# Eccentric orbits on Schwarzschild: Transforming metric perturbations from Regge-Wheeler to Lorenz gauge

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# Outline

Choosing a gauge and solving the Einstein equations

Transforming from Regge-Wheeler to Lorenz gauge

Odd-parity gauge transformation

Even-parity gauge transformation

Conclusions

# Outline

# Choosing a gauge

- In GR, gauge freedom is coordinate freedom
- At lowest order, we use Schwarzschild coordinates
- At first order, we choose between Regge-Wheeler and Lorenz gauge

- We choose RW gauge and transform our solutions to Lorenz gauge

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| 2 equations for each harmonic mode | 10 equations for each harmonic mode |
| Solutions are $C^{-1}$ / singular at particle | Solutions are $C^0$ at particle |
| Asymptotically grows | Asymptotically flat |
| ? | Regularization procedure & Eqs. of motion |

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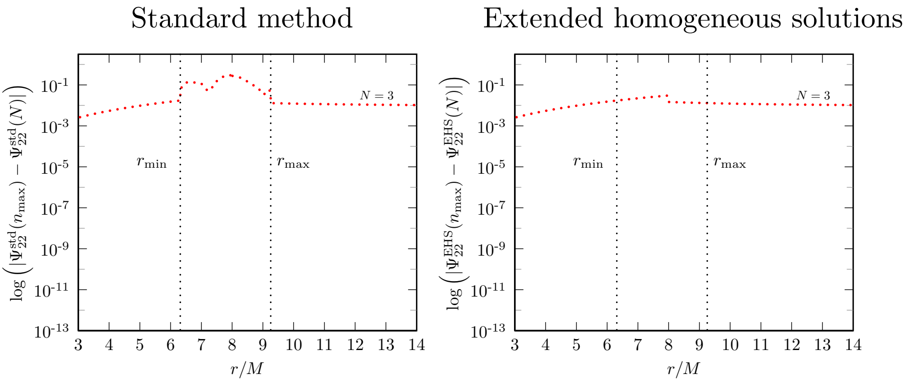
# Fourier convergence of the master functions




$p = 7.50478$

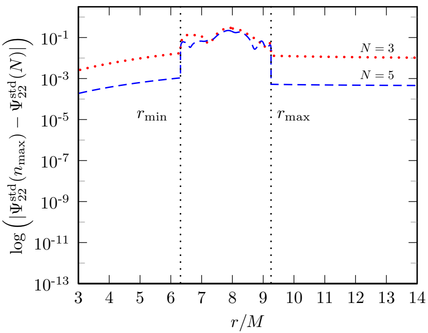
$e = 0.188917$

$t = 80.62M$

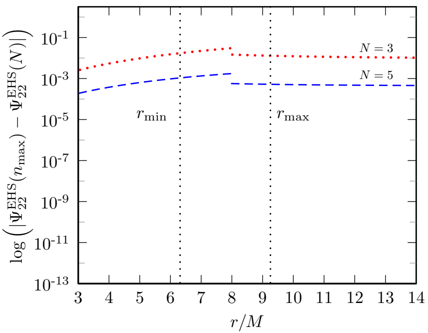
$$\Psi_{\ell m}(t_p, r) = \sum_{n=-3}^{3} R_{\ell mn}(r) e^{-i\omega t}$$

# Fourier convergence of the master functions

Standard method




Extended homogeneous solutions




$$p = 7.50478$$
$$e = 0.188917$$
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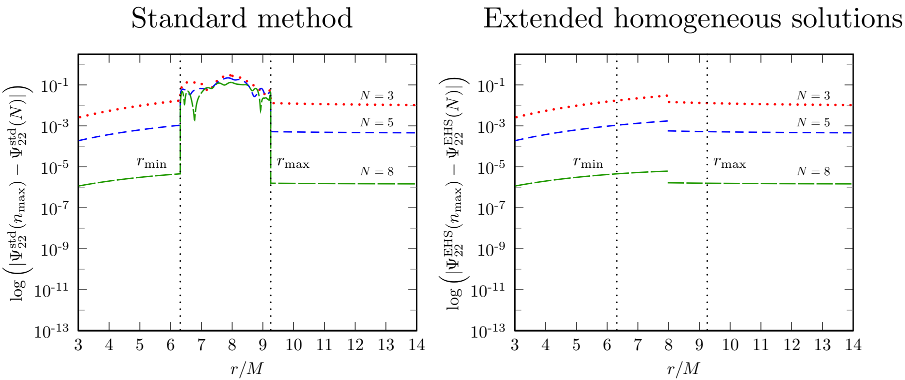
$$\Psi_{\ell m}(t_p, r) = \sum_{n=-5}^{5} R_{\ell mn}(r) e^{-i\omega t}$$

# Fourier convergence of the master functions




$p = 7.50478$

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$$\Psi_{\ell m}(t_p, r) = \sum_{n=-8}^{8} R_{\ell mn}(r) e^{-i\omega t}$$

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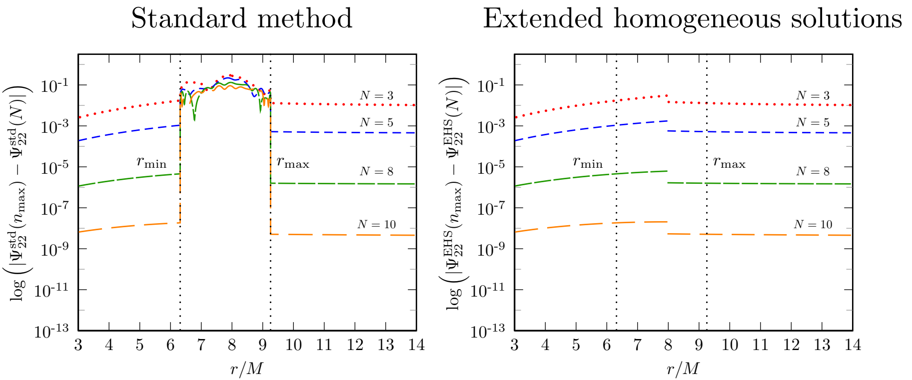


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$$\Psi_{\ell m}(t_p, r) = \sum_{n=-10}^{10} R_{\ell m n}(r) e^{-i\omega t}$$

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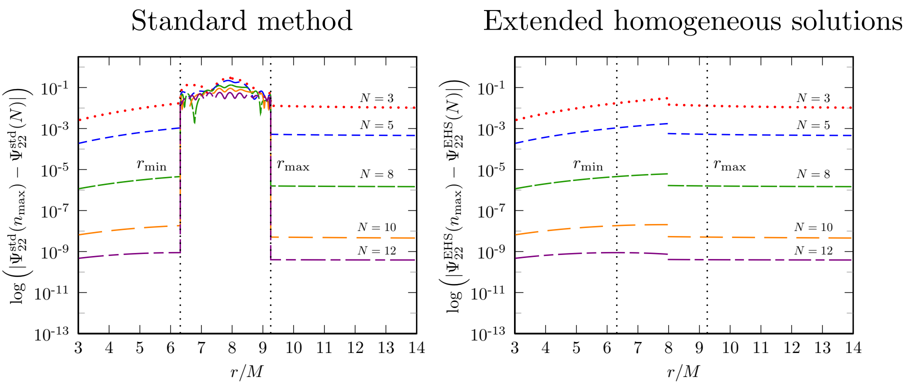


$p = 7.50478$

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$$\Psi_{\ell m}(t_p, r) = \sum_{n=-12}^{12} R_{\ell mn}(r) e^{-i\omega t}$$

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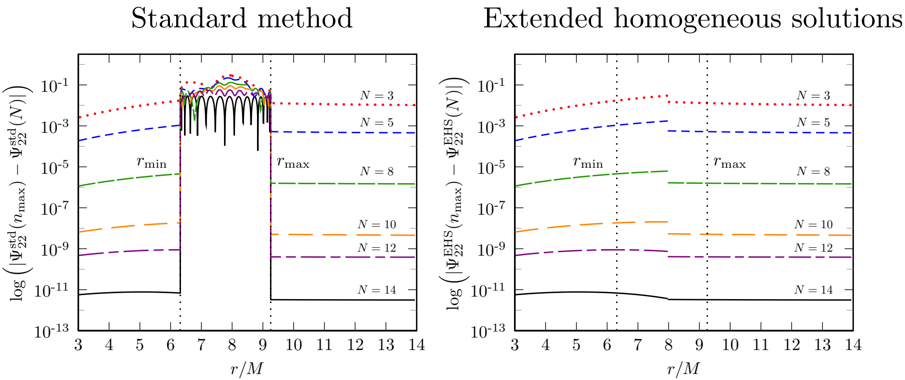


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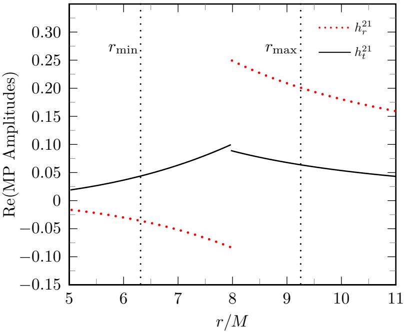
# Metric perturbation reconstruction

$p^{21}_{\mu\nu}(80.62M, r)$

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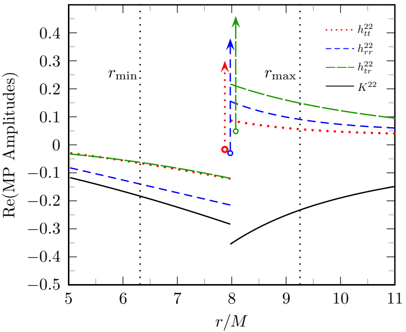


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- 3 of 4 even parity metric perturbations are singular.
- We compute analytical values of those singularities as funcs of $t$.

# Outline

# Transforming to Lorenz gauge

- Gauge transformation from Regge-Wheeler (RW) to Lorenz (L)

$$x^\mu_{\rm RW} \to x^\mu_{\rm L} = x^\mu_{\rm RW} + \Xi^\mu, \qquad |\Xi^\mu| \sim |p_{\mu\nu}| \ll 1$$

- Metric perturbation transforms as

$$p^{\rm RW}_{\mu\nu} \to p^{\rm L}_{\mu\nu} = p^{\rm RW}_{\mu\nu} - \Xi_{\mu|\nu} - \Xi_{\nu|\mu}$$

- Demand $p^{\rm L}_{\mu\nu}$ satisfy the Lorenz gauge condition, $\bar{p}^{\rm L}_{\mu\nu}{}^{|\nu} = 0$
- Therefore

$$\Xi_{\mu|\nu}{}^{\nu} = p^{\rm RW|\nu}_{\mu\nu} - \frac{1}{2} g^{\alpha\beta} p^{\rm RW}_{\alpha\beta|\mu}$$

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# Equations for the gauge generator amplitudes

- Decompose gauge vector $\Xi_\mu$ in scalar and vector harmonics

$$\Xi_t = \xi_t^{\ell m}(t,r) Y^{\ell m}$$
$$\Xi_r = \xi_r^{\ell m}(t,r) Y^{\ell m}$$
$$\Xi_A = \xi_{(e)}^{\ell m}(t,r) Y_A^{\ell m} + \xi_{(o)}^{\ell m}(t,r) X_A^{\ell m}$$

- One, separate odd-parity wave equation

$$\frac{1}{f}\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V_1\right]\xi_{(o)}^{\ell m} = 2\frac{f}{r} h_r^{\ell m} + p_{\ell m}\delta\left[r - r_p(t)\right]$$

- Three, coupled even-parity wave equations

$$\Box \xi_t^{\ell m} + M_t(\xi_t, \xi_r) = F_t(\Psi_{\rm ZM}) + \text{singular term}$$
$$\Box \xi_r^{\ell m} + M_r(\xi_r, \xi_t, \xi_{(e)}) = F_r(\Psi_{\rm ZM}) + \text{singular term}$$
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# Partial annihilator method

- An inhomogeneous wave equation

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- Original Regge-Wheeler variable $\Psi^{\ell m}=fh_r^{\ell m}/r$
- Satisfies the equation

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}-V_2\right]\Psi^{\ell m}=S_{\rm RW}$$

- Act with Regge-Wheeler wave operator on both sides

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$$=2S_{\rm RW}+\text{Other singular terms}$$

- Now a 4th-order PDE, but *the source is point-singular*

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# An alternative: 2nd-order solutions

- A method for finding solutions without relying on annihilators
- Consider again

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V_1\right]\xi_{\ell m} = 2f\Psi^{\mathrm{RW}}_{\ell m} + P_{\mathrm{Singular}}$$

- Or, in the FD:

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_1\right]\tilde{\xi}_{\ell mn} = 2fR^{\mathrm{RW}}_{\ell mn} + Z_{\mathrm{Singular}}$$

- The solution to the $Z_{\mathrm{Singular}}$ part can always be found using EHS
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# Finding causal solutions

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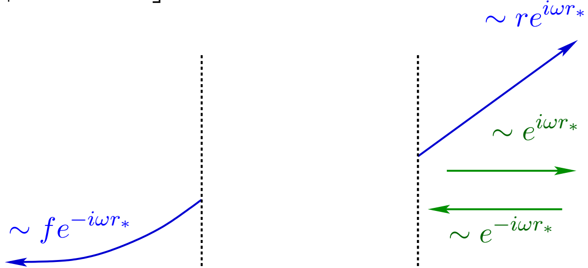
Integrate from left to right



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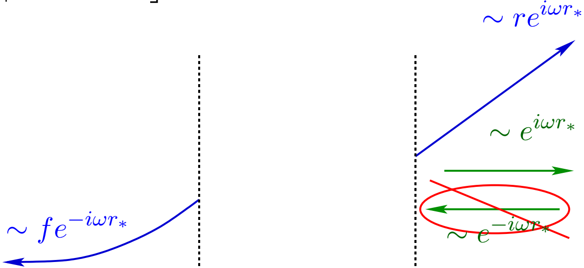

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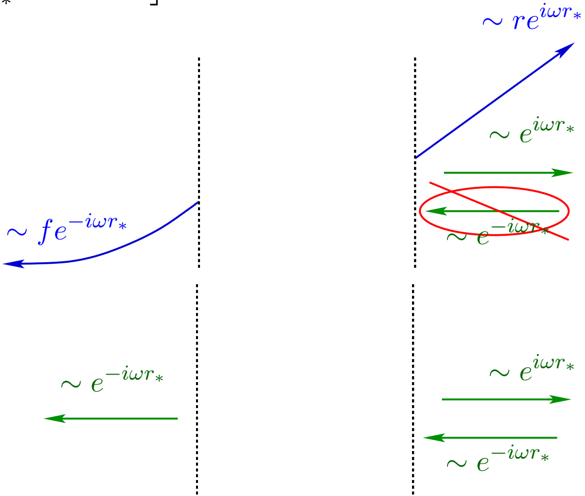
Subtracte to
remove acausality



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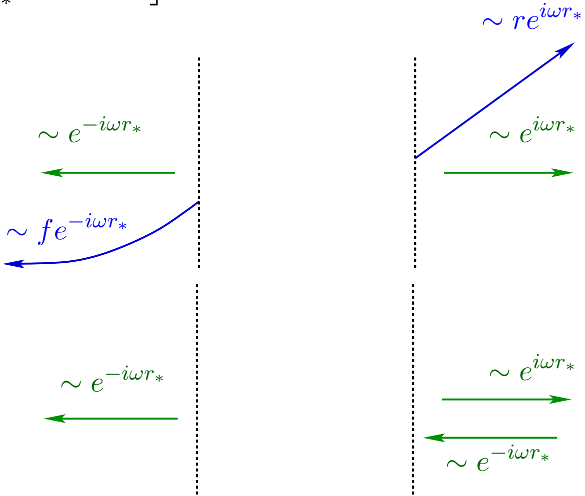

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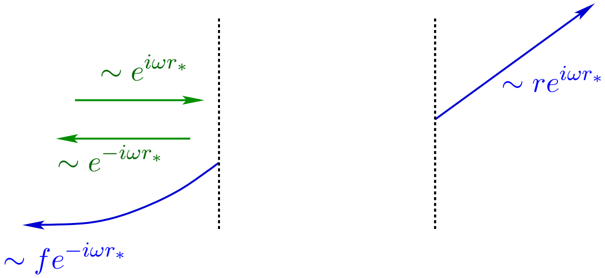

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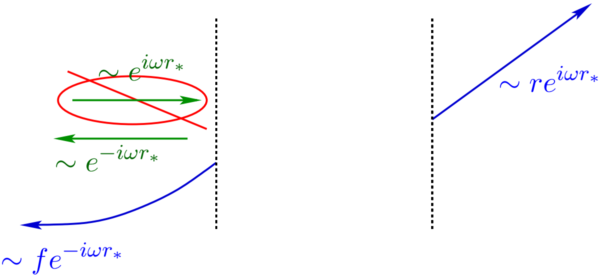

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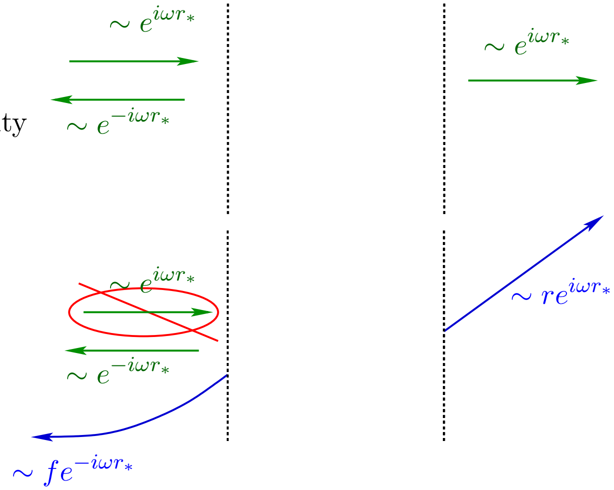

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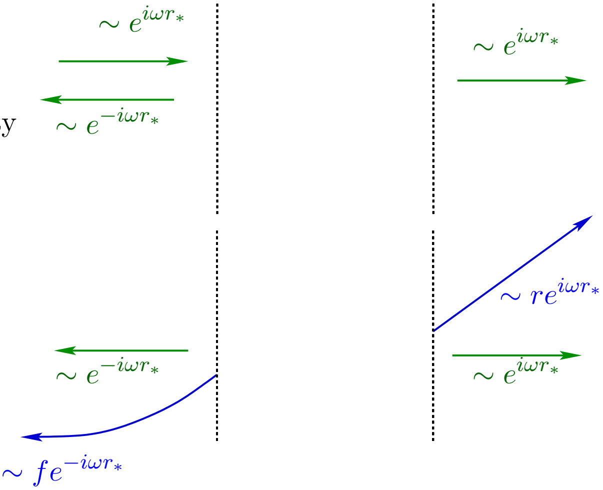

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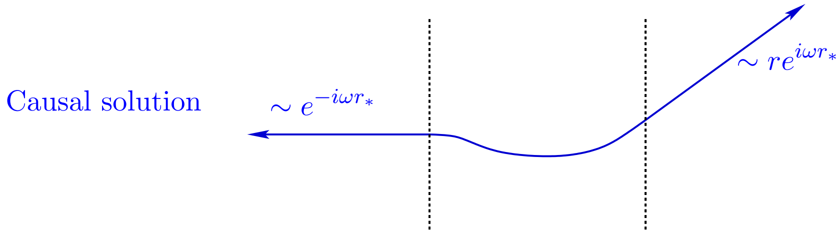


- TD reconstruction

$$\xi(t,r) = \sum_n \tilde{\xi}(r)e^{-i\omega t}$$

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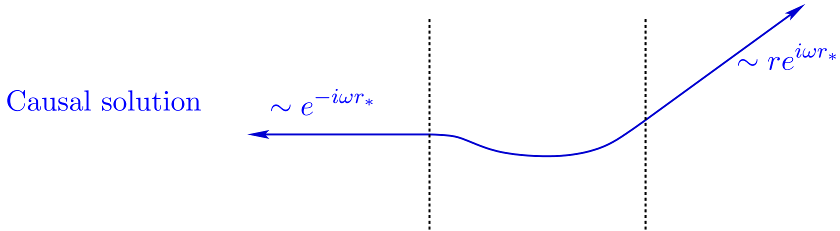


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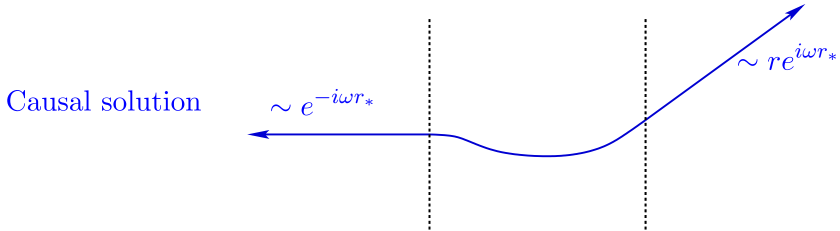


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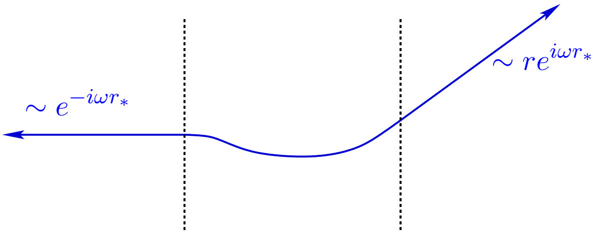


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$$\xi(t,r) = \xi^{+}(t,r)\ \theta\left[r - r_p(t)\right] + \xi^{-}(t,r)\ \theta\left[r_p(t) - r\right]$$

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# 2nd-order EPS summary

- Std. source
- Std. particular solutions: $\tilde{\xi}_p^{\infty/H}$
- Causality gives homog. sols: $\tilde{\xi}_h^{\pm}$

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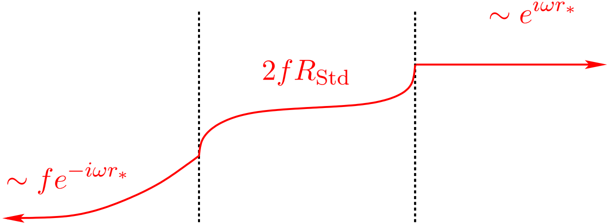

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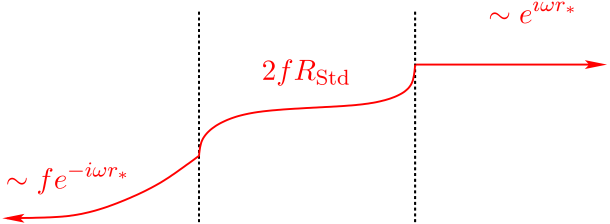

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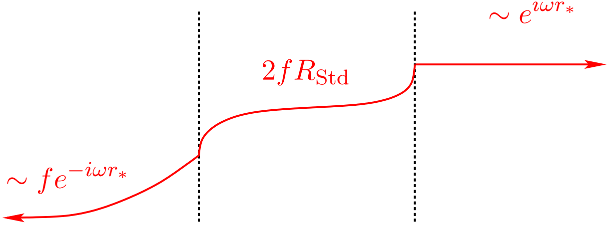


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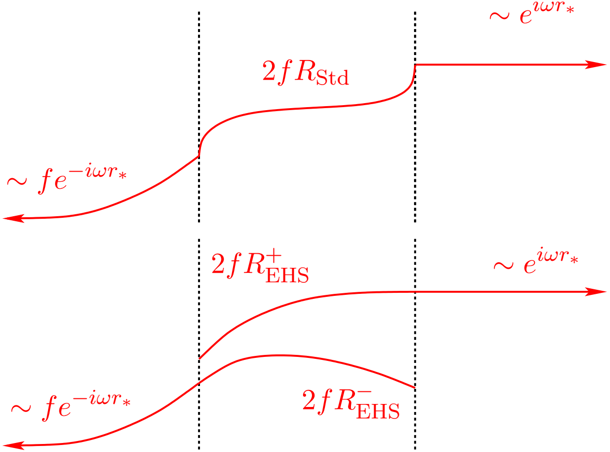

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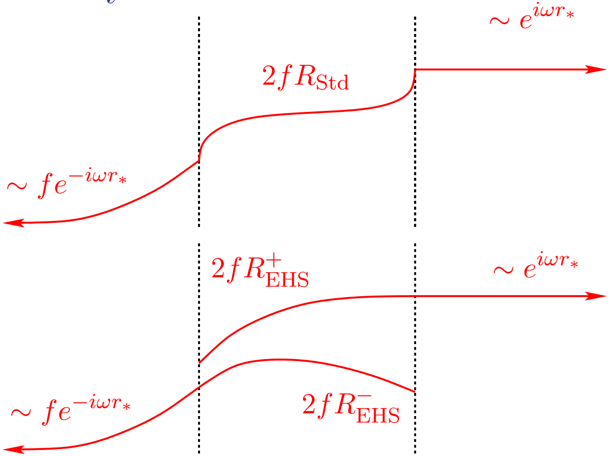

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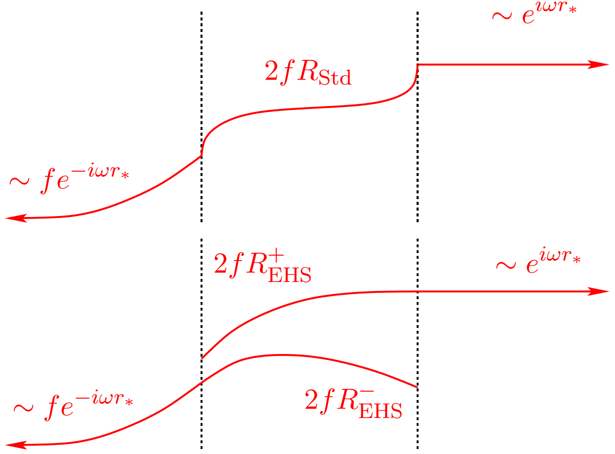


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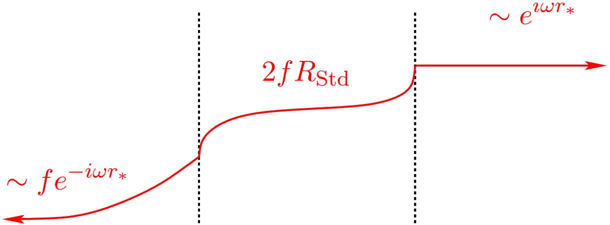


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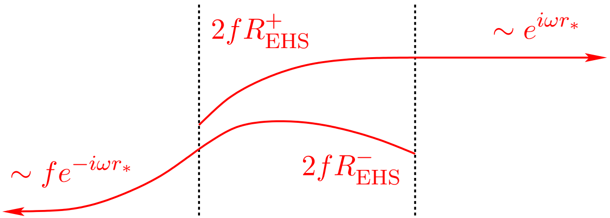


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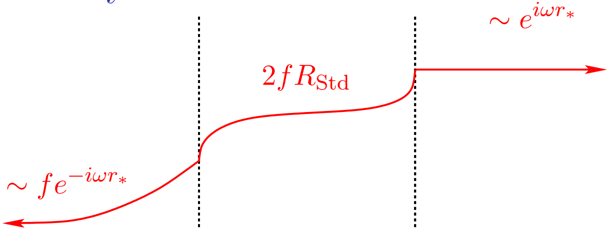


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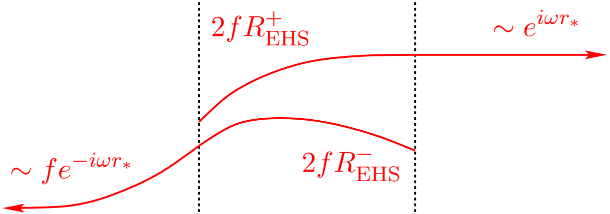


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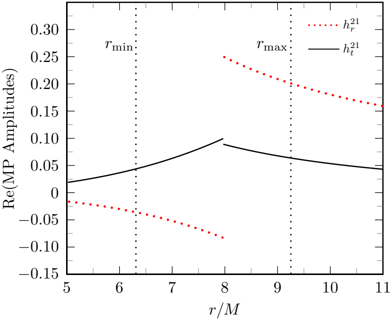


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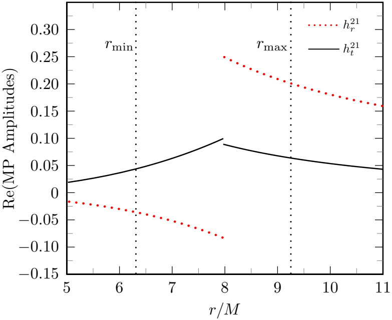


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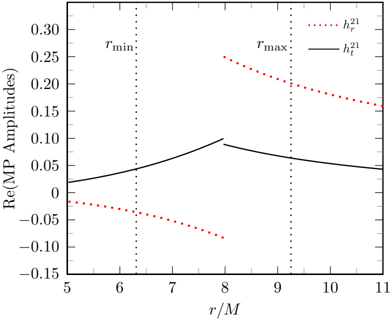


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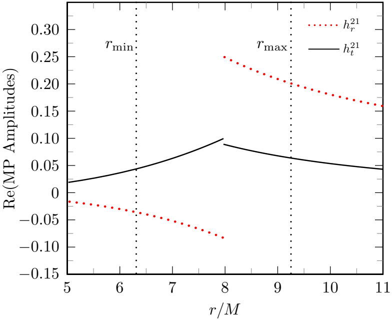


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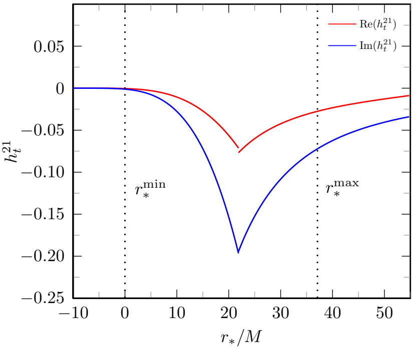
$h_t^{21}(t_\circ, r_*)$ locally

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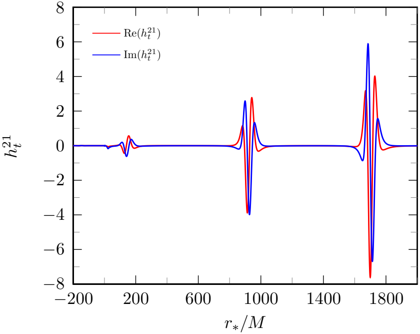


$$p = 8.75455$$
$$e = 0.764124$$
$$t_\circ = 143.45M$$
$$-50 \leq n \leq 50$$

- Now $C^{-1}$ at the particle
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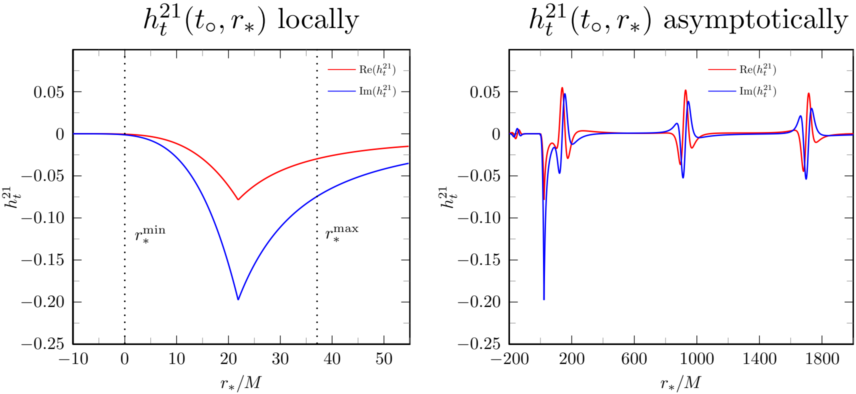
# $h_t^{\ell m}$ in Lorenz gauge

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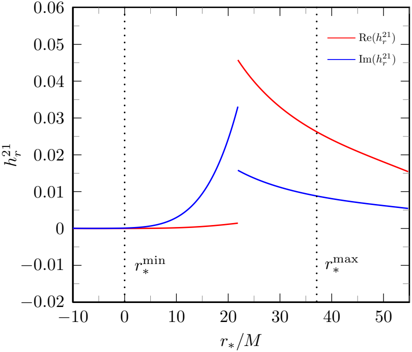


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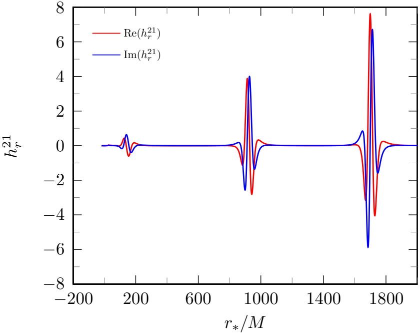


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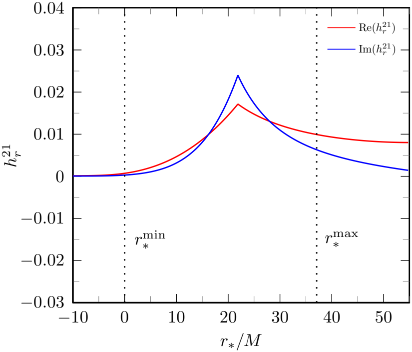


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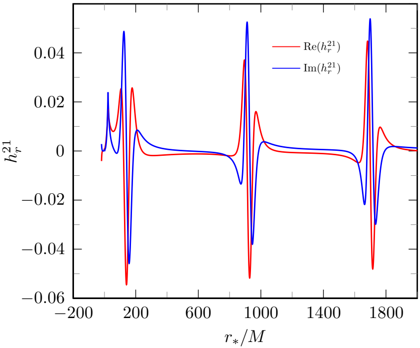


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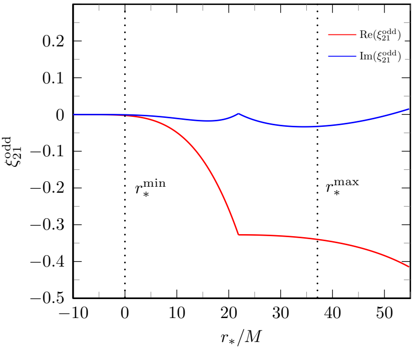
# $\xi^{\text{odd}}_{\ell m}$ - numerical results

$\xi_{21}(t_\circ, r_*)$ locally

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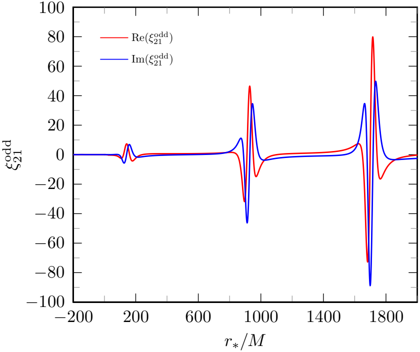


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- We see the expected local and asymptotic behavior following the partial sum.

# Outline

Choosing a gauge and solving the Einstein equations

Transforming from Regge-Wheeler to Lorenz gauge

Odd-parity gauge transformation

Even-parity gauge transformation

Conclusions

# Even-parity gauge transformations: Direct approach

- We choose between the "Direct approach" and the "SNS approach" (Sago, Nakano, and Sasaki)
- As we saw earlier, in the direct approach the natural decomposition of $\Xi_\mu$ is

$$\Xi_t = \xi_t^{\ell m}(t,r) Y^{\ell m}$$
$$\Xi_r = \xi_r^{\ell m}(t,r) Y^{\ell m}$$
$$\Xi_A = \xi_{(e)}^{\ell m}(t,r) Y_A^{\ell m} + \xi_{(o)}^{\ell m}(t,r) X_A^{\ell m}$$

- This leads to the **coupled** even-parity equations

$$\Box \xi_t^{\ell m} + M_t(\xi_t, \xi_r) = F_t(\Psi_{\mathrm{ZM}}) + \text{singular terms}$$
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- We choose between the "Direct approach" and the "SNS approach" (Sago, Nakano, and Sasaki)
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# Even-parity gauge transformations: SNS approach

- Formalism devised by Sago, Nakano, and Sasaki (2002)
- The odd-parity part same as before
- The even-parity splits further into scalar part and divergence-free vector part

$$\Xi^{\mu}_{\text{even}} = \Xi_{(s)}{}^{|\mu} + \Xi^{\mu}_{(v)}$$

- Leads to a set of **decoupled** equations

$$\begin{aligned}
\mathcal{W}^4 \xi^{\ell m}_{\text{scalar}} &= F_{\text{scalar}}(\Psi_{\text{ZM}}) + \text{singular terms} \\
\mathcal{W}^2 \psi^{\ell m}_{1} &= F_1(\Psi_{\text{ZM}}) + G_1(\xi_{\text{scalar}}) + \text{singular terms} \\
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# Even-parity gauge transformations: SNS approach

- These equations are wave equations with sources that include extended terms with jumps and singular terms
- Can be solved via the EHS and EPS methods
- Having solved these equations gives

$$\xi^{\ell m}_{\text{scalar}}, \quad \xi^{\ell m}_{v_t}, \quad \xi^{\ell m}_{v_r},$$

  and derivatives to high accuracy everywhere.
- Finally, we recover the gauge push variables

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\xi^{\ell m}_{t} &= F_t(\xi_{v_t}) + G_t(\xi_{v_r}) + H_t(\xi_{\text{scalar}}) \\
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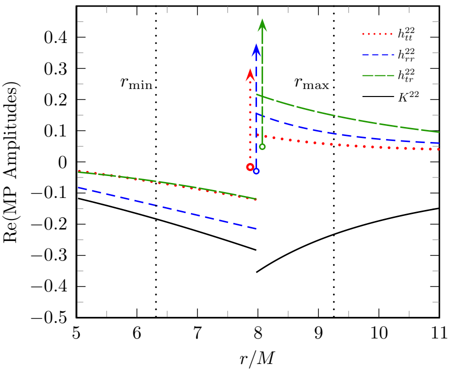
# Even-parity gauge transformations: Pushing the MP

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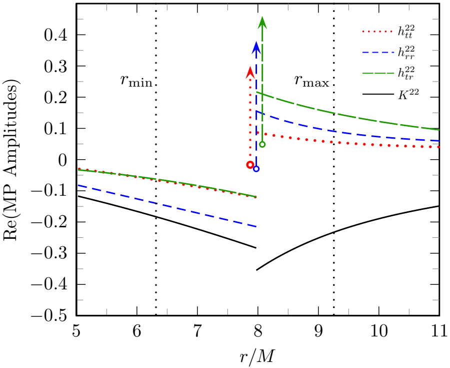


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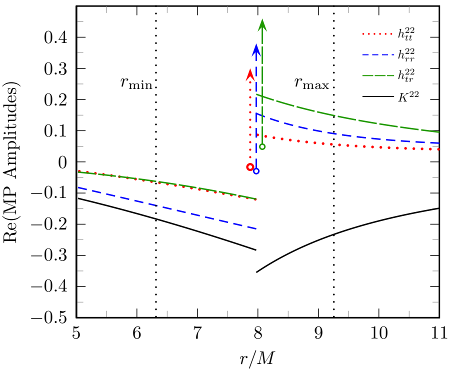


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# Even-parity gauge transformations: Current status

- We have adopted the SNS approach
- The equations have extended source terms and singular source terms
- We have solutions for all extended source terms
- Presently computing expressions for singular sources
- Will give high accuracy solutions to the first-order Einstein equations for “all” radiative modes.

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# Outline

# Conclusions

- High accuracy MPs in RW gauge for eccentric orbits on Schwarzschild
- Performed the odd-parity gauge transformation taking the metric perturbation from Regge-Wheeler to Lorenz gauge
- Partial annihilators and extended particular solutions give the same result
- Even-parity very close to being done
- With low-order modes, will allow for high-accuracy SF calcs

- We have developed two useful solution techniques for wave equations with periodic extended source terms with discontinuities
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- These are exact analogies to the EHS method for extended sources.

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