Uniform asymptotics for the effect of small inhomogeneities

Collaborator: H.M. Nguyen

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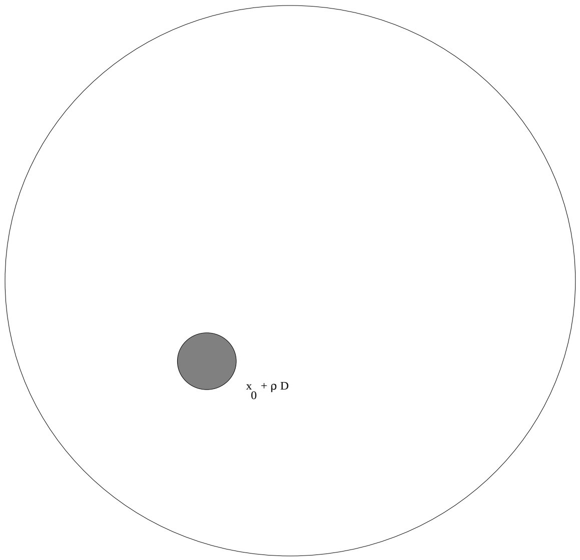

$$\sigma_\rho(x) = \begin{cases} \sigma_0 & \text{in } \Omega \setminus D_\rho \\ \sigma & \text{in } D_\rho \end{cases}$$

$$D_\rho = x_0 + \rho D \subset \Omega$$

$$0 < \sigma_0 \ , \quad \sigma < \infty$$

$$\begin{cases} \nabla \cdot (\sigma_\rho \nabla u_\rho) = 0 & \text{in } \Omega \\ (\sigma_\rho \nabla u_\rho) \cdot \nu = \psi & \text{on } \partial\Omega. \end{cases}$$

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$N(x,y)$ is the Neumann function for $\nabla \cdot (\sigma_0 \nabla\ )$:

$$\nabla_x \cdot \big(\sigma_0 \nabla_x N(x,y)\big) = \delta_y \text{ in } \Omega$$

$$(\sigma_0 \nabla_x N) \cdot \nu_x = \tfrac{1}{|\partial\Omega|} \text{ on } \partial\Omega.$$

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GIVEN A FIXED σ_0 (FOR EXAMPLE $\sigma_0 = I$) THERE ARE TWO REMARKABLE FACTS

1. M is bounded uniformly in σ.

2. $o(\rho^n)/\rho^n \to 0$ as $\rho \to 0$ uniformly in σ and ψ, provided $\|\psi\|_{H^{-1/2}(\partial\Omega)} \leq 1$.

For $\sigma_0 = I$, M is defined as follows

$$M_{i,k} = \frac{1}{|D|} \int_D \left(\delta_{i,j} - \sigma_{ij}(\rho z)\right) \frac{\partial}{\partial z_j} \phi_k \, dz \quad ,$$

where

$$\nabla_z \cdot (\gamma(z) \nabla_z \phi_k) = 0 \quad \text{in } \mathbb{R}^n \quad ,$$

$$\phi_k - z_k \to 0 \quad \text{as } |z| \to \infty \quad , \qquad \text{with}$$

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We note that γ, ϕ_k and M generically depend on ρ.

Let $\Lambda_{\sigma_\rho}^{-1}$ denote the Neumann-to-Dirichlet data map (i.e., Λ_{σ_ρ} is the Dirichlet-to-Neumann data map) then as a consequence

$$\|\Lambda_{\sigma_\rho}^{-1} - \Lambda_{\sigma_0}^{-1}\|_{H^{-1/2} \to H^{1/2}} \leq C\rho^n$$

with C completely independent of the conductivity, σ, inside the inhomogeneity ρD.

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Consider

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Define the energy

$$E_\rho(v) = \frac{1}{2} \int_\Omega < \sigma_\rho \nabla v, \nabla v > \, dx + \int_\Omega F v \, dx - \int_{\partial\Omega} f v \, d\sigma$$

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Suppose supp$F \subset\subset \Omega \setminus \rho D$. Then

$$|E_\rho(v_\rho) - E_0(v_0)| \leq C\rho^n \left(\|F\|^2_{L^2} + \|f\|^2_{H^{-1/2}} \right)$$

with C independent of the conductivity σ (inside ρD).

Proof: suppose $E_\rho(v_\rho) \geq E_0(v_0)$, then

$$\begin{aligned} |E_\rho(v_\rho) - E_0(v_0)| &= E_\rho(v_\rho) - E_0(v_0) \\ &\leq E_\rho(v^*) - E_0(v_0) \qquad \text{for any } v^* \in H^1(\Omega) \end{aligned}$$

suppose $x_0 = 0$, and select

$$v^*(x) = \chi_\rho(x) v_0(0) + (1 - \chi_\rho(x)) v_0(x)$$

with $\chi_\rho \equiv 1$ in B_ρ, $\chi_\rho \equiv 0$ outside $B_{2\rho}$ (where $\rho D \subset B_\rho$).

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Then

$$\begin{aligned} &2(E_\rho(v^*) - E_0(v_0)) \\ &= \int_\Omega < \sigma_\rho \nabla v^*, \nabla v^* > \; - \int_\Omega < \sigma_0 \nabla v_0, \nabla v_0 > \\ &= \int_{B_{2\rho} \backslash B_\rho} < \sigma_\rho \nabla v^*, \nabla v^* > \; - \int_{B_{2\rho}} < \sigma_0 \nabla v_0, \nabla v_0 > \end{aligned}$$

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&\quad \leq C\rho^n \left(\|F\|^2_{L^2} + \|f\|^2_{H^{-1/2}} \right)
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with C independent of the conductivity σ (inside ρD). Similarly for the case $E_\rho(v_\rho) < E_0(v_0)$ we use the dual variational characterization

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$$A_\rho \;:\; (F, f) \to \left((v_\rho - v_0)|_{\Omega\setminus B_\delta}, -(v_\rho - v_0)|_{\partial\Omega} \right)$$

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| < A_\rho(F,f),(F,f) > | &= \left| \int_\Omega F(v_\rho - v_0) - \int_{\partial\Omega} f(v_\rho - v_0) \right| \\
&= 2\left| E_\rho(v_\rho) - E_0(v_0) \right| \le C\rho^n \left(\|F\|^2_{L^2} + \|f\|^2_{H^{-1/2}} \right)
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and so by “polarization”

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or

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in particular

$$\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \leq C\rho^n \|\psi\|_{H^{-1/2}(\partial\Omega)}$$

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We have a more general Representation Formula (with Y. Capdeboscq)

$$\forall y \in \partial\Omega,\ u_\rho(y) - u_0(y) = |D_\rho| \int_\Omega \qquad\qquad\qquad\qquad + o(|D_\rho|)$$

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but in this case we do not in general (for $D_\rho \neq x_0 + \rho D$) get that $\|\Lambda_{\sigma_\rho}^{-1} - \Lambda_{\sigma_0}^{-1}\|_{H^{-1/2} \to H^{1/2}}$ approaches 0 uniformly with respect to σ, as $|D_\rho|$ approaches 0.

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as a example take the thin filament: $D_\rho = (-1,1) \times (-\rho, \rho)$!!

For the Helmholtz problem

$$\begin{cases} div(A_\rho \nabla u_\rho) + \omega^2 q_\rho u_\rho = 0 & \text{in } \Omega \\ \dfrac{\partial u_\rho}{\partial \nu} = \psi & \text{on } \partial\Omega \end{cases},$$

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On the one side: if $-\omega^2$ is not an eigenvalue corresponding to A_0, q_0, then for any **fixed** parameters A and q (inside ρD) there exists ρ_0 such that $-\omega^2$ is not an eigenvalues corresponding to A_ρ, q_ρ, for $\rho < \rho_0$, and

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$$\|u_{s,\rho}\|_{L^2(\partial\Omega)} \leq C\sqrt{\rho}$$

for a "unit-sized" incident wave, with C independent of ω.

On the other hand: even if $-\omega^2$ is not an eigenvalue corresponding to A_0, q_0, it may be an eigenvalue for some A_ρ, q_ρ (with A_ρ, q_ρ very large inside ρD) for ρ arbitrarily small. To remedy this situation, and obtain estimates we introduce an absorbing (lossy) layer. For example

$$\begin{cases} A_\rho = q_\rho = 1 & \text{in } \Omega \setminus B_{2\rho} \\ A_\rho = 1,\ q_\rho = 1 + i\beta & \text{in } B_{2\rho} \setminus B_\rho \\ A_\rho, q_\rho \text{ arbitrary, real} & \text{in } B_\rho \end{cases} .$$

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$$||u_\rho - u_0||_{H^{\frac{1}{2}}(\partial\Omega)} \leq C||\psi||_{H^{-\frac{1}{2}}(\partial\Omega)} \begin{cases} \frac{1}{|\log\rho|} , & n = 2 , \\ \rho^{1/2} , & n = 3 , \end{cases}$$

for $0 < \rho < \rho_0$.

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This study we are initially conducting in the context of the scattering problem ($\Omega = I\!R^n$) to avoid some of the eigenvalue issues.