Global existence and stability results for shear flows of viscoelastic fluids

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Global existence of solutions

Existence proofs for initial value problems have two parts:

1. An argument for local existence, typically based on proving convergence of some approximation scheme.
2. A priori estimates showing solutions do not blow up and can be continued.

Example: Navier-Stokes equations

Assume, for simplicity, periodic boundary conditions.

$$\rho(\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}) = \eta \Delta \mathbf{v} - \nabla p,$$

$$\operatorname{div} \mathbf{v} = 0.$$

If we multiply by **v** and integrate we find

$$\frac{d}{dt} \int_\Omega \frac{1}{2} \rho |\mathbf{v}|^2 \, d\mathbf{x} = - \int_\Omega \eta |\nabla \mathbf{v}|^2 \, d\mathbf{x}.$$

This is enough to guarantee global existence (but not uniqueness) of a weak solution. In two dimensions, we can do more. Take the curl of the equation of motion, and let ω be the vorticity. We find

$$\rho(\omega_t + (\mathbf{v} \cdot \nabla)\omega) = \eta \Delta \omega,$$

and hence

$$\frac{d}{dt} \int_\Omega \frac{1}{2} \rho \omega^2 \, d\mathbf{x} = - \int_\Omega \eta |\nabla \omega|^2 \, d\mathbf{x}.$$

This suffices to prove global existence of smooth solutions.

How about non-Newtonian fluids?

Win an arguement against Prof
Plums and he'd prove you didn't exist.

Global existence results in viscoelastic fluids are restricted to simple flows

Simple flows, Palouse Falls, Washington -- Extensive sheetflows of the Columbia River floodbasalt province are exposed in southeastern Washington at Palouse falls. *Photo by Vic Camp.*

Viscoelastic fluids

Only one-dimensional results for global existence (other than for small data).
We shall consider parallel shear flows (reduces to the heat equation for Newtonian case).

Governing equations:

$$\rho u_t = \tau_y + \eta u_{yy} + f(y,t),$$

$$\mathbf{T} = \begin{pmatrix} \sigma & \tau \\ \tau & \psi \end{pmatrix},$$

$$\mathbf{T}_t = \mathbf{G}(\mathbf{T}, u_y).$$

Initial conditions for u and **T**, Dirichlet boundary conditions for u.

A local existence result is easy, for instance by using an iterative construction like the following:

$$\rho u_t^{n+1} = \tau_y^n + \eta u_{yy}^{n+1} + f(y,t),$$

$$\mathbf{T}_t^{n+1} = \mathbf{G}(\mathbf{T}^n, u_y^{n+1}).$$

Global continuation (of a solution as smooth as the data will allow) is possible if we can get a priori bounds on the L^1 norms of u_y and **T**. This is possible if we make certain assumptions that are satisfied for a number of constitutive laws.

Positive definiteness conditions for the stress tensor play an essential role in the arguments.

Assumptions sufficient for global existence

(A1) There is p<1 such that

$$|\mathbf{G}(\mathbf{T}, u_y)| \le C(|u_y| + |\mathbf{T}|)^p.$$

(A2) There is q<1 and ν<1 such that every solution of

$$\mathbf{T}_t = \mathbf{G}(\mathbf{T}, u_y)$$

satisfies the bounds

$$|\tau| \le C(1 + \max_{s \in [0,t]} |u_y(s)|^q),$$

$$|\sigma| + |\psi| \le C(1 + \max_{s \in [0,t]} |u_y(s)|^\nu),$$

where C depends only on t and the initial data.

White-Metzner model

$$\tau_t = u_y - \frac{\lambda}{\mu(u_y)}\tau,$$

$$\sigma_t = 2\tau u_y - \frac{\lambda}{\mu(u_y)}\sigma,$$

$$\psi = 0,$$

$$\lambda > 0,\ \ \mu(u_y) > 0,\ \ \mu(u_y) \sim |u_y|^{-\gamma}$$

$$\text{for large } |u_y|,\ \ 0 < \gamma < 1.$$

Assumption (A2) holds with q=1-γ and ν=2-2γ.

Phan-Thien Tanner model

$$\sigma_t = 2\tau u_y - \lambda\sigma - \kappa\sigma^2,$$

$$\tau_t = -\lambda\tau - \kappa\sigma\tau + \mu u_y,$$

$$\psi = 0.$$

We can derive that

$$\frac{d}{dt}(\mu\sigma - \tau^2) = -(\lambda + \kappa\sigma)(\mu\sigma - \tau^2) + (\lambda + \kappa\sigma)\tau^2.$$

This implies positive definiteness:

$$\begin{vmatrix} \sigma & \tau \\ \tau & \mu \end{vmatrix} = \mu\sigma - \tau^2 \geq 0.$$

It follows that

$$\frac{d}{dt}(\tau)^2 \leq -2\lambda\tau^2 - 2\kappa\tau^4/\mu + 2\mu\tau u_y.$$

This implies (A2) with q=1/3.

Johnson-Segalman model

$$\begin{aligned} \sigma_t &= -\lambda\sigma + (1+a)\tau u_y, \\ \tau_t &= -\lambda\tau + (\frac{a}{2}(\sigma+\psi) + \frac{1}{2}(\psi-\sigma) + \mu)u_y, \\ \psi_t &= -\lambda\psi + (a-1)\tau u_y. \end{aligned}$$

Here $\lambda, \mu > 0$ and $-1 < a < 1$. It is convenient to introduce new variables:

$$Y = (1-a)\sigma + (1+a)\psi, \; Z = \frac{a}{2}(\sigma+\psi) + \frac{1}{2}(\sigma-\psi).$$

The equations transform to $Y_t + \lambda Y = 0$ and

$$\begin{aligned} \tau_t &= -\lambda\tau + (Z+\mu)u_y, \\ Z_t &= -\lambda Z + (a^2-1)\tau u_y. \end{aligned}$$

With

$$\Phi = \frac{1}{2}Z^2 + \mu Z + \frac{1}{2}(1 - a^2)\tau^2,$$

we find

$$\Phi_t = -2\lambda\Phi + \lambda\mu Z.$$

This leads to a priori bounds on τ and Z.

Note:

$$\begin{vmatrix} \sigma + \mu/a & \tau \\ \tau & \psi + \mu/a \end{vmatrix} = -\frac{2}{1 - a^2}\Phi + \frac{\mu^2}{a^2}.$$

Giesekus model

$$\begin{aligned}
\sigma_t &= -\lambda\sigma - \kappa(\sigma^2 + \tau^2) + 2\tau u_y, \\
\tau_t &= -\lambda\tau - \kappa(\sigma + \psi)\tau + (\mu + \psi)u_y, \\
\psi_t &= -\lambda\psi - \kappa(\tau^2 + \psi^2).
\end{aligned}$$

Here $\lambda,\mu,\kappa>0$, $\kappa\mu<\lambda$.
We set

$$\chi = \kappa\mu(\sigma - \psi) + \kappa(\sigma\psi - \tau^2) + \lambda\psi,$$

and find

$$\chi_t = -(\lambda + \kappa(\sigma + \psi))\chi.$$

We shall now assume $\chi=0$.

Next, consider

$$d = \sigma(\psi + \mu) - \tau^2.$$

It can be shown that $\sigma \geq 0$, $d \geq 0$ if this is the case initially, since

$$d_t = \frac{d^2\kappa^2}{\lambda - \kappa\mu} - d(2\lambda + \kappa(\sigma - \mu)) + \mu\sigma(\lambda - \kappa\mu).$$

Moreover,

$$d = \frac{(\lambda - \kappa\mu)(\mu\sigma - \tau^2)}{\lambda - \kappa\mu + \kappa\sigma}.$$

Hence $\tau^2 \leq \mu\sigma$. Moreover, from $\chi = 0$ and $\sigma \geq 0$ it follows that $\psi > -\mu$.

Now consider the equation

$$\tau_t = -\lambda\tau - \kappa(\sigma + \psi)\tau^2 + (\mu + \psi)u_y.$$

We have $\sigma \geq \tau^2/\mu$ and $0 \geq \psi > -\mu$. We can conclude (A2) with q=1/3.

Nonlinear dumbbell models

These do not fit into the preceding framework, but for creeping flow a priori bounds can be found by other means.

Creeping flow:

$$\tau_y + \eta u_{yy} = 0.$$

An immediate consequence is

$$|u_y(y,t)| \leq C + \frac{1}{\eta} \max_{y\in[0,L]} |\tau(y,t)|,$$

where the constant depends only on the boundary conditions.

Constitutive law:

$$\mathbf{C}_t = (\nabla \mathbf{u})\mathbf{C} + \mathbf{C}(\nabla \mathbf{u})^T + \gamma \mathbf{I} - \delta f(\operatorname{tr} \mathbf{C})\mathbf{C},$$

$$\mathbf{T} = f(\operatorname{tr} \mathbf{C})\mathbf{C}.$$

Here **C** is called a configuration tensor. γ and δ are positive constants. For the function f, we assume it is monotone and one of the following:

$$f(c) \sim c^{\mu}, \;\; f'(c) \sim c^{\mu-1}, \;\; \text{for } c \to \infty, \;\; \mu > 0;$$

$$f(c) \sim (L-c)^{-\mu}, \; f'(c) \sim (L-c)^{-\mu-1}, \; \text{for } c \to L, \; \mu > 0.$$

In shear flow, we have

$$\mathbf{C} = \begin{pmatrix} A & D & 0 \\ D & E & 0 \\ 0 & 0 & E \end{pmatrix},$$

$$\begin{aligned} A_t &= 2Du_y + \gamma - \delta f(A+2E)A, \\ D_t &= Eu_y - \delta f(A+2E)D, \\ E_t &= \gamma - \delta f(A+2E)E. \end{aligned}$$

For physically acceptable initial data, C is positive definite, and E < $\gamma/(\delta f(0))$.
We have

$$(A + 2E)_t = 2Du_y + 3\gamma - \delta f(A + 2E)(A + 2E).$$

Now let

$$Q = \max_{y \in [0,L]} (A + 2E), \;\; R = \max_{y \in [0,L]} |D|,$$

$$S = \max_{y \in [0,L]} |u_y|.$$

We find

$$Q_t \leq 2RS + 3\gamma - \delta f(Q)Q,$$

$$R \leq \sqrt{Q\gamma/(\delta f(0))},$$

$$S \leq C + \frac{f(Q)R}{\eta}.$$

By combining these, we obtain

$$Q_t \leq f(Q)Q(\frac{2\gamma}{\eta\delta f(0)} - \delta) + 3\gamma + C\sqrt{Q}.$$

This yields an a priori bound for Q if

$$\eta > \frac{2\gamma}{\delta^2 f(0)}.$$

Global stability of the rest state

Assume a constitutive law of the form

$$\mathbf{T}_t = \mathbf{G}(\mathbf{T}, u_y).$$

Make the following assumptions:

1. G(0)=0 and polynomial growth of G and its derivatives.
2. A priori estimates which imply (for some p>=1)

$$\lim_{t \to \infty} \|T\|_p = 0.$$

3. Assumption (A2) for global existence as before.

Then $\|\tau + \eta u_y\|_\infty$ tends to zero.

PTT model

$$\sigma_t = 2\tau u_y - \lambda\sigma - \kappa\sigma^2,$$

$$\tau_t = -\lambda\tau - \kappa\sigma\tau + \mu u_y,$$

$$\rho u_t = \tau_y + \eta u_{yy}.$$

This yields (assuming homogeneous Dirichlet conditions for u)

$$\frac{d}{dt}\int \sigma + \frac{\tau^2}{2} + \rho u^2(1+\frac{\mu}{2})\,dy$$

$$= -\int \lambda\sigma + \kappa\sigma^2 + (\lambda+\kappa\sigma)\tau^2 + (2+\mu)\eta u_y^2\,dy.$$

Consequently $\|\sigma\|_1 + \|\tau\|_2 \to 0.$

Note that the a priori information that σ is positive is essential here!

Similar arguments work for Johnson-Segalman and Giesekus. For the nonlinear dumbbell model and large enough η, we can prove global stability of the rest state by exploiting a refinement of the positive definiteness condition. Recall the set of equations

$$\begin{aligned} A_t &= 2Du_y + \gamma - \delta f(A+2E)A, \\ D_t &= Eu_y - \delta f(A+2E)D, \\ E_t &= \gamma - \delta f(A+2E)E. \end{aligned}$$

We derive

$$\frac{d}{dt}((A-E)E-D^2) = -2\delta f(A+2E)((A-E)E-D^2)+\gamma(A-E).$$

Positive definiteness of the conformation tensor implies that A,E and $AE-D^2$ are nonnegative. We find the stronger condition that A-E and $(A-E)E-D^2$ are nonnegative.

Questions?