left(\left(\frac{dr}{dt}\right)^2\right) = \frac{\kappa e_0}{r^{d-1}}\frac{dr}{dt} = \kappa e_0 \frac{dN(r)}{dt}, \qquad N(r) := \begin{cases} \ln r, & d=2 \\ \dfrac{1}{2-d} r^{2-d}, & d>2. \end{cases}$$

It follows that

$$\left(\frac{dr}{dt}\right)^2 - R_0^2(\alpha) = 2\kappa e_0 (N(r) - N(\alpha)).$$

Following [6], we restrict ourselves to the case $R_0 > 0$. Since $\dfrac{du}{dt} = \dfrac{\kappa e}{r^{d-1}} > 0$, we have $u(\cdot,t) > R_0(\alpha) > 0$, which in turn implies that $r(\cdot,t)$ is increasing, $r(\alpha,t) > r(\alpha,0) = \alpha$. It follows, since the Newtonian potential $N(r)$ is increasing, that $N(r(\cdot,t))$ is increasing in time; i.e., $N(r(\alpha,t)) > N(r(\alpha,0)) = N(\alpha)$. Therefore

$$\frac{dr}{dt} = [2\kappa e_0 \big(N(r) - N(\alpha)\big) + R_0^2(\alpha)]^{1/2},$$

or

$$\frac{dr}{[2\kappa e_0 \big(N(r) - N(\alpha)\big) + R_0^2(\alpha)]^{1/2}} = dt. \tag{2.5}$$

Integrating both sides we find

$$\int_\alpha^{r(\alpha,t)} \frac{1}{[2\kappa e_0 \big(N(s) - N(\alpha)\big) + R_0^2(\alpha)]^{1/2}} ds = t. \tag{2.6}$$

Taking the α derivative of (2.6) yields

$$\frac{\partial r(\alpha,t)}{\partial \alpha} \frac{1}{[2\kappa e_0(\alpha)\big(N(r(\alpha,t)) - N(\alpha)\big) + R_0^2(\alpha)]^{1/2}} - \frac{1}{R_0(\alpha)}$$
$$-\frac{1}{2}\int_\alpha^{r(\alpha,t)} \frac{2\kappa \rho_0(\alpha)\big(N(s) - N(\alpha)\big) - 2\kappa e_0(\alpha)\alpha^{1-d} + 2R_0(\alpha)R_0'(\alpha)}{[2\kappa e_0(\alpha)\big(N(s) - N(\alpha)\big) + R_0^2(\alpha)]^{3/2}} ds = 0.$$

Therefore

$$\frac{\partial r(\alpha,t)}{\partial \alpha}=\Big(2\kappa e_0(\alpha)\big(N(r(\alpha,t))-N(\alpha)\big)+R_0^2(\alpha)\Big)^{1/2}\times\psi_\alpha(r(\alpha,t)), \tag{2.7a}$$

where

$$\psi_\alpha(y):=\left(\frac{1}{R_0(\alpha)}+\int_\alpha^y\frac{\kappa\rho_0(\alpha)\big(N(s)-N(\alpha)\big)-\kappa e_0(\alpha)\alpha^{1-d}+R_0(\alpha)R_0'(\alpha)}{[2\kappa e_0(\alpha)\big(N(s)-N(\alpha)\big)+R_0^2(\alpha)]^{3/2}}ds\right). \tag{2.7b}$$

Thus, a global smooth solution exists as long as $\psi_\alpha(r(\alpha,t))$ remains positive, so that $\partial_\alpha r(\alpha,t)>0$ for all $\alpha>0$. This leads us to the main theorem of this section, which specifies the precise critical thresholds for radial solutions of the repulsive Euler-Poisson equations.

THEOREM 2.1 (Radial solutions without swirl). *Consider the d-dimensional repulsive Euler-Poisson equations* (2.1) *subject to the spherically symmetric initial conditions, $\rho_0(r)$ and $\mathbf{u}_0(\mathbf{x})=R_0(r)\frac{\mathbf{x}}{r}$, with $R_0>0$. Then, they admit a globally smooth radial solution $\left(\rho(r,t),u(r,t)\frac{\mathbf{x}}{r}\right)$ if and only if*

$$R_0'(\alpha)>\sup_{y>\alpha}F_\alpha(y;\rho_0,e_0,R_0); \tag{2.8a}$$

here $F_\alpha(y)\equiv F_\alpha(y;\rho_0,e_0,R_0)$ is given by

$$F_\alpha(y):=\frac{-\dfrac{1}{R_0(\alpha)}-\displaystyle\int_\alpha^y\frac{\kappa\rho_0(\alpha)\big(N(s)-N(\alpha)\big)-\kappa e_0(\alpha)\alpha^{1-d}}{[2\kappa e_0(\alpha)\big(N(s)-N(\alpha)\big)+R_0^2(\alpha)]^{3/2}}ds}{R_0(\alpha)\displaystyle\int_\alpha^y\frac{1}{[2\kappa e_0(\alpha)\big(N(s)-N(\alpha)\big)+R_0^2(\alpha)]^{3/2}}ds}. \tag{2.8b}$$

Proof. Recall that $u(\alpha,t)>R_0(\alpha)>0$. Therefore, every $y>\alpha$ takes the form $y=r(\alpha,t)$ for some $t>0$. According to (2.7), therefore, the radial solutions of the repulsive equations (2.1) remains smooth if and only if $\psi_\alpha(y)$ is positive for all $y(>\alpha)$'s:

$$\frac{1}{R_0(\alpha)}+\int_\alpha^y\frac{\kappa\rho_0(\alpha)\big(N(s)-N(\alpha)\big)-\kappa e_0(\alpha)\alpha^{1-d}+R_0(\alpha)R_0'(\alpha)}{[2\kappa e_0(\alpha)\big(N(s)-N(\alpha)\big)+R_0^2(\alpha)]^{3/2}}ds>0,\quad \forall y>\alpha>0. \tag{2.9}$$

The critical threshold condition (2.9) is equivalent to

$$\begin{aligned}&R_0(\alpha)R_0'(\alpha)\int_\alpha^y\frac{1}{[2\kappa e_0(\alpha)N(s)-2\kappa e_0(\alpha)N(\alpha)+R_0^2(\alpha)]^{3/2}}ds\\&>-\frac{1}{R_0(\alpha)}-\int_\alpha^y\frac{\kappa\rho_0(\alpha)\big(N(s)-N(\alpha)\big)-\kappa e_0(\alpha)\alpha^{1-d}}{[2\kappa e_0(\alpha)\big(N(s)-N(\alpha)\big)+R_0^2(\alpha)]^{3/2}}ds.\end{aligned}$$

That is, since $R_0(\alpha)>0$,

$$R_0'(\alpha)>\frac{-\dfrac{1}{R_0(\alpha)}-\displaystyle\int_\alpha^y\frac{\kappa\rho_0(\alpha)\big(N(s)-N(\alpha)\big)-\kappa e_0(\alpha)\alpha^{1-d}}{[2\kappa e_0(\alpha)\big(N(s)-N(\alpha)\big)+R_0^2(\alpha)]^{3/2}}ds}{R_0(\alpha)\displaystyle\int_\alpha^y\frac{1}{[2\kappa e_0(\alpha)\big(N(s)-N(\alpha)\big)+R_0^2(\alpha)]^{3/2}}ds},\quad \forall y>\alpha, \tag{2.10}$$

and (2.8) follows. □

Remark 2.2. $\sup_{y>\alpha} F_\alpha(y)$ is finite since $F_\alpha(y) < \kappa e_0(\alpha)\alpha^{1-d}/R_0(\alpha)$, $\forall y > \alpha$.

Example 2.3 (The two- and three-dimensional cases). *Critical thresholds for the two- and three-dimensional radial Euler-Poisson were worked out in [6, Section 5]: they take the form of a* sub-critical *threshold, $R_0'(\alpha) > F_\alpha^+(\rho_0,e_0,R_0)$, which guarantees the existence of a global smooth solution, and a* super-critical threshold *$R_0'(\alpha) > F_\alpha^-(\rho_0,e_0,R_0)$ which leads to a finite-time breakdown. Here, we close the gap $F_\alpha^+ > F_\alpha^-$: Theorem 2.1 provides the precise description of the critical threshold F_α in the two- and three-dimensional spaces of initial configurations.*

Example 2.4 (The four-dimensional case). *When $d=4$, then $N(s) = -s^{-2}/2$, and the integral (2.6) admits the explicit form,*

$$\begin{aligned}
&\int_\alpha^r \frac{1}{[2\kappa e_0(\alpha)N(s) - 2\kappa e_0(\alpha)N(\alpha) + R_0^2(\alpha)]^{1/2}} ds \\
&= \int_\alpha^r \frac{1}{[-\kappa e_0(\alpha)s^{-2} + \kappa e_0(\alpha)\alpha^{-2} + R_0^2(\alpha)]^{1/2}} ds \\
&= \int_\alpha^r \frac{s}{\left[-\kappa e_0(\alpha) + \left(\kappa e_0(\alpha)\alpha^{-2} + R_0^2(\alpha)\right)s^2\right]^{1/2}} ds \\
&= \frac{1}{2}\int_\alpha^r \frac{1}{\left[-\kappa e_0(\alpha) + \left(\kappa e_0(\alpha)\alpha^{-2} + R_0^2(\alpha)\right)s^2\right]^{1/2}} d(s^2) \\
&= \frac{\left[-\kappa e_0(\alpha) + \left(\kappa e_0(\alpha)\alpha^{-2} + R_0^2(\alpha)\right)r^2\right]^{1/2} - \alpha R_0(\alpha)}{\left(\kappa e_0(\alpha)\alpha^{-2} + R_0^2(\alpha)\right)}.
\end{aligned}$$

Thus

$$\begin{aligned}
r(\alpha,t) &= \sqrt{\frac{\left(t\left(\kappa e_0(\alpha)\alpha^{-2} + R_0^2(\alpha)\right) + \alpha R_0(\alpha)\right)^2 + \kappa e_0(\alpha)}{\kappa e_0(\alpha)\alpha^{-2} + R_0^2(\alpha)}} \\
&= \sqrt{\left(\kappa e_0(\alpha)\alpha^{-2} + R_0^2(\alpha)\right)t^2 + 2\alpha R_0(\alpha)t + \alpha^2}.
\end{aligned}$$

This is the same as [6, equation 5.46]. Taking the α derivative

$$\frac{\partial r}{\partial \alpha} = \frac{\alpha + (R_0 + \alpha R_0')t + (R_0 R_0' - \kappa e_0 \alpha^{-3} + \frac{1}{2}\kappa\rho_0\alpha^{-2})t^2}{\sqrt{\left(\kappa e_0(\alpha)\alpha^{-2} + R_0^2(\alpha)\right)t^2 + 2\alpha R_0(\alpha)t + \alpha^2}}.$$

We conclude that $\partial_\alpha r(\cdot,t)$ remains positive for all $t > 0$ if and only if both (i) and (ii) hold:

(i) $R_0 R_0' - \kappa e_0\alpha^{-3} + \frac{1}{2}\kappa\rho_0\alpha^{-2} > 0$;

(ii) $[R_0 - \alpha R_0']^2 < 4\alpha[-\kappa e_0\alpha^{-3} + \frac{1}{2}\kappa\rho_0\alpha^{-2}]$ (so that $\partial_\alpha r(\cdot,t) = 0$ has no real solution), or $R_0 + \alpha R_0' > 0$ (so that $\partial_\alpha r(\cdot,t) = 0$ has two negative solutions, $t_1 < 0$ and $t_2 < 0$).

We set

$$f_1(\alpha) := \frac{\kappa e_0 \alpha^{-3} - \frac{1}{2}\kappa \rho_0 \alpha^{-2}}{R_0}, \tag{2.11a}$$

and

$$f_2(\alpha) = \begin{cases} \frac{1}{\alpha}\min\{-R_0, R_0 - \sqrt{-4\alpha R_0 f_1(\alpha)}\}, & f_1(\alpha) \le 0, \\ -\infty, & f_1(\alpha) > 0. \end{cases} \tag{2.11b}$$

Then the critical threshold condition (2.8) for 4*-dimensional radial repulsive solutions reads*

$$R_0'(\alpha) > \max\{f_1(\alpha), f_2(\alpha)\}. \tag{2.11c}$$

Starting from (2.10) will yield the same result; that is,

$$\sup_{y>\alpha} F_\alpha(y) = \max\{f_1(\alpha), f_2(\alpha)\},$$

which proves the 4*-dimensional critical threshold condition of the form* $R_0'(\alpha) > \sup_{y>\alpha} F_\alpha(y)$. *This extends the critical threshold result of [6, theorem 5.10] which uses the further restriction that* R_0' *needs to be upper-bounded for global smooth solutions: the reason is that the authors of [6] ignored case (i) and the second part of case (ii).*

3. Two-dimensional radial Euler-Poisson solutions with swirl

Consider the two dimensional Euler-Poisson equations

$$n_t + \nabla\cdot(n\mathbf{u}) = 0, \qquad \mathbf{u}(\cdot,t) = (u(\cdot,t), v(\cdot,t)) : \mathbb{R}^2 \mapsto \mathbb{R}^2, \tag{3.1a}$$

$$(n\mathbf{u})_t + \nabla\cdot(n\mathbf{u}\otimes\mathbf{u}) = \kappa n \nabla\phi, \qquad \Delta\phi = n = \rho/r, \tag{3.1b}$$

subject to spherically symmetric initial data with swirl,

$$\rho_0(\mathbf{x}) = \rho_0(r),\ \mathbf{u}_0(\mathbf{x}) = R_0(r)\frac{\mathbf{x}}{r} + \Theta_0(r)\frac{\mathbf{x}^\perp}{r}, \quad \mathbf{x} = (x,y),\ \mathbf{x}^\perp = (-y,x),\ r = |\mathbf{x}|. \tag{3.2}$$

Here, R_0 and Θ_0 are the radial and tangential components of the initial velocity field $\mathbf{u}_0$. Due to the radial symmetry, the solution propagates along circles; starting with a circle of radius α at time 0, the particle path $\frac{d}{dt}\mathbf{x} = \mathbf{u}(\mathbf{x},t)$ such that $|\mathbf{x}|(\alpha,0) = \alpha$ will form a circle at time t with radius $|\mathbf{x}| = r(\alpha,t)$. Indeed, the precise evolution of $r(\alpha,t)$ will be worked out in (3.8) below. To trace the solution along these circles, we can therefore pick *any* sampling point on the initial circle with radius α and evolve it along its particle path to discover $r(\alpha,t)$. Without loss of generality, we choose the particle located at $(x,y)|_{t=0} = (\alpha,0)$. Observe that the x- and y-components of the velocity at this initial position, $(x,y) = (\alpha,0)$, coincide with the polar components, $u_0(\alpha,0) = R_0(\alpha)$ and $v_0(\alpha,0) = \Theta_0(\alpha)$. Since the charge $e = r\phi_r$ remains constant along these paths, then $\mathbf{u}(t) \equiv (u(\mathbf{x}(t),t), v(\mathbf{x}(t),t))$, is therefore governed by

$$\frac{du}{dt} = \frac{\kappa e_0(\alpha) x}{r^2}, \quad \frac{dv}{dt} = \frac{\kappa e_0(\alpha) y}{r^2}, \tag{3.3}$$

subject to initial data $u((\alpha,0),0) = R_0(\alpha), v((\alpha,0),0) = \Theta_0(\alpha)$. This implies that

$$\frac{d^2x}{dt^2} = \kappa e_0 \frac{x}{r^2}, \tag{3.4a}$$

$$\frac{d^2y}{dt^2} = \kappa e_0 \frac{y}{r^2}. \tag{3.4b}$$

Multiplying (3.4a) by $2\frac{dx}{dt}$, (3.4b) by $2\frac{dy}{dt}$, and adding, we obtain

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{dy}{dt}\frac{d^2y}{dt^2} = 2\kappa e_0 \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{r^2},$$

or

$$\frac{d}{dt}\Big[\Big(\frac{dx}{dt}\Big)^2 + \Big(\frac{dy}{dt}\Big)^2\Big] = \frac{d}{dt}(2\kappa e_0 \ln r).$$

Therefore

$$\Big(\frac{dx}{dt}\Big)^2 + \Big(\frac{dy}{dt}\Big)^2 = 2\kappa e_0 \ln\frac{r}{\alpha} + R_0^2(\alpha) + \Theta_0^2(\alpha). \tag{3.5}$$

Another useful equality is

$$x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} = \kappa e_0. \tag{3.6}$$

We combine (3.5) and (3.6) to find

$$\frac{d^2(r^2)}{dt^2} = \frac{d^2(x^2+y^2)}{dt^2} = 2\kappa e_0 + 4\kappa e_0 \ln\frac{r}{\alpha} + 2R_0(\alpha)^2 + 2\Theta_0^2(\alpha). \tag{3.7}$$

Multiplying (3.7) by $d(r^2)/dt$, we obtain

$$\frac{d(r^2)}{dt}\frac{d^2(r^2)}{dt^2} = 2r\frac{dr}{dt}[2\kappa e_0 + 4\kappa e_0 \ln\frac{r}{\alpha} + 2R_0^2(\alpha) + 2\Theta_0^2(\alpha)].$$

That is,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\frac{d(r^2)}{dt}\Big]^2 \\ &= (2\kappa e_0 + 2R_0^2(\alpha) + 2\Theta_0^2(\alpha))\frac{d(r^2)}{dt} + \frac{d}{dt}(4\kappa e_0 r^2 \ln r - 2\kappa e_0 r^2) - \frac{d}{dt}(4\kappa e_0 r^2 \ln\alpha),\end{aligned}$$

and hence

$$\begin{aligned}&\frac{1}{2}\Big[\frac{d(r^2)}{dt}\Big]^2 - \frac{1}{2}\big(2\alpha R_0(\alpha)\big)^2 \\ &= (2\kappa e_0 + 2R_0^2(\alpha) + 2\Theta_0^2(\alpha))(r^2-\alpha^2) + (4\kappa e_0 r^2 \ln r - 2\kappa e_0 r^2) \\ &\quad - (4\kappa e_0 \alpha^2 \ln\alpha - 2\kappa e_0 \alpha^2) - (4\kappa e_0 r^2 \ln\alpha) + (4\kappa e_0 \alpha^2 \ln\alpha) \\ &= 2(R_0^2(\alpha) + \Theta_0^2(\alpha))(r^2-\alpha^2) + 4\kappa e_0 r^2 \ln\frac{r}{\alpha}.\end{aligned}$$

Therefore,

$$2r^2\Big[\frac{dr}{dt}\Big]^2 = 2r^2(R_0^2(\alpha) + \Theta_0^2(\alpha)) - 2\alpha^2\Theta_0^2(\alpha) + 4\kappa e_0 r^2 \ln\frac{r}{\alpha},$$

which implies

$$\frac{dr}{dt} = \Big[R_0^2(\alpha) + \Theta_0^2(\alpha) - \frac{\alpha^2}{r^2}\Theta_0^2(\alpha) + 2\kappa e_0 \ln\frac{r}{\alpha}\Big]^{1/2} =: \zeta(r,\alpha)^{-1}. \tag{3.8}$$

Integrating $\zeta(r,\alpha)dr = dt$ we obtain

$$\int_{\alpha}^{r(\alpha,t)} \zeta(s,\alpha)ds = t, \tag{3.9}$$

and taking the α-derivative of (3.9) yields

$$\frac{\partial r(\alpha,t)}{\partial \alpha} = \frac{1}{\zeta(r(\alpha,t),\alpha)} \times \varphi_\alpha(r(\alpha,t)), \tag{3.10a}$$

where

$$\varphi_\alpha(y) = \zeta(\alpha,\alpha) - \int_{\alpha}^{y} \frac{\partial}{\partial \alpha}\zeta(s,\alpha)ds = 0, \qquad \zeta(\alpha,\alpha) = \frac{1}{R_0(\alpha)}. \tag{3.10b}$$

Thus, (3.1) admits global smooth solutions as long as $\varphi_\alpha(r(\alpha,t))$ remains positive. Similar to our analysis of the case without a swirl in Section 2, we derive the following theorem.

THEOREM 3.1 (Radial solutions with swirl). *Consider the two dimensional repulsive Euler-Poisson equations* (3.1) *with radial initial data with swirl* (3.2). *They admit global in time smooth solutions if and only if*

$$P_\alpha(y)R_0'(\alpha) + Q_\alpha(y)\Theta_0'(\alpha) > S_\alpha(y), \qquad \forall y > \alpha > 0, \tag{3.11a}$$

where P_α, Q_α *and* S_α *are given by*

$$P_\alpha(y) = \int_{\alpha}^{y} \frac{R_0}{\left[R_0^2 + \Theta_0^2 - (\alpha^2/s^2)\Theta_0^2 + 2\kappa e_0 \ln(s/\alpha)\right]^{3/2}} ds,$$

$$Q_\alpha(y) = \int_{\alpha}^{y} \frac{\Theta_0\left(1-(\alpha^2/s^2)\right)}{\left[R_0^2 + \Theta_0^2 - (\alpha^2/s^2)\Theta_0^2 + 2\kappa e_0 \ln(s/\alpha)\right]^{3/2}} ds,$$

$$S_\alpha(y) = \int_{\alpha}^{y} \frac{(\alpha/s^2)\Theta_0^2 - \kappa\rho_0 \ln(s/\alpha) + (\kappa e_0/\alpha)}{\left[R_0^2 + \Theta_0^2 - (\alpha^2/s^2)\Theta_0^2 + 2\kappa e_0 \ln(s/\alpha)\right]^{3/2}} ds - \frac{1}{R_0}.$$

4. Radial Euler-Poisson equations with pressure

We now return to the Euler-Poisson equations with pressure. The one-dimensional critical threshold in this case [22] states that there exists a constant $K_0 = K_0(k) > 0$ such that

$$R_0'(\alpha) > -K_0\sqrt{\rho_0(\alpha)} + \sqrt{A\gamma}|\rho_0'(\alpha)|\big(\rho_0(\alpha)\big)^{\frac{\gamma-3}{2}}, \qquad \gamma \geq 1. \tag{4.1}$$

When $A=0$, (4.1) recovers the critical threshold of the one-dimensional pressureless case with $K_0 = \sqrt{2k}$. Otherwise, the inequality (4.1) quantifies the *competition* between the destabilizing pressure effects, as the range of sub-critical initial configurations shrinks with the growth of the amplitude of the pressure A, while the stabilizing effect of the Poisson forcing increases the sub-critical range with a growing k. Similarly, we expect that pressure will have a similar "competitive" role with multi-dimensional radial solutions of the Euler-Poisson equations (1.2a). Namely, if the amplitude of the pressure is not "too large" relative to k then (1.2a) admits global smooth solutions for a large set of sub-critical initial configurations. The precise form of the multidimensional radial critical threshold is left for a future work.

5. Euler-Poisson in $\mathbb{R}^2$ – beyond the radial case

In this section we discuss the difficulties in addressing the question of global smooth solutions vs. finite time breakdown of the Euler-Poisson equation in the general non-radial case. The main difficulty lies with the nonlocal term $\nabla\phi$ in the system (2.1). This feature was emphasized in [15], and was the main motivation for studying the restricted model, where the nonlocal term $\nabla\phi$ is replaced by a local term. Here is a brief overview.

We start with the local well-posedness and the blowup criterion for the system (2.1). To simplify matters, we restrict our attention to the two-dimensional case, governing the velocity field $\mathbf{u}:=(\mathbf{u}_1,\mathbf{u}_2)$. Standard energy method arguments lead to the local well-posedness for the system (2.1) in energy spaces H^s, $s>2$, and the blowup criterion in terms of $\nabla\mathbf{u}$. Then, we will refine the blowup condition in terms of the divergence $d:=\nabla\cdot\mathbf{u}$.

Proposition 5.1 (Local existence). *Fix $k>0$ and consider the Euler-Poisson equations*

$$n_t+\nabla\cdot(n\mathbf{u})=0, \tag{5.1a}$$

$$\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}=k\nabla\Delta^{-1}n, \tag{5.1b}$$

subject to initial conditions $n_0\in H^s(\mathbb{R}^2)$ and $\nabla\mathbf{u}_0\in H^s(\mathbb{R}^2)$, $s>1$. Then there exists $T>0$ and a unique solution of (5.1), $(n,\nabla\mathbf{u})\in C([0,T];H^s)$. Moreover, we have the following blow-up criterion: if $t_c>0$ is the maximal time for the existence of such a smooth solution, then

$$t_c<\infty \Rightarrow \int_0^{t_c}\|\nabla\cdot\mathbf{u}(t)\|_{L^\infty}dt=\infty.$$

Proof. The proof consists of three steps.

Step 1. We begin with standard energy method arguments to obtain the usual energy estimates in Sobolev spaces, $H^s:=\{f \mid \|f\|_{H^s}=\|D^sf\|_{L^2}\}$, where D stands for the pseudo-differential operator, $D:=(I-\Delta)^{1/2}$. We differentiate the momentum equation (5.1b) by acting with D^s and integrate by parts against $D^s\mathbf{u}$ to find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}(\cdot,t)\|^2_{H^s}\\ &=-\sum_j\Big(D^s\mathbf{u},[D^s,\mathbf{u}_j]\partial_j\mathbf{u}\Big)+\frac{1}{2}\Big(D^s\mathbf{u},(\nabla\cdot\mathbf{u})D^s\mathbf{u}\Big)+k\Big(D^s\mathbf{u},D^s\nabla\Delta^{-1}n\Big).\end{aligned}$$

The commutator on the first term on the right does not exceed [11]

$$\|[D^s,\mathbf{u}_j]\partial_j\mathbf{u}\|_{L^2}\lesssim\|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{u}\|_{H^s},$$

which yields

$$\|\mathbf{u}(\cdot,t)\|_{H^s}\lesssim\|\mathbf{u}_0\|_{H^s}+\int_0^t\|\nabla\mathbf{u}(\cdot,\tau)\|_{L^\infty}\|\mathbf{u}(\cdot,\tau)\|_{H^s}d\tau+\int_0^t\|n(\tau)\|_{H^{s-1}}d\tau. \tag{5.2}$$

Similarly, an energy estimate of the mass Equation (5.1a) yields [13]

$$\|n(\cdot,t)\|_{H^s}\lesssim\|n_0\|_{H^s}+\int_0^t\Big(\|\nabla\mathbf{u}(\cdot,\tau)\|_{L^\infty}\|n(\cdot,\tau)\|_{H^s}\Big)d\tau. \tag{5.3}$$

Let $Y(T) := \sup_{0\le t\le T}\Big(\|\mathbf{u}(t)\|_{H^s} + \|n(t)\|_{H^s}\Big)$. Since $H^s(\mathbb{R}^2)\subset L^\infty(\mathbb{R}^2)$ for $s>1$, we have

$$Y(T)\lesssim Y(0)+TY^2(T),$$

which implies the local well-posedness for the system (1.2).

Step 2. We prove the blow-up criterion in terms of $\nabla\mathbf{u}$. By (5.2), (5.3), and with the aid of Gronwall's inequality, we obtain

$$Y(T)\lesssim Y(0)\exp\Big[\int_0^T \Big(\|\nabla\mathbf{u}(\cdot,t)\|_{L^\infty} + \|n(\cdot,t)\|_{L^\infty} + 1\Big)dt\Big]. \tag{5.4}$$

The mass equation tells us that

$$n_t + \mathbf{u}\cdot\nabla n = -(\nabla\cdot\mathbf{u})n,$$

and hence

$$\sup_{0\le t\le T}\|n(\cdot,t)\|_{L^\infty}\lesssim \|n_0\|_{L^\infty}\exp\Big[\int_0^T ||\nabla\cdot\mathbf{u}(\cdot,t)||_{L^\infty}dt\Big].$$

Therefore, we can replace (5.4) by

$$Y(T)\lesssim \exp\Big[Y(0)\exp\int_0^T(\|\nabla\mathbf{u}(\cdot,t)\|_{L^\infty}+1)dt\Big], \tag{5.5}$$

which implies the blow-up criterion in terms of $\nabla\mathbf{u}$.

Step 3. Next, we express $\nabla\mathbf{u}$ in terms of the vorticity $\omega := \partial_1\mathbf{u}_2 - \partial_2\mathbf{u}_1$ and the divergence $d=\nabla\cdot\mathbf{u}$,

$$\frac{\partial\mathbf{u}_i}{\partial x_j} = \mathcal{R}_i\mathcal{R}_j(d)\pm\mathcal{R}_j\mathcal{R}_{3-i}(\omega),\quad i,j=1,2,\qquad \omega=\nabla\times\mathbf{u},\quad d=\nabla\cdot\mathbf{u}.$$

Here, $\mathcal{R}_i$'s are the singular Riesz transforms, $\mathcal{R}_i=\partial_i\Delta^{-1/2}$. These singular integral operators do not map L^∞ to L^∞, yet the estimate $\nabla\mathbf{u}$ in terms of ω and d can be saved using a logarithmic correction,

$$\|\nabla\mathbf{u}\|_{L^\infty}\lesssim(\|\omega\|_{L^\infty}+\|d\|_{L^\infty})\log(\|\mathbf{u}\|_{H^s}+1),\quad s>2. \tag{5.6}$$

Finally, we recall that the two-dimensional vorticity is transported

$$\omega_t + \mathbf{u}\cdot\nabla\omega + \omega d = 0,$$

from which we can estimate the vorticity ω in terms of d as

$$\sup_{0\le t\le T}\|\omega(\cdot,t)\|_{L^\infty}\lesssim\|\omega_0\|_{L^\infty}\exp\Big[\int_0^T\|d(\cdot,t)\|_{L^\infty}dt\Big]. \tag{5.7}$$

Therefore, we only need to control the divergence d in L^∞ to determine whether a smooth solution exists globally in time. The final regularity result, in the form of a double exponential bound on $Y(T)$ in terms of $\int_0^T\|d(\cdot,t)\|_{L^\infty}dt$, then follows from (5.5), (5.6), and (5.7). □

To proceed with the global regularity, we obtain an evolution equation for the divergence d: taking the divergence of (5.1b), we find,

$$\begin{aligned}(\partial_t+\mathbf{u}\cdot\nabla)\,d &= kn-\sum_{i,j=1,2}\partial_i\mathbf{u}_j\partial_j\mathbf{u}_i\\ &= kn+2(\partial_1\mathbf{u}_1\partial_2\mathbf{u}_2-\partial_1\mathbf{u}_2\partial_2\mathbf{u}_1)-(\nabla\cdot\mathbf{u})^2\\ &= kn-d^2+2\lambda_1\lambda_2=kn-\frac{1}{2}d^2-\frac{1}{2}\eta^2.\end{aligned} \tag{5.8}$$

Here, λ_1 and λ_2 are eigenvalues of the 2×2 gradient matrix, $\nabla\mathbf{u}:=\{\partial\mathbf{u}_i/\partial x_j\}_{i,j=1,2}$, and $\eta:=\lambda_2-\lambda_1$, is the *spectral gap*. One possible approach for controlling d in (5.8) is therefore to estimate η globally-in-time. To this end, we first differentiate the momentum equation, (5.1b), which yields that the gradient matrix $\nabla\mathbf{u}$ satisfies the Riccati-type equation

$$(\partial_t+\mathbf{u}\cdot\nabla)(\nabla\mathbf{u})+(\nabla\mathbf{u})^2=k\mathcal{R}(n),\quad \mathcal{R}(n)_{ij}:=\mathcal{R}_i\mathcal{R}_j(n),\ i,j=1,2. \tag{5.9}$$

The *spectral dynamics* associated with this system [14, Lemma 3.1] tells us that the eigenvalues λ_i, associated with left and right eigenvectors l_i and r_i, are governed by

$$(\partial_t+\mathbf{u}\cdot\nabla)\lambda_i+\lambda_i^2=k\langle l_i,\mathcal{R}(n)r_i\rangle. \tag{5.10}$$

Taking the difference, we find that the spectral gap satisfies the non-local evolution equation

$$(\partial_t+\mathbf{u}\cdot\nabla)\eta+d\eta=k\langle l_2,\mathcal{R}(n)r_2\rangle-k\langle l_1,\mathcal{R}(n)r_1\rangle. \tag{5.11}$$

The right-hand side of (5.11) is highly nonlinear and non-local and it therefore seems rather difficult to control the spectral gap globally in time.

To avoid this difficulty, the authors of [15] introduced the following *restricted* Euler-Poisson system for the 2×2 matrix $M:\mathbb{R}^2\mapsto\mathbb{R}^2\times\mathbb{R}^2$,

$$\begin{cases} n_t+\nabla\cdot(n\mathbf{u})=0,\\ (\partial_t+\mathbf{u}\cdot\nabla)M+M^2=\dfrac{k}{2}nI_{2\times 2}.\end{cases} \tag{5.12}$$

This is similar to the system of equations satisfied by the 2×2 velocity gradient of the non-restricted Euler-Poisson equations (5.9),

$$\begin{cases} n_t+\nabla\cdot(n\mathbf{u})=0,\\ (\partial_t+\mathbf{u}\cdot\nabla)(\nabla\mathbf{u})+(\nabla\mathbf{u})^2=k\mathcal{R}(n),\quad \mathcal{R}(n)_{ij}:=\mathcal{R}_i\mathcal{R}_j(n),\ i,j=1,2.\end{cases} \tag{5.13}$$

Thus, compared with the restricted model (5.12), we see that the non-local Riesz matrix $\mathcal{R}(n)$ is replaced here by the local matrix, $\frac{1}{2}nI_{2\times 2}$ while keeping the same trace:

$$\mathrm{trace}\big(\mathcal{R}(n)\big)=\mathrm{trace}\left(\frac{1}{2}nI_{2\times 2}\right).$$

This simplification of the restricted model yields a spectral dynamics, $(\partial_t+\mathbf{u}\cdot\nabla)\lambda_i+\lambda_i^2=kn/2$, which in turn implies the following evolution equation for the spectral gap:

$$(\partial_t+\mathbf{u}\cdot\nabla)\eta+d\eta=0.$$

This should be contrasted with the nonlocal terms on the right-hand-side of (5.11). Using this local version of the spectral gap, one is able to derive a complete description of the critical threshold in the two dimensional restricted Euler-Poisson equations [15, Theorem 1.1], expressed in terms of the relative sizes of three quantities: the initial density, the initial divergence, and the initial spectral gap.

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