Isotropic Transformation Optics
and
Approximate Cloaking

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  At least one eigenvalue $\longrightarrow 0$ **or** ∞ **at some points**

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- $F : \Omega \longrightarrow \Omega$ **a smooth transformation:**

$\sigma(x)$ **pushes forward to a new conductivity,** $\tilde{\sigma} = F_*\sigma$**,**

$$(F_*\sigma)^{jk}(y) = \frac{1}{\det[\frac{\partial F^j}{\partial x^k}]} \sum_{p,q=1}^{n} \frac{\partial F^j}{\partial x^p} \frac{\partial F^k}{\partial x^q} \sigma^{pq}$$

with the RHS evaluated at $x = F^{-1}(y)$

- F **is a diffeomorphism, then**

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- **For many TO designs,** F **is singular**
- **Removable singularity theory** *can* $\implies \exists$ **one-to-one correspondence**

$$\{\,\textbf{Solutions of } \nabla \cdot (\tilde{\sigma}\nabla\tilde{u}) = 0\,\} \leftrightarrow \{\,\textbf{Solutions of } \nabla \cdot (\sigma\nabla u) = 0\,\}$$

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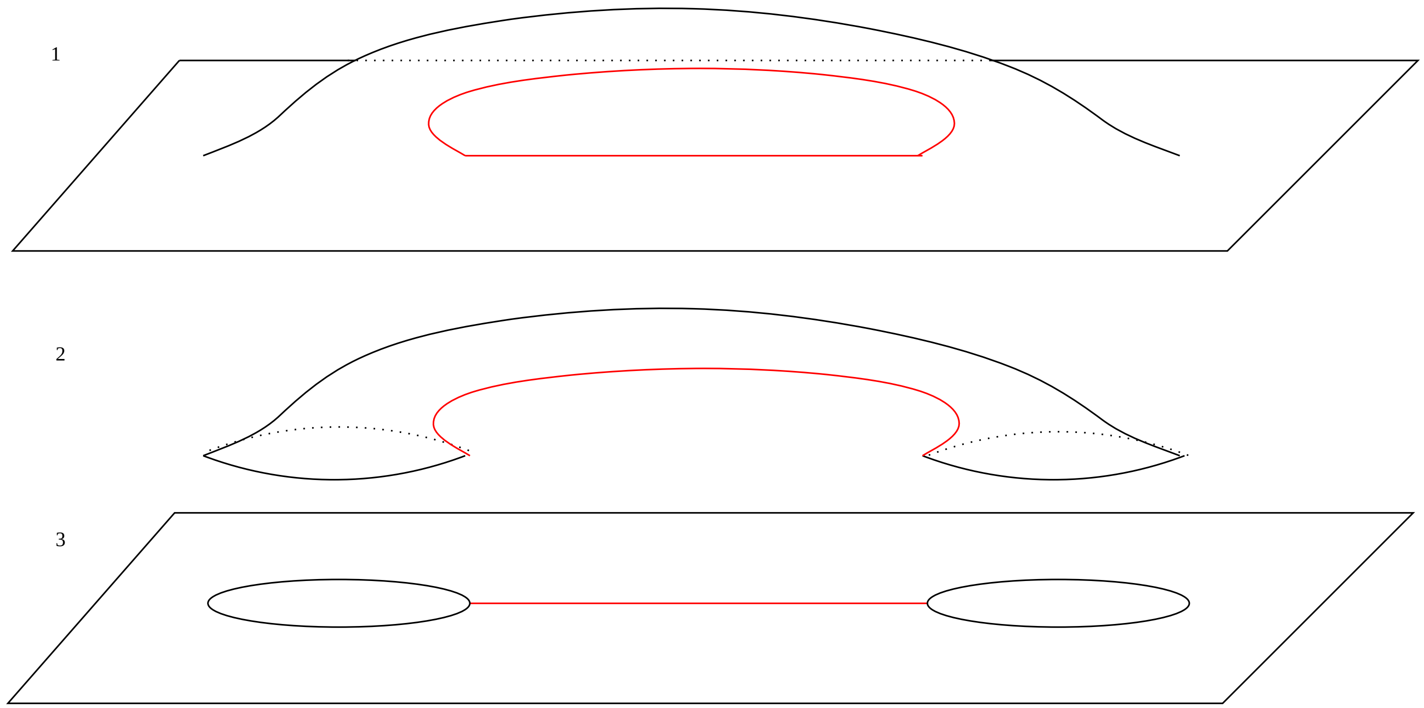
- **Based on "blowing up a curve" rather than "blowing up a point"**
- **Inside of wormhole can be varied to get different effects**
- **Produces global effect on waves encountering the WH**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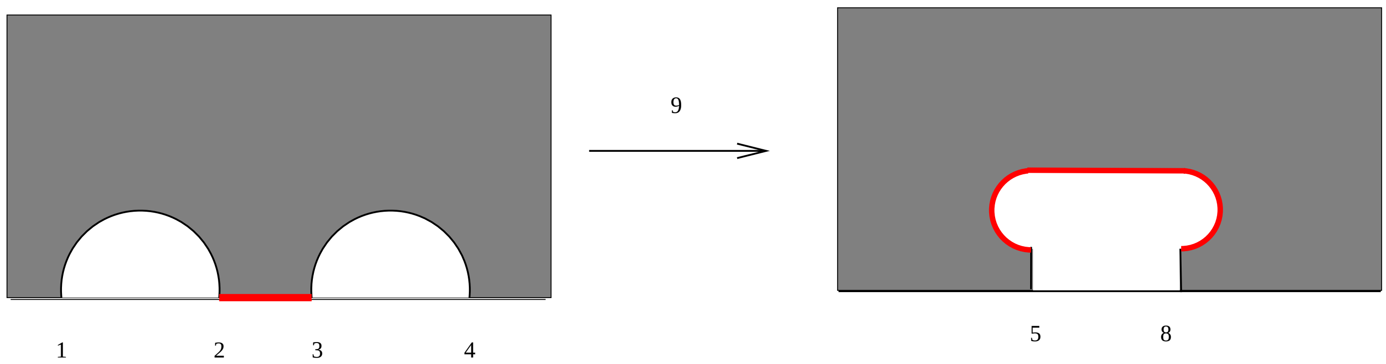
- **Missing:** K = **thickened wall of tunnel, where impose SHS condition**
- $\tilde{g} \leftrightarrow \tilde{\epsilon} = \tilde{\mu} = |\tilde{g}|^{1/2} \tilde{g}^{-1}$ **: anisotropic, and singular at surfaces of tunnel**



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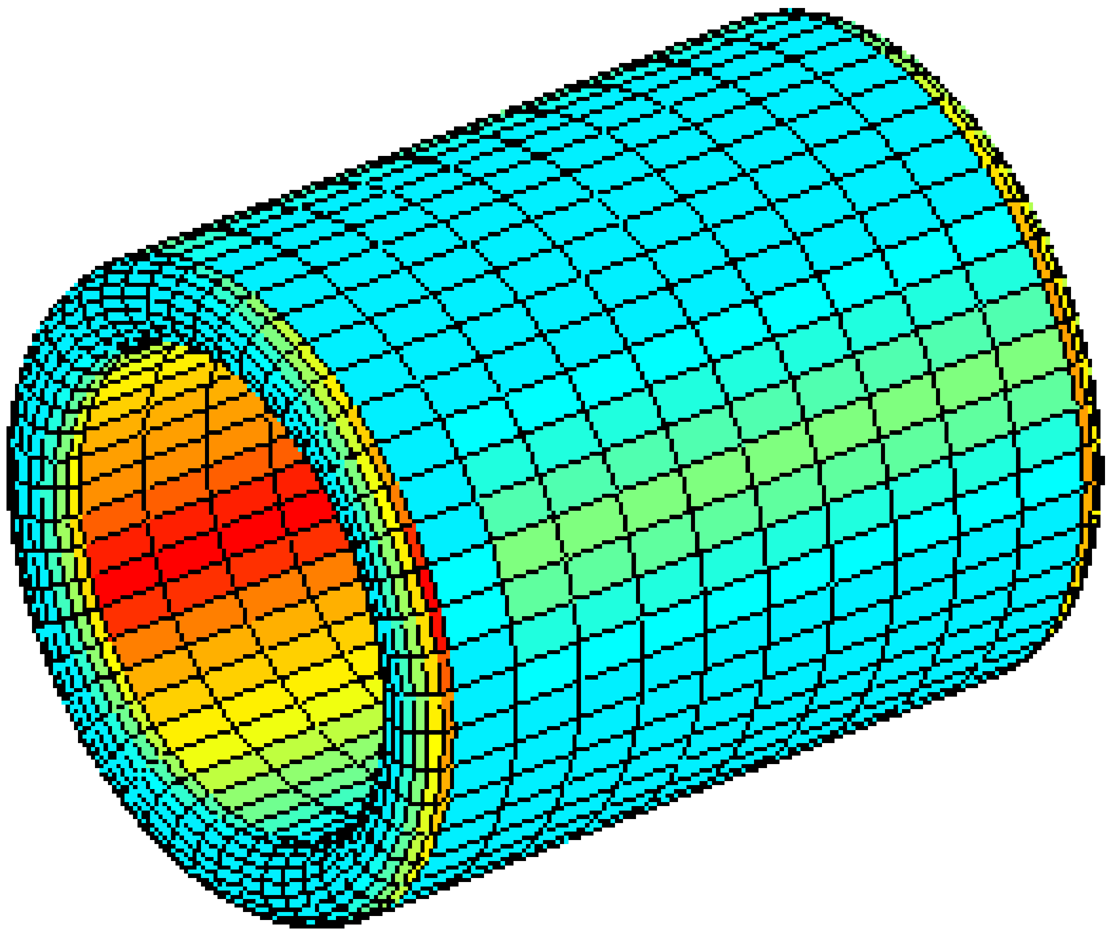

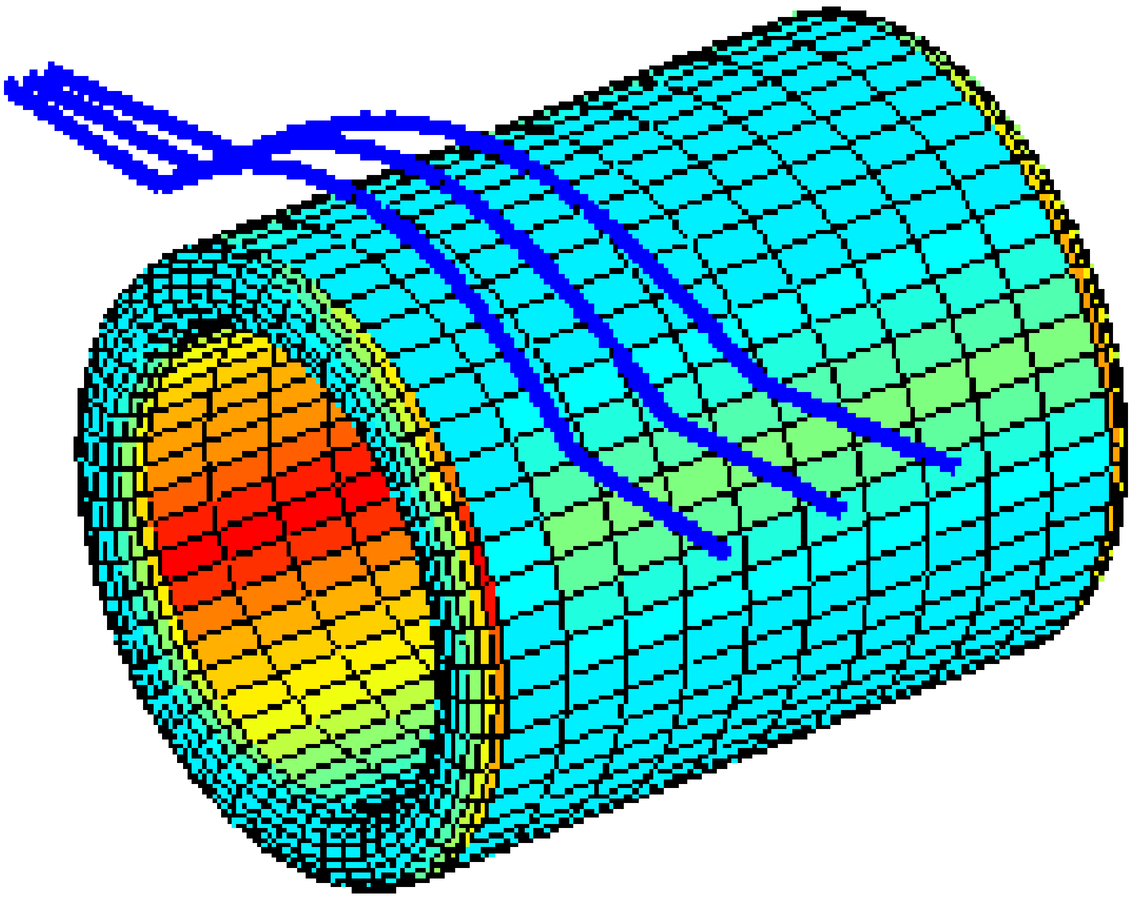

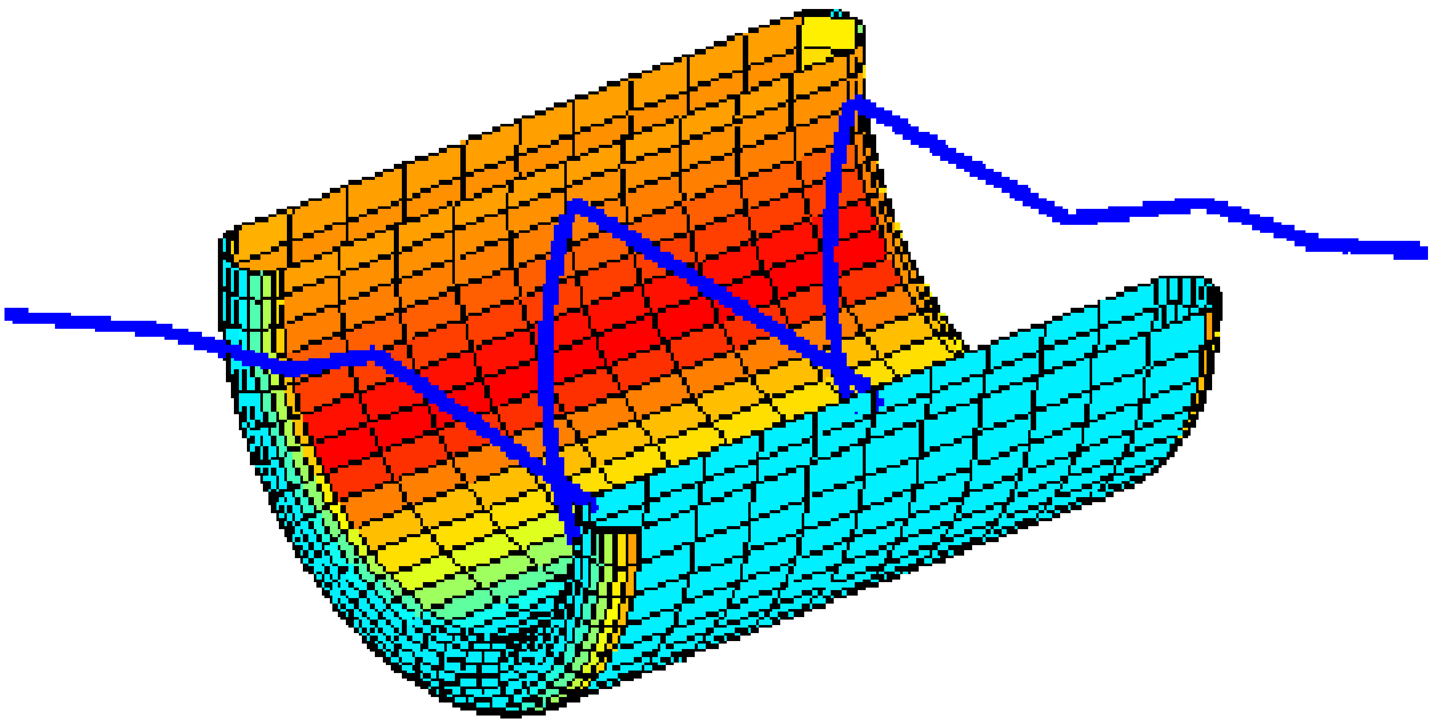

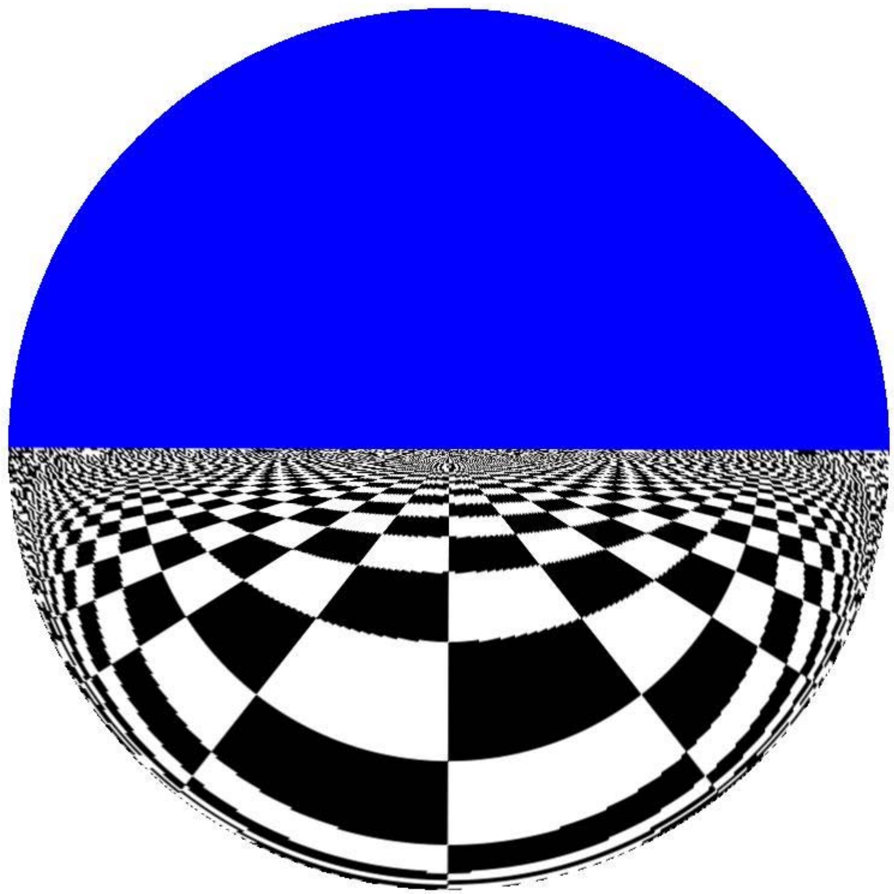

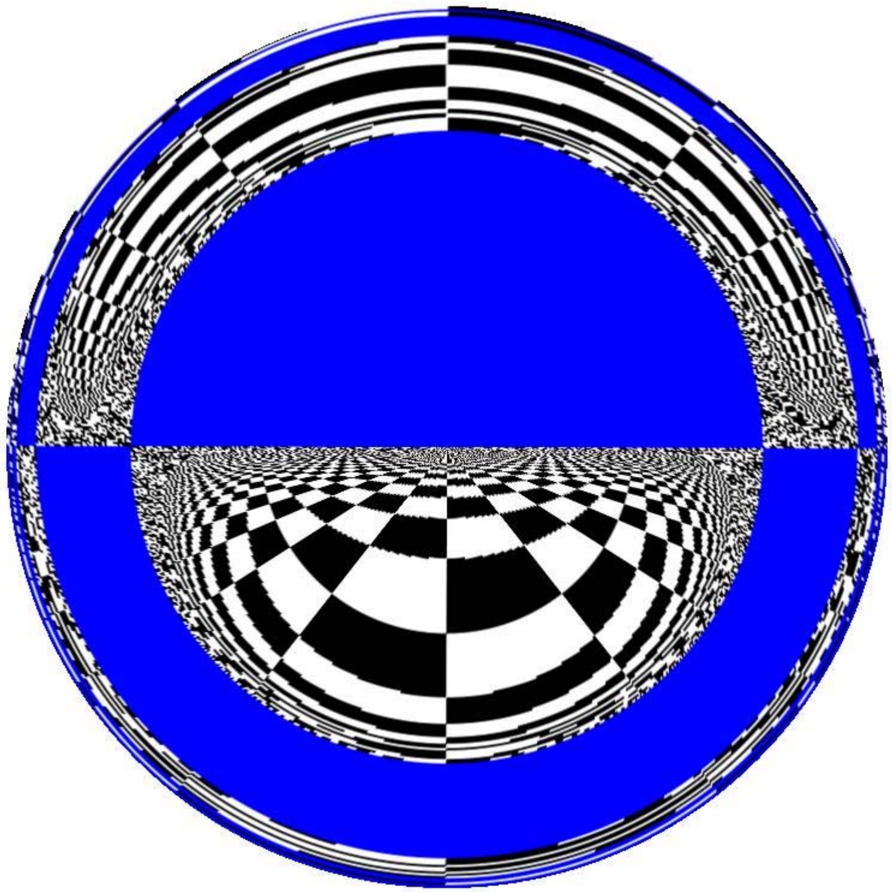

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$$\sigma^{jk} = |g|^{1/2} g^{jk} \leftrightarrow \quad g^{jk} = |\sigma|^{-1/(n-2)} \sigma^{jk}$$

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- $(\Delta_g + \omega^2) u(x) = h(x)$ **, with source** h

- **Cloaking manifold (virtual space)**

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$$(M_2, g_2) = (B_1, g_{any})$$

- **Singular transformation**

$$F_1 : M_1 \setminus \{0\} \longrightarrow N_1 = B_2 \setminus B_1 \subset \mathbb{R}^3, \quad F_1(x) = (1 + \tfrac{|x|}{2})\tfrac{x}{|x|}$$

$$F_2 : M_2 \longrightarrow N_2 = B_1, \quad F_2(x) = x \textbf{ (or any diffeom.)}$$

- **Cloaking device (physical space)**

  $N = N_1 \cup N_2 = B_2$ **with** $\tilde{g} = (\tilde{g}_1, \tilde{g}_2) = ((F_1)_* g_1, (F_2)_* g_2)$

  Cloaking surface $\Sigma = \{|x| = 1\}$

  $\tilde{g}$ **nonsingular on** Σ^-**, singular on** Σ^+**:** $\lambda_1, \lambda_2 \sim 1, \lambda_3 \sim (r-1)^2$

Thm. (3D Cloaking for Helmholtz) Let $\tilde{h} = (\tilde{h}_1, \tilde{h}_2)$ be supported away from Σ. Then there is a 1-1 correspondence between [finite energy] [distributional] solutions $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ of

$$(\Delta_{\tilde{g}} + \omega^2)\tilde{u} = \tilde{h} \textbf{ on } N$$

and solutions $u = (u_1, u_2) = \tilde{u} \circ F = (\tilde{u}_1 \circ F_1, \tilde{u}_2 \circ F_2)$ of

$$(\Delta_{g_1} + \omega^2)u_1 = h_1 := \tilde{h}_1 \circ F_1 \textbf{ on } M_1$$

$$(\Delta_{g_2} + \omega^2)u_2 = h_2 := \tilde{h}_2 \textbf{ on } M_2, \quad \partial_\nu u_2 = 0 \textbf{ on } \partial M_2$$

- **“Virtual surface” at Σ: acts as a perfectly reflector**

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- **Dichotomy: cloaking vs. trapped states**

  (I) If ω^2 is not a Neumann eigenvalue of (M_2, g_2), waves cannot penetrate Σ, and $\tilde{u}_2 \equiv 0$ on B_1:

  Cloaking works as advertised

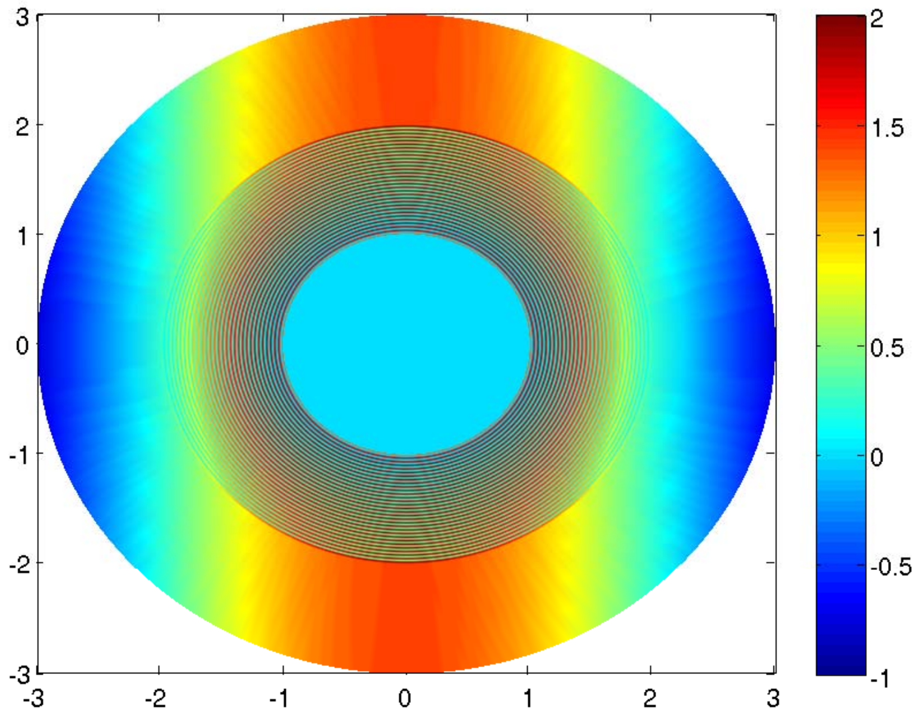
or

  (II) If ω^2 is an eigenvalue, then $\exists$ waves $\equiv 0$ on $B_2 \setminus B_1$

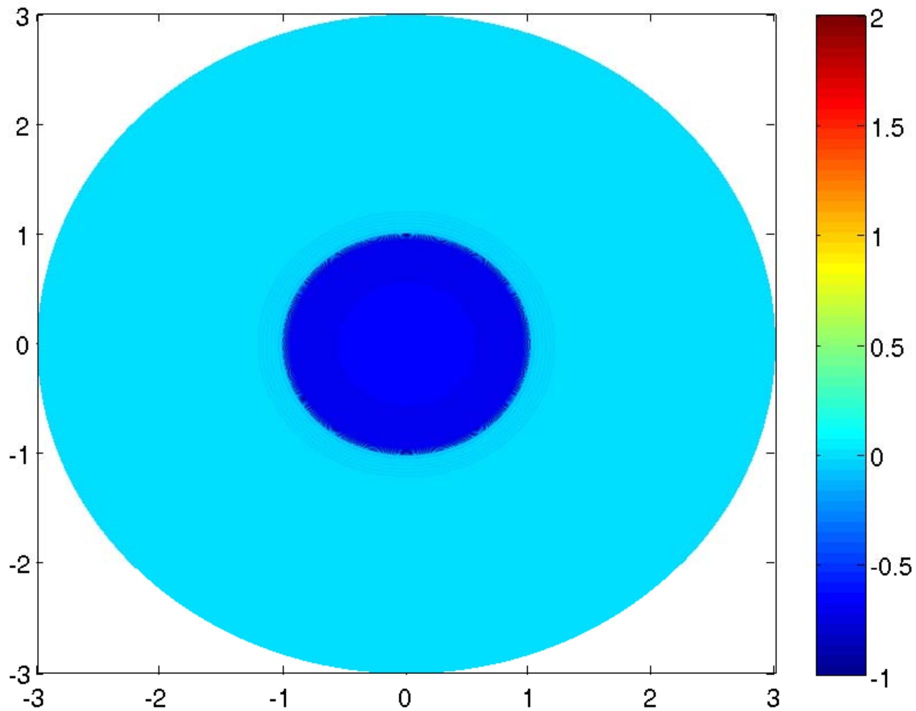
  and $=$ a Neumann eigenfunction on B_1:

  Trapped states

Wave passing cloak (ω^2 not an eigenvalue)



Trapped state (ω^2 an eigenvalue)



3D Acoustic cloak

(Helmholtz) $$|g|^{-1/2} \sum_{j,k} \partial_j (|g|^{1/2} g^{jk} \partial_k u) + \omega^2 u = 0$$

$$\Longleftrightarrow$$

(Acoustic) $$\sum_{j,k} \partial_j (|g|^{1/2} g^{jk} \partial_k u) + \omega^2 |g|^{1/2} u = 0$$

with *mass density* $\rho^{jk} = |g|^{1/2} g^{jk}$**,** *bulk modulus* $\lambda = |g|^{1/2}$**.**

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• **Same as H. Chen and C.T. Chan,** *Appl. Phys. Lett.* **91 (2007), 183518, and S. Cummer, et al.,** *Phys. Rev. Lett.* **100 (2008), 024301.**

• Σ^- **acts as a sound-hard virtual surface, and dichotomy holds...**

Quantum Mechanical Cloak for Matter Waves

At energy E, let $\omega = \sqrt{E}$:

(Schrödinger) $$-\sum_{j,k} \partial_j(|g|^{1/2} g^{jk} \partial_k \psi) + E(1 - |g|^{1/2})\psi = E\psi$$

with *effective mass* $\widehat{m}^{jk} = |g|^{1/2} g^{jk}$, $\quad \widehat{V} = E(1 - |g|^{1/2}$

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- **Same as Zhang, et al.,** *Phys. Rev. Lett.* **100 (2008), 123002.**
- **Ditto, ditto, ...**

Approximate Cloaking

- **Avoid anisotropic, singular material parameters**
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Approximate Cloaking

- **Avoid anisotropic, singular material parameters**
- **Replace with isotropic, nonsingular parameters**
- **Price: cloaks only** *approximately* **(but to arbitrary accuracy)**
- **Believe: should work for other singular TO designs**

General acoustic-like equations

- **Incorporate magnetic potential b into eqn.** $\longrightarrow \nabla_b = \nabla + ib$

$$(*) \qquad \nabla_b \cdot \sigma_1 \nabla_b u + q|g|^{1/2} u = h$$

- **Truncated equations: For $1 < R \leq \frac{3}{2}$, replace σ_1 by**

$$\sigma_R(x) = \begin{cases} (F_1)_*(\delta^{jk}), & x \in B_2 \setminus B_R \\ 2\delta^{jk}, & x \in B_R \end{cases}$$

- **Quadratic forms** a_1 **and** a_R

- **Monotonicity:** $a_R[u] \searrow$ **as** $R \searrow 1$ **(NOT TRUE FOR** $n = 2$ **)**

- **Lemma** $\Gamma - lim_{R \longrightarrow 1} a_R = a_1$ **on** L^2_g

- **Then truncate** $|g|^{1/2}$**, ... , get nonsingular, anisotropic acoustic eqns whose solutions approximate those of the original eqn.**

- **Homogenization: approximate these by isotropic equations, ditto**

Approximate quantum cloaking

- **Fix** V_0 **with** $\mathbf{supp}(V_0) \subset B_1$**, and magnetic potential** $b(x)$

Then, if $E \notin Spec_D(-\nabla_b^2; B_2) \cup Spec_N(-\nabla_b^2 + V_0; B_1)$**, there exist approximate cloaking potentials** $\{V_n^E\}_{n=1}^{\infty}$ **such that**

$$\lim \Lambda_{V_0 + V_n^E} f = \Lambda_0 f, \quad \forall f \in H^{1/2}(\partial B_2)$$

Approximate dichotomy:

• **(I) If** E **is far from a** $Spec_N(-\nabla_b^2 + V_0; B_1)$**, then the** V_n^E **act as** *approximate quantum cloaks*: **matter waves at energy** E **will pass by roughly undisturbed;**

or

• **(II) If** E **is close to an eigenvalue, then** V_n^E **supports** *almost trapped states*, **largely concentrated in** B_1**.**

• **Magnetically tunable: switch between (I) and (II) by varying** $b(x)$

Red: wave passing cloak. Blue: almost trapped state