High-order Expansions of the Detweiler-Whiting Singular Field

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Outline

- Self Force
- Singular Field Calculation
- Effective Source
- Mode Sum Decomposition
- Results
- Future Work

Self Force

- Smaller mass does not follow a geodesic of the background
- Its mass curves the space-time itself - it follows a geodesic of an effective space-time
- 3 main approaches all require subtraction of an appropriate singular component from the retarded field to leave a finite regular field which is solely responsible for self-force.

$$\psi_R^A = \psi_{ret}^A - \psi_S^A$$

Self Force

Retarded field satisfies

$$\mathcal{D}^A{}_B\varphi^B = -4\pi\mathcal{Q}\int u^A\delta_4\left(x, z(\tau')\right)d\tau' \qquad \text{where} \qquad \mathcal{D}^A{}_B = \delta^A{}_B(\Box - m^2) - P^A{}_B$$

The retarded solution to the above equation gives rise to a field which gives the self-force,

$$f^a = p^a{}_A\varphi^A_{(R)}$$

Given the Detweiler-Whiting Green Function,

$$G_{(\mathrm{S})}{}^A{}_{B'}\left(x, x'\right) = \frac{1}{2}\left\{U^A{}_{B'}\left(x, x'\right)\delta\left[\sigma\left(x, x'\right)\right] + V^A{}_{B'}\left(x, x'\right)\theta\left[\sigma\left(x, x'\right)\right]\right\},$$

We may define the Detweiler-Whiting singular field,

$$\varphi^A_{(\mathrm{S})} = \int_{\tau_{(\mathrm{adv})}}^{\tau_{(\mathrm{ret})}} G_{(\mathrm{S})}{}^A{}_{B'}\left(x, z(\tau')\right)u^{B'}d\tau'.$$

Singular Field

The scalar singular field and self-force are

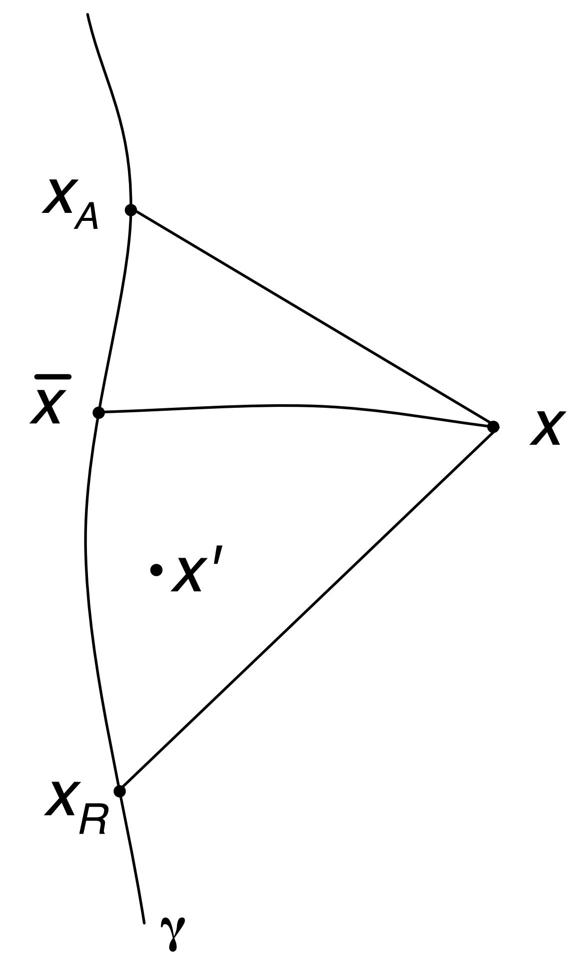
$$\Phi^{(\mathrm{S})}(x) = \left[\frac{U(x,x')}{2\sigma_{a'}u^{a'}}\right]_{x'=x_-}^{x'=x_+} + \int_{x_-}^{x_+} V(x,z(\tau))d\tau$$

$$f^a = g^{ab}\Phi^{(\mathrm{R})}{}_{,b}.$$

The EM singular field and self-force are

$$A_a^{\mathrm{S}} = \left[\frac{u^{a'}U^a{}_{a'}(x,x')}{2\sigma_{a'}u^{a'}}\right]_{x'=x_-}^{x'=x_+} + \int_{\tau_-}^{\tau_+} V^a{}_{a'}(x,z(\tau))u^{a'}d\tau$$

$$f^a = g^{ab}u^c A^{(\mathrm{R})}{}_{[c,b]}.$$



The gravitational singular field and self-force are

$$\bar{h}_{ab}^{S} = \left[\frac{u^{a'}u^{b'}U^{ab}{}_{a'b'}(x,x')}{2\sigma_{a'}u^{a'}}\right]_{x'=x_-}^{x'=x_+} + \int_{\tau_-}^{\tau_+} V^{ab}{}_{a'b'}(x,z(\tau))u^{a'}u^{b'}d\tau \qquad \text{and} \qquad f^a = k^{abcd}\bar{h}^{(R)}_{bc;d},$$

where

$$k^{abcd} \equiv \frac{1}{2}g^{ad}u^b u^c - g^{ab}u^c u^d - \frac{1}{2}u^a u^b u^c u^d + \frac{1}{4}u^a g^{bc}u^d + \frac{1}{4}g^{ad}g^{bc}.$$

Singular Field

For coordinate expansion, we introduce the notation

$$\Delta x^a = x^a - x^{\bar{a}}, \qquad \delta x^{a'} = x^{a'} - x^a = x^{a'} - \Delta x^a - x^{\bar{a}}$$

We expand the Synge world function

$$\sigma(x,x') = \tfrac{1}{2} g_{ab}(x)\delta x^{a'}\delta x^{b'} + A_{abc}(x)\delta x^{a'}\delta x^{b'}\delta x^{c'} + B_{abcd}(x)\delta x^{a'}\delta x^{b'}\delta x^{c'}\delta x^{d'} + \dots$$

We use $2\sigma = \sigma_{a'}\sigma^{a'}$ to solve for coefficients

$$A_{abc}(x) = \tfrac{1}{4} g_{(ab,c)}(x)$$

$$B_{abcd}(x) = \tfrac{1}{12} g_{(ab,cd)}(x) - \tfrac{1}{384} g^{pq}(x)\left(g_{(ab,|p|}(x) g_{cd),q}(x) - 12 g_{(ab,|p|}(x) g_{|q|c,d)}(x) + 36 g_{p(a,b}(x) g_{|q|c,d)}(x)\right)$$

We can differentiate this easily to obtain σ_a

Once we know $\sigma(x,x')$, we can calculate the Van Vleck Determinant

$$\Delta^{\frac{1}{2}}(x,x') = \left(-\left[-g(x)\right]^{-\frac{1}{2}} \left|-\sigma_{a'b}(x,x')\right| \left[-g(x')\right]^{-\frac{1}{2}}\right)^{\frac{1}{2}}$$

Singular Field

We expand x' around the point $\bar{x}$. Representing the worldline in terms of proper time gives us

$$x^{a'}(\tau) = x^{\bar{a}} + u^{\bar{a}}\Delta\tau + \frac{1}{2!}\dot{u}^{\bar{a}}\Delta\tau^2 + \frac{1}{3!}\ddot{u}^{\bar{a}}\Delta\tau^3 + \cdots$$

We want to determine the points on the world line that are connected by a null geodesic. That is we want to solve

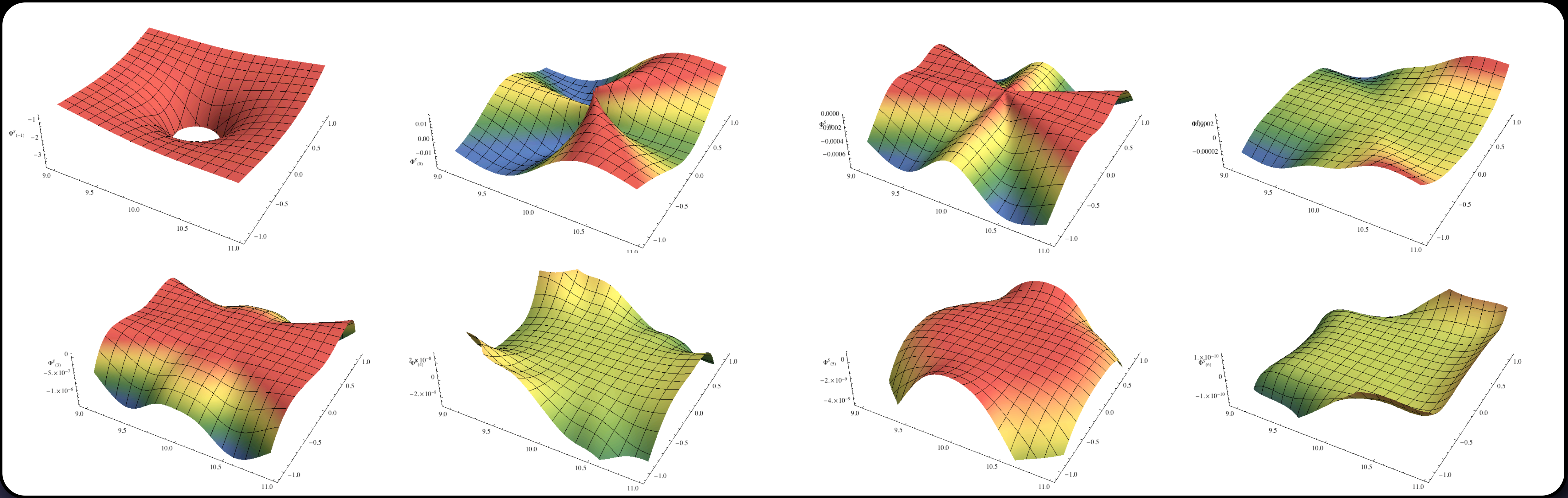
$$\begin{aligned}\sigma\big(x, x^{a'}(\tau)\big) = 0 &= \tfrac{1}{2}g_{\bar{a}\bar{b}}(u^{\bar{a}}\Delta\tau - \Delta x^a)(u^{\bar{b}}\Delta\tau - \Delta x^b) \\ &+ \big[\tfrac{1}{2}g_{\bar{a}\bar{b}}(u^{\bar{a}}\Delta\tau - \Delta x^a)\dot{u}^{\bar{b}}\Delta\tau^2 + \tfrac{1}{2}g_{\bar{a}\bar{b},\bar{c}}(u^{\bar{a}}\Delta\tau - \Delta x^a)(u^{\bar{b}}\Delta\tau - \Delta x^b)\Delta x^c \\ &+ \tfrac{1}{4}g_{\bar{a}\bar{b},\bar{c}}(\bar{x})(u^{\bar{a}}\Delta\tau - \Delta x^a)(u^{\bar{b}}\Delta\tau - \Delta x^b)(u^{\bar{c}}\Delta\tau - \Delta x^c)\big] + O(\epsilon^4)\end{aligned}$$

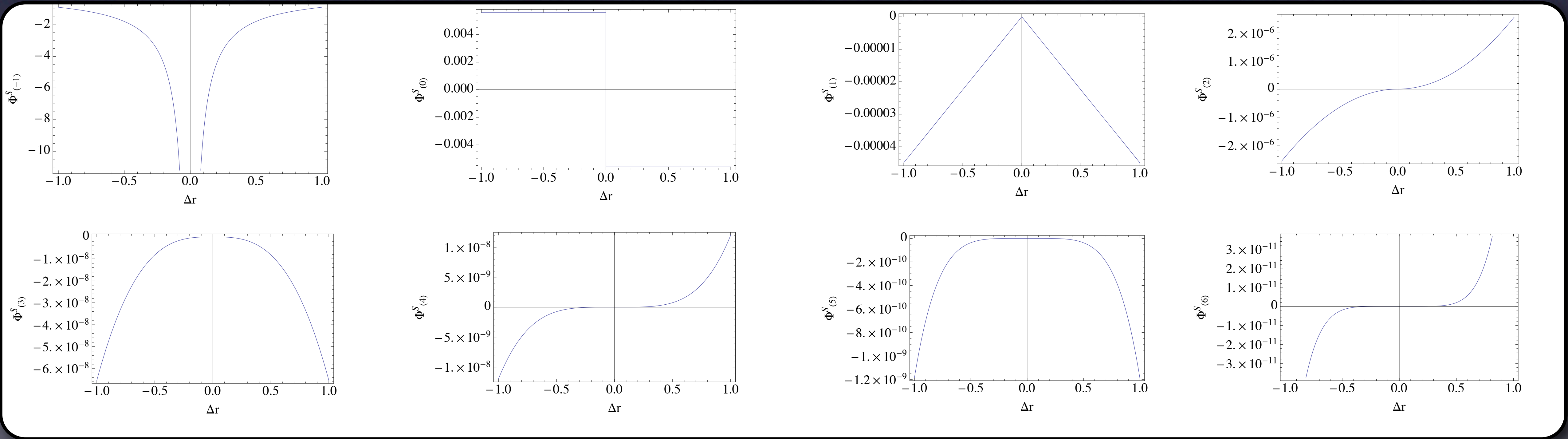
Use $g_{ab';c}\sigma^c = 0$ calculate bivector of parallel transport.

Writing $\Delta\tau = \tau_1\epsilon + \tau_2\epsilon^2 + \tau_3\epsilon^3 + \cdots$, this gives us

$$\tau_{1\pm} = g_{\bar{a}\bar{b}}u^{\bar{a}}\Delta x^b \pm \sqrt{(g_{\bar{a}\bar{b}}u^{\bar{a}}\Delta x^b)^2 + g_{\bar{a}\bar{b}}\Delta x^a\Delta x^b} \equiv g_{\bar{a}\bar{b}}u^{\bar{a}}\Delta x^b \pm \rho$$

Singular Field - Schwarzschild





Effective Source

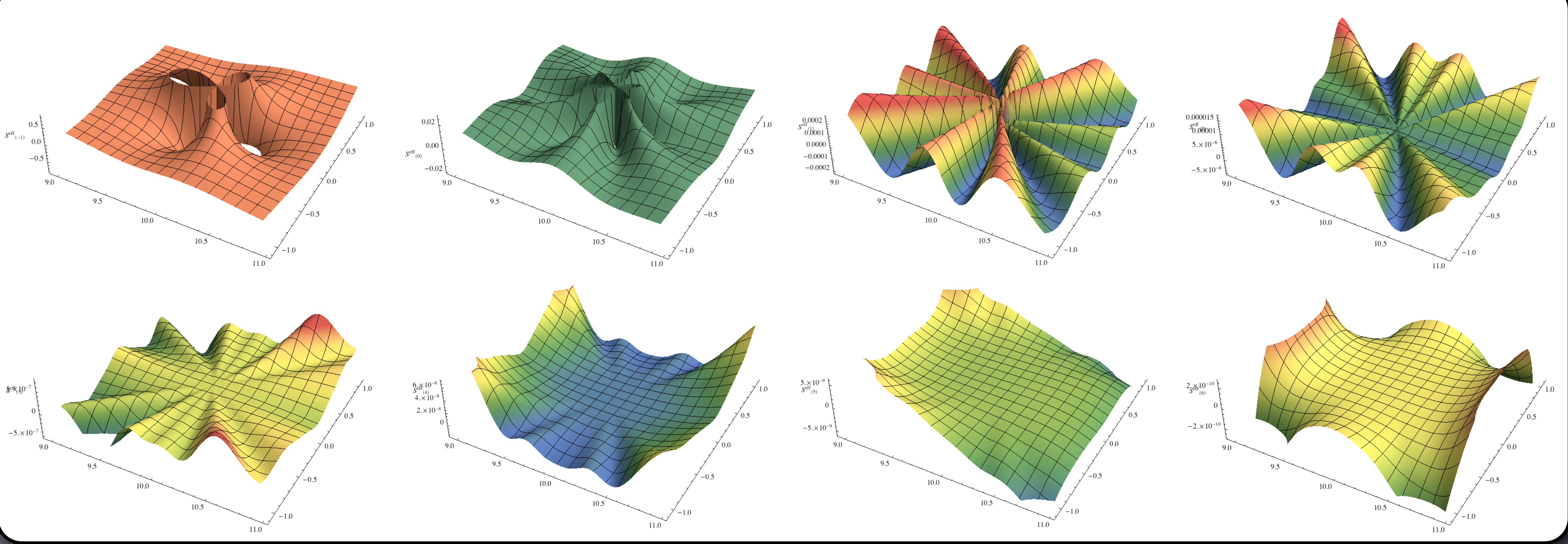
Splitting the retarded field into approximate singular and regularized parts

$$\varphi^A_{(\mathrm{ret})} = \tilde{\varphi}^A_{(\mathrm{S})} + \tilde{\varphi}^A_{(\mathrm{R})}$$

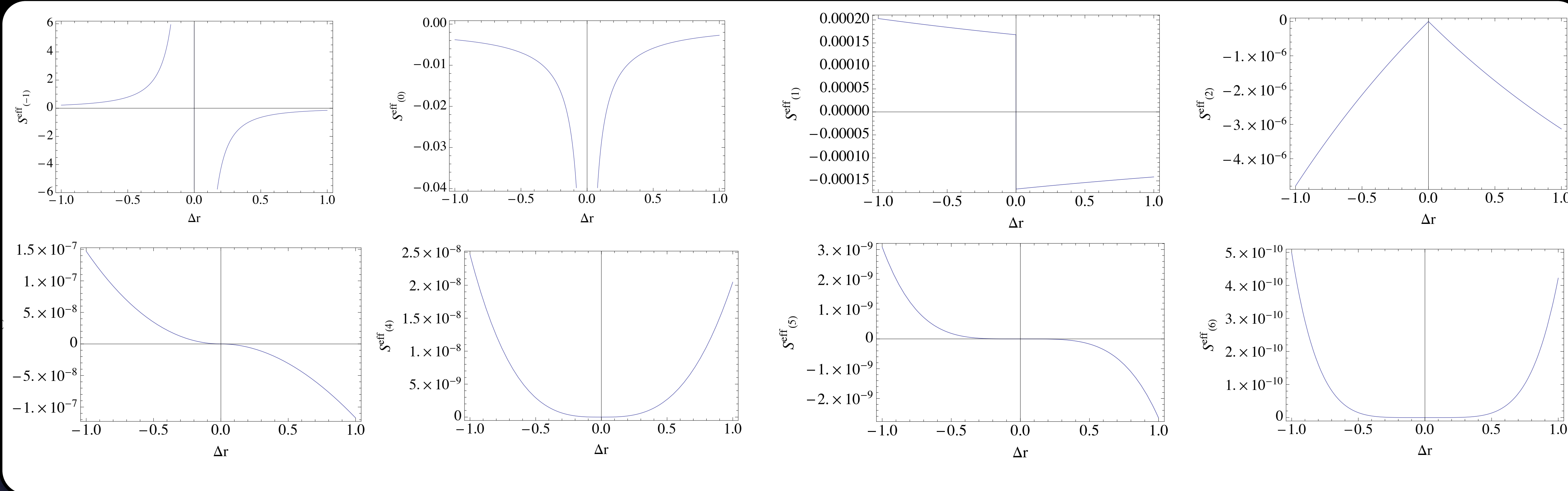
Substituting into the wave equation gives $\mathcal{D}^A{}_B \tilde{\varphi}^B_{(\mathrm{R})} = S^A_{(\mathrm{eff})}$ with an effective source,

$$S^A_{(\mathrm{eff})} = \mathcal{D}^A{}_B \tilde{\varphi}^B - 4\pi \mathcal{Q} \int u^A \delta_4 \left(x, z(\tau')\right) d\tau'.$$

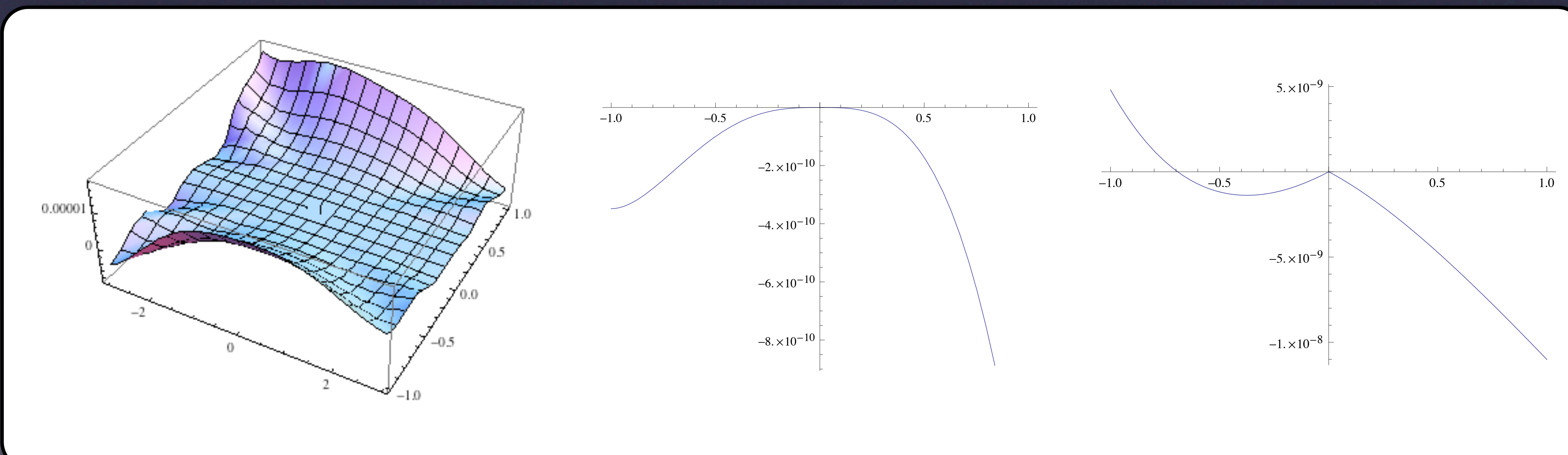
Schwarzschild



Effective Source



Kerr



Mode Sum

To obtain expressions which are readily expressed as mode sums, it is useful to work in a rotated coordinate frame. We introduce Riemann normal coordinates on the 2-sphere at $\bar{x}$, in the form

$$w_1 = 2\sin\left(\frac{\alpha}{2}\right)\cos\beta \qquad w_2 = 2\sin\left(\frac{\alpha}{2}\right)\sin\beta$$

where

$$\sin\theta\cos\phi = \cos\alpha$$
$$\sin\theta\sin\phi = \sin\alpha\sin\beta$$
$$\cos\theta = \sin\alpha\cos\beta$$

The Schwarzschild metric is now given by the line element

$$ds^2 = -\left(\frac{r-2m}{r}\right)dt^2 + \left(\frac{r}{r-2m}\right)dr^2 + r^2\left\{\left[\frac{16 - w_2^2\left(8 - w_1^2 - w_2^2\right)}{4\left(4 - w_1^2 - w_2^2\right)}\right]dw_1^2 + 2dw_1dw_2\left[\frac{w_1w_2\left(8 - w_1^2 - w_2^2\right)}{4\left(4 - w_1^2 - w_2^2\right)}\right] + \left[\frac{16 - w_1^2\left(8 - w_1^2 - w_2^2\right)}{4\left(4 - w_1^2 - w_2^2\right)}\right]dw_2^2\right\}$$

Mode Sum

Barack and Ori (2000, 2002) first looked at the multipole decomposition of the self-force, f_a

$$f_a(r,t,\alpha,\beta) = \sum_{lm} f_a^{lm}(r,t)\, Y^{lm}(\alpha,\beta) \quad \Rightarrow \quad f_a^{lm}(r,t) = \int f_a(r,t,\alpha,\beta)\, Y^{lm*}(\alpha,\beta)\, d\Omega.$$

The l mode contribution at $\bar{x} = (t_0, r_0, \alpha_0, \beta_0)$ is given by

$$f_a^l(r_0,t_0) = \lim_{\Delta r \to 0} \sum_m f_a^{lm}(r_0 + \Delta r, t_0)\, Y^{lm}(\alpha_0, \beta_0).$$

With particle on the pole, $Y^{lm}(\alpha_0 = 0, \beta_0) = 0$ for all $m \neq 0$, so

$$f_a^l(r_0,t_0) = \lim_{\Delta r \to 0} \sqrt{\frac{2l+1}{4\pi}}\, f_a^{l,m=0}(r_0 + \Delta r, t_0) = \frac{2l+1}{4\pi} \lim_{\Delta r \to 0} \int f_a(r_0 + \Delta r, t_0, \alpha, \beta)\, P_l(\cos\alpha)\, d\Omega.$$

We find the self-force has the form

$$f_a(r,t,\alpha,\beta) = \sum_{n=1} \frac{\mathcal{B}_a^{(3n-2)}}{\rho^{2n+1}} \epsilon^{n-3}$$

where $\mathcal{B}_a^{(k)} = b_{a_1 a_2 \ldots a_k}(\bar{x}) \Delta x^{a_1} \Delta x^{a_2} \ldots \Delta x^{a_k}$

and $\rho = \sqrt{(g_{\bar{a}\bar{b}} u^{\bar{b}} \Delta x^b)^2 + g_{\bar{a}\bar{b}} \Delta x^a \Delta x^b}$

Mode Sum

Explicitly in our coordinates, $\rho = \sqrt{(g_{\bar{a}\bar{b}} u^{\bar{b}} \Delta x^b)^2 + g_{\bar{a}\bar{b}} \Delta x^a \Delta x^b}$ takes the form

$$\rho\left(r, t_0, \alpha, \beta\right)^2 = \frac{E^2 r_0^4}{\left(L^2 + r_0^2\right)\left(r_0 - 2M\right)^2} \Delta r^2 + \left(L^2 + r_0^2\right)\left(\Delta w_1 + \frac{L r_0 \dot{r}_0}{\left(r_0 - 2M\right)\left(L^2 + r_0^2\right)} \Delta r\right)^2 + r_0^2 \Delta w_2^2$$

$$\rho_0\left(\alpha, \beta\right)^2 = \rho\left(r_0, t_0, \alpha, \beta\right)^2 = \left(L^2 + r_0^2\right) \Delta w_1^2 + r_0^2 \Delta w_2^2$$

We use our definition of ρ_0 to rewrite our Δw's in an alternate form

$$\Delta w_1^2 = 2\left(1 - \cos\alpha\right)\cos^2\beta = \frac{\rho_0^2}{(r_0^2 + L^2)\chi}\cos^2\beta = \frac{\rho_0^2}{L^2\chi}\left(k - \left(1 - \chi\right)\right),$$

$$\Delta w_2^2 = 2\left(1 - \cos\alpha\right)\sin^2\beta = \frac{\rho_0^2}{(r_0^2 + L^2)\chi}\sin^2\beta = \frac{\rho_0^2}{L^2\chi}(1 - \chi)$$

where $\chi(\beta) = 1 - k\sin^2\beta, \qquad k = \frac{L^2}{r_0^2 + L^2}.$

As $\lim_{\Delta r \to 0} f_a\left(r, t, \alpha, \beta\right) = \rho_0^{n-3}\epsilon^{n-3} c_{a(n)}(r_0, \beta)$, it can now be shown that

$$f_a^l\left(r_0, t_0, \alpha, \beta\right) = \frac{2l+1}{4\pi}\left[\epsilon^{-2}\lim_{\Delta r \to 0}\int \frac{B_a^{(1)}}{\rho^3} P_l\left(\cos\alpha\right) d\Omega + \epsilon^{n-3}\sum_{n=2}\int \rho_0^{n-3} c_{a(n)}\left(r_0, \beta\right) P_l\left(\cos\alpha\right) d\Omega\right]$$

$$\equiv A_a^l\left(r_0, t_0\right)\epsilon^{-2} + B_a^l\left(r_0, t_0\right)\epsilon^{-1} + C_a^l\left(r_0, t_0\right)\epsilon^0 + D_a^l\left(r_0, t_0\right)\epsilon^1 + E_a^l\left(r_0, t_0\right)\epsilon^2 + F_a^l\left(r_0, t_0\right)\epsilon^3 +$$

Mode Sum

For the higher order terms we can write

$$\rho_0\left(r_0, t_0, \alpha, \beta\right)^n = \zeta^n\left(1-\cos\alpha\right)^{n/2} = \zeta^n \sum_{l=0} A_l^{n/2}(0) P_l\left(\cos\alpha\right)$$

where

$$\zeta\left(\beta\right)^2 = 2\chi(\beta)\left(L^2 + r_0^2\right)$$

and

$$A_l^{k+\frac{1}{2}}(0) = \mathcal{P}_{k+\frac{1}{2}} = (2l+1)\frac{(-1)^{k+1}\, 2^{k+\frac{3}{2}}\left[(2k+1)!!\right]^2}{(2l-2k-1)(2l-2k+1)\ldots(2l+2k+1)(2l+2k+3)}$$

We note that $P_l(\cos\alpha)$ will simply integrate to 1

We note that integrating over β is the same as averaging over the angles β, so

$$\frac{1}{2\pi}\int\frac{d\beta}{\chi(\beta)^n} = \left\langle\chi^{-n}(\beta)\right\rangle = {}_2F_1\left(n, \frac{1}{2}, 1, k\right)$$

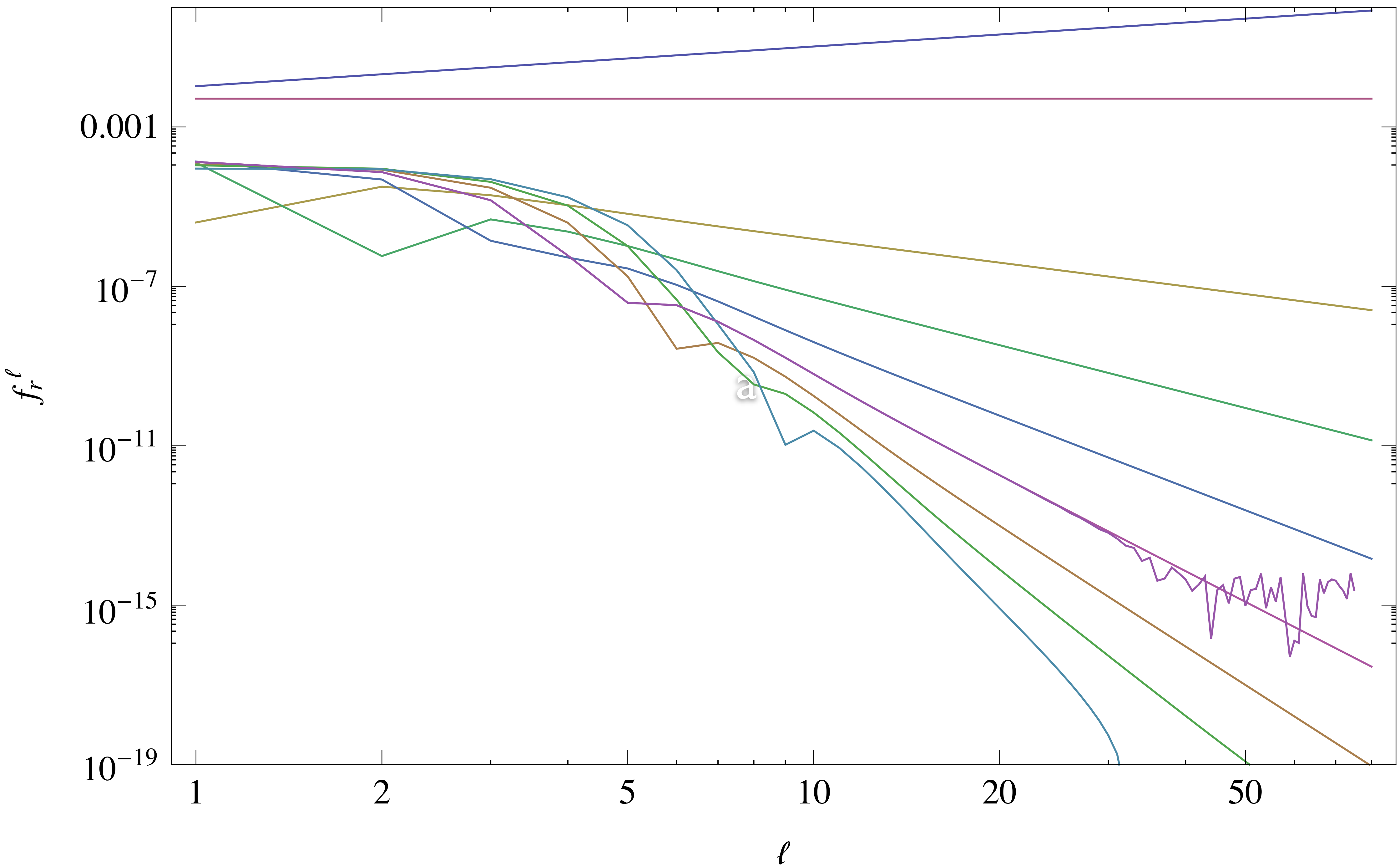
Tidy up resulting equations with

$$\left\langle\chi^{-\frac{1}{2}}\right\rangle = F_{\frac{1}{2}} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k\right) = \frac{2}{\pi}K\left(k\right) = \mathcal{K}$$

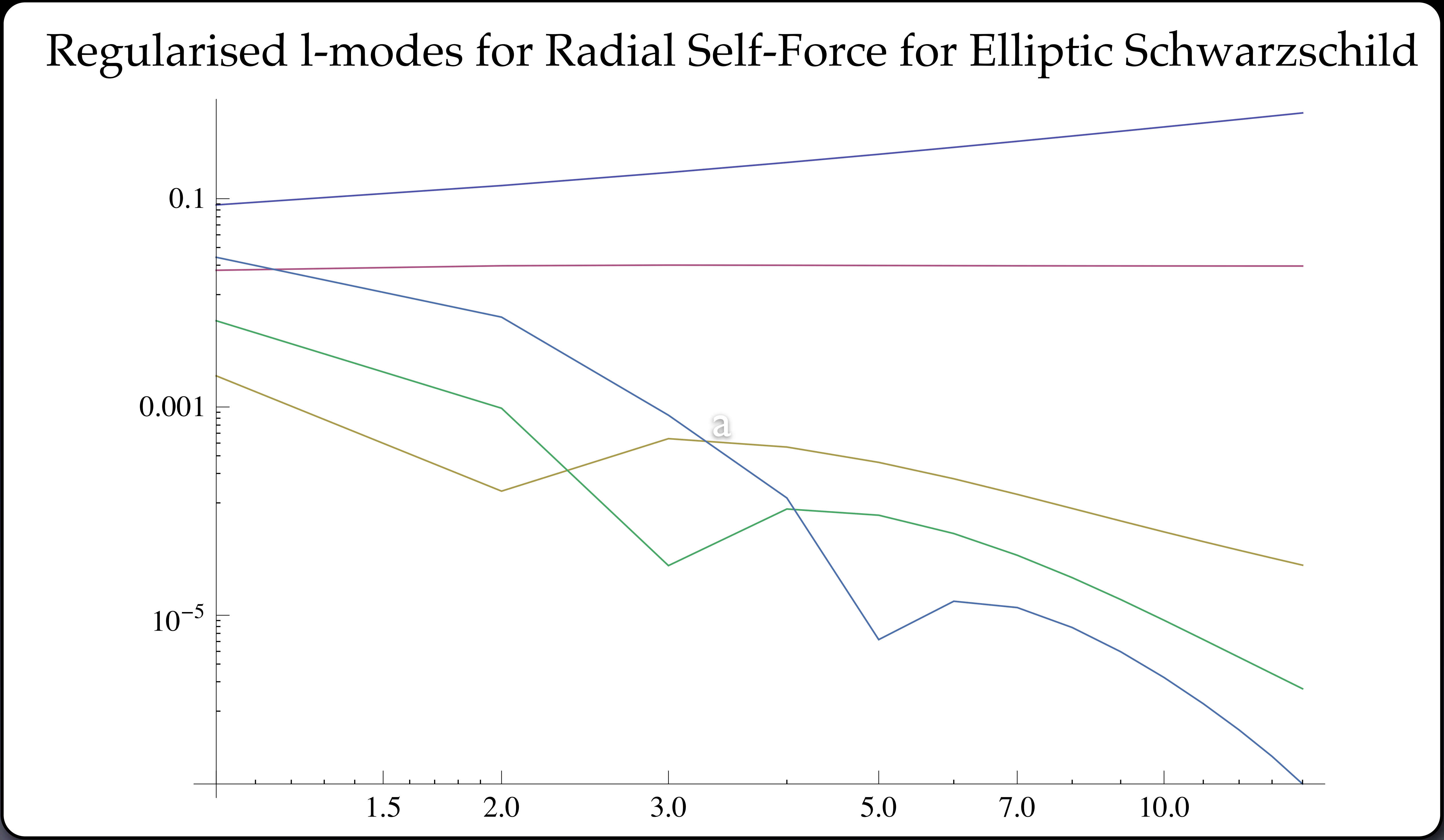
$$\left\langle\chi^{\frac{1}{2}}\right\rangle = F_{-\frac{1}{2}} = {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; k\right) = \frac{2}{\pi}E\left(k\right) = \mathcal{E}$$

Results - Scalar

Regularised l-modes for Radial Self-Force for Circular Schwarzschild



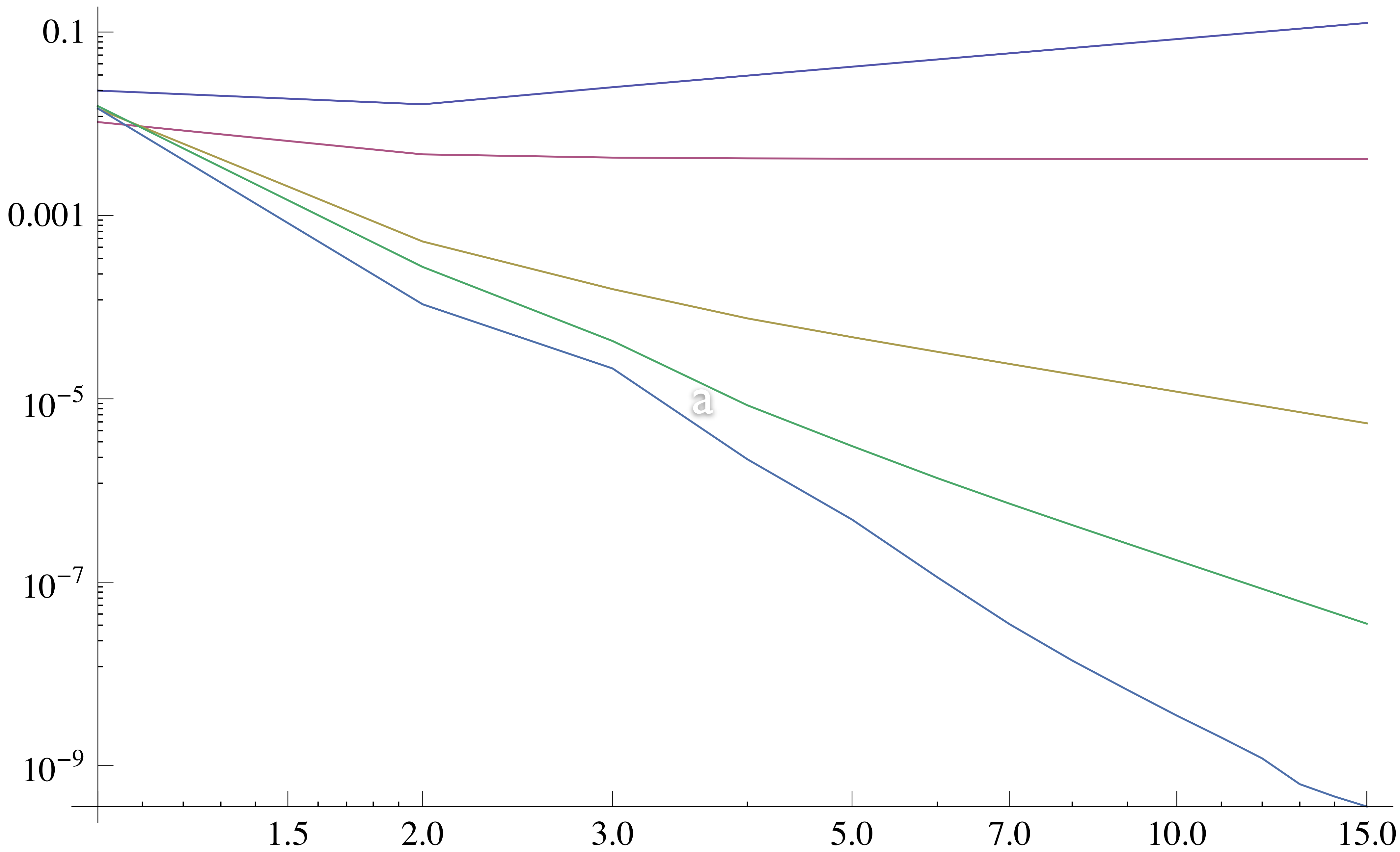
Results - EM



Data supplied by Roland Haas

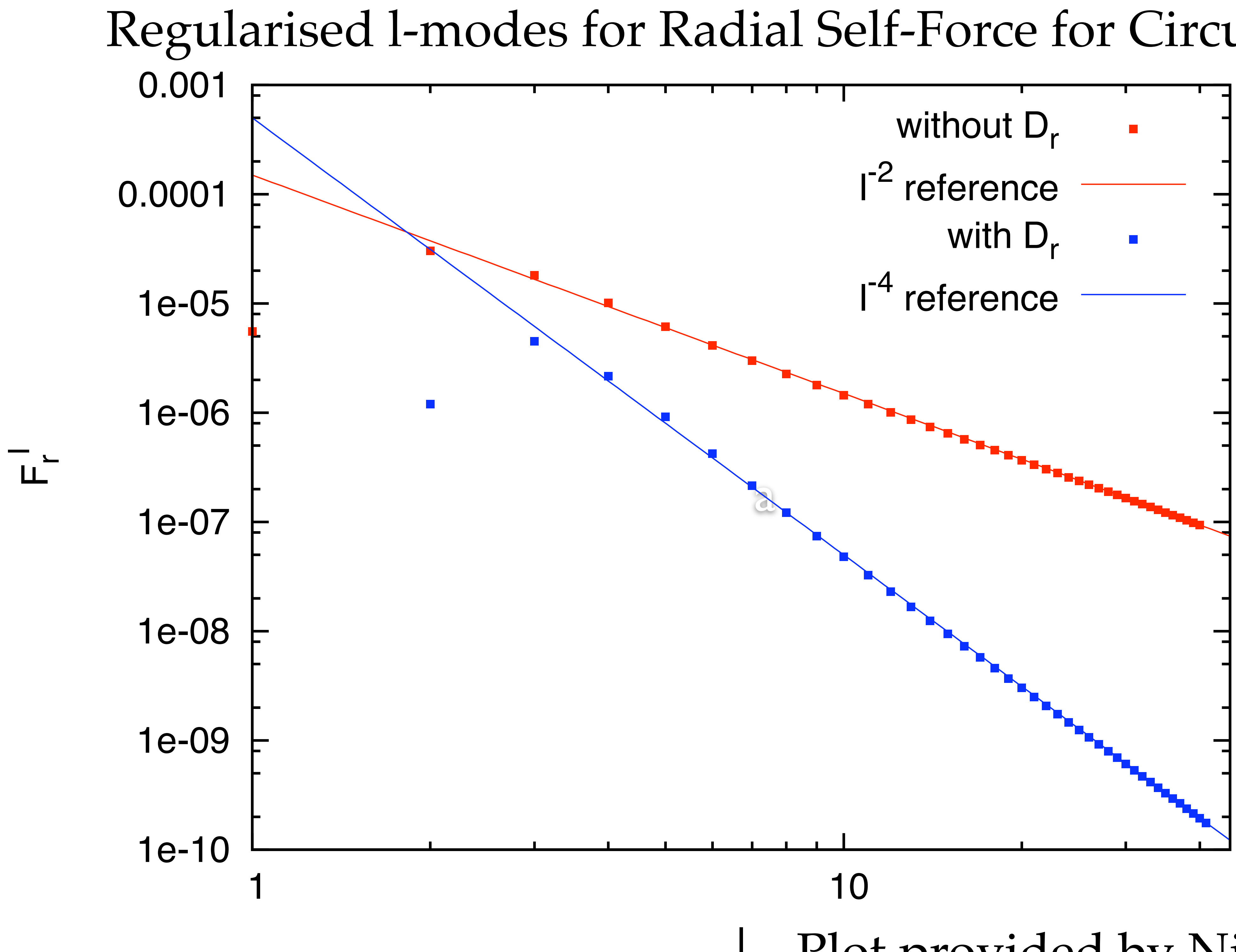
Results - Gravity

Regularised l-modes for Radial Self-Force for Circular Schwarzschild



Data supplied by Niels Warburton

Results - Kerr



Plot provided by Niels Warburton

Future Work

- Eccentric orbits in EM and Gravity Kerr
- Second order Effective Source
- Spheroidal harmonic decomposition into spherical harmonics for Kerr
- Radiation gauge
- Kerr gravity????