Representations and Translations of Functions in the FMM

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Function Representations and FMM Operations

What do we need in the FMM?

- Sum up functions;
- Translate functions (or represent them in different bases);
- In computations we can operate only with finite vectors.

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Finite Approximations

Let

$$f : \mathbb{R}^d \to \mathbb{C} \quad (f = f(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^d).$$

We consider *approximations* of $f(\mathbf{y})$ inside or outside a sphere $\Omega_a(\mathbf{x}_*)$ of radius a centered at $\mathbf{y} = \mathbf{x}_*$. We say that function $L_P(\mathbf{y};\mathbf{x}_*)$ *uniformly* approximate $f(\mathbf{y})$ inside a sphere $\Omega_a(\mathbf{x}_*)$ if

$$\exists \epsilon_P > 0, \quad \forall \mathbf{y} \in \Omega_a(\mathbf{x}_*) \subset \mathbb{R}^d, \quad |f(\mathbf{y}) - L_P(\mathbf{y};\mathbf{x}_*)| < \epsilon_P,$$

and function $F_P(\mathbf{y})$ uniformly approximate $f(\mathbf{y})$ outside a sphere $\Omega_a(\mathbf{x}_*)$ if

$$\exists \epsilon_P > 0, \quad \forall \mathbf{y} \notin \Omega_a(\mathbf{x}_*), \quad |f(\mathbf{y}) - F_P(\mathbf{y};\mathbf{x}_*)| < \epsilon_P.$$

The subscript P near functions $L_P(\mathbf{y};\mathbf{x}_*)$ and $F_P(\mathbf{y};\mathbf{x}_*)$ means that these functions can be determined by specification of a vector $\mathbf{C}$ in the complex P dimensional space $\mathbb{C}^P$, which we call *representing vector*.

So we have a one-to-one mapping of the space of functions $L_P(\mathbf{y};\mathbf{x}_*)$ to $\mathbf{C}(\mathbf{x}_*)$ and the space of functions $F_P(\mathbf{y};\mathbf{x}_*)$ to $\mathbf{C}(\mathbf{x}_*)$:

$$L_P(\mathbf{y},\mathbf{x}_*) \rightleftarrows \mathbf{C}(\mathbf{x}_*) = (c_1,\ldots,c_P), \quad \mathbf{C} \in \mathbb{C}^P,$$
$$F_P(\mathbf{y},\mathbf{x}_*) \rightleftarrows \mathbf{C}(\mathbf{x}_*) = (c_1,\ldots,c_P), \quad \mathbf{C} \in \mathbb{C}^P.$$

The representing vector $\mathbf{C}(\mathbf{x}_*)$ for $L_P(\mathbf{y};\mathbf{x}_*)$ we will identify as *local representation*. In the case when $\mathbf{C}(\mathbf{x}_*)$ corresponds to $F_P(\mathbf{y};\mathbf{x}_*)$ we call it as *far-field* representation.

Examples:

P=*p* (real and complex functions)

Taylor expansion (for differentiable functions):

$$L_p(y;x_*) = \sum_{n=0}^{p-1} c_n (y - x_*)^n,$$

$$c_n = \frac{1}{n!} \frac{d^n f}{dy^n}\bigg|_{y=x_*}, \quad n = 0,\ldots,p-1.$$

Asymptotic expansion (for some decaying functions):

$$F_p(y;x_*) = \sum_{n=0}^{p-1} c_n (y-x_*)^{-n-1},$$

$$c_n = \lim_{y\to\infty}\left\{(y-x_*)^{n+1}\left[f(y) - \sum_{m=0}^{n-1} c_m (y-x_*)^{-m-1}\right]\right\}, \quad n = 0,\ldots,p-1.$$

Examples:

$P=p^2$ (Solutions of the Laplace equation in 3D)

$$L_p(\mathbf{y};\mathbf{x}_*) = \sum_{n=0}^{p-1}\sum_{m=-n}^{n} c_n r^n Y_n^m(\theta,\varphi),$$

$$F_p(\mathbf{y};\mathbf{x}_*) = \sum_{n=0}^{p-1}\sum_{m=-n}^{n} c_n r^{-n-1} Y_n^m(\theta,\varphi),$$

$$\mathbf{y}-\mathbf{x}_* = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

Examples:

$P=4N$ (Sum of Green's functions for Laplace equation in 3D)

$$L_P(\mathbf{y};\mathbf{x}_*) = \sum_{i=1}^{N} \frac{Q_i}{4\pi|\mathbf{y}-\mathbf{x}_i|}, \quad |\mathbf{x}_i-\mathbf{x}_*| > a,$$

$$F_P(\mathbf{y};\mathbf{x}_*) = \sum_{i=1}^{N} \frac{Q_i}{4\pi|\mathbf{y}-\mathbf{x}_i|}, \quad |\mathbf{x}_i-\mathbf{x}_*| > a,$$

$$\mathbf{C} = (x_{11}, x_{12}, x_{13}, Q_1, \ldots, x_{N1}, x_{N2}, x_{N3}, Q_N), \quad P = 4N.$$

Examples:

P=*N* (Regular solution of the Helmholtz equation in 3D)

$$L_P(\mathbf{y};\mathbf{x}_*) = \sum_{j=1}^{N} w_j \Psi(\mathbf{s}_j) e^{ik\mathbf{s}_j\cdot(\mathbf{y}-\mathbf{x}_*)}, \; P = N,$$

$$\mathbf{C} = (w_1\Psi(\mathbf{s}_1), \ldots, w_N\Psi(\mathbf{s}_N))$$

follows from integral representation

$$f(\mathbf{y}) = \int_{S_u} e^{ik\mathbf{s}\cdot(\mathbf{y}-\mathbf{x}_*)}\Psi(\mathbf{s})dS.$$

Consolidation Operation

Linear operators (easy summation)

$$R_{P1}(\mathbf{y};\mathbf{x}_*) + R_{P2}(\mathbf{y};\mathbf{x}_*) \rightleftarrows \mathbf{C}_1(\mathbf{x}_*) + \mathbf{C}_2(\mathbf{x}_*).$$

Non-linear mapping (difficult summation):

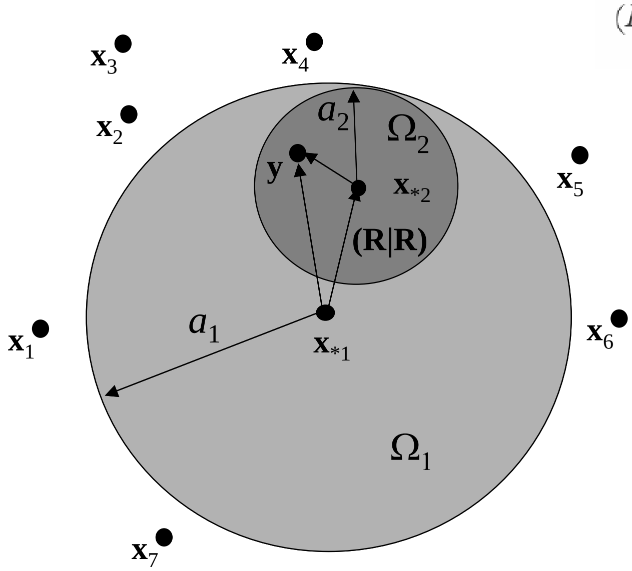
$$R_{P1}(\mathbf{y};\mathbf{x}_*) + R_{P2}(\mathbf{y};\mathbf{x}_*) \rightleftarrows \mathbf{C}(\mathbf{x}_*) = \mathbf{C}_1(\mathbf{x}_*)[+]\mathbf{C}_2(\mathbf{x}_*).$$

Consolidation operation

We usually focus on linear operators

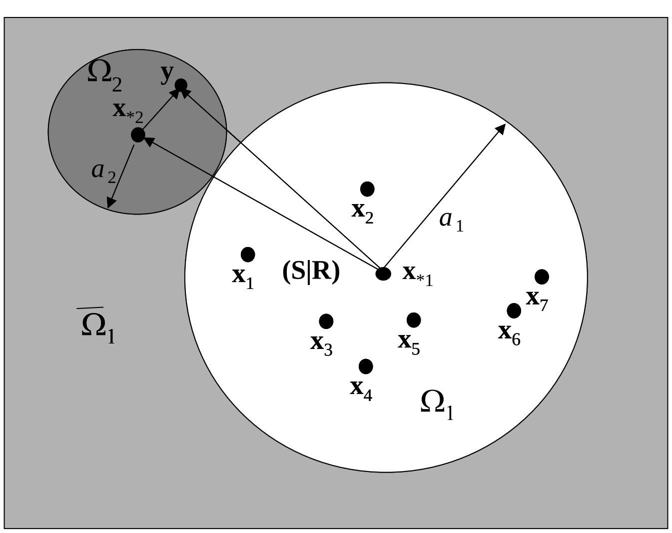
Translations: Local-to-local

$$(R|R)(\mathbf{x}_{*2} - \mathbf{x}_{*1})[\mathbf{C}_1(\mathbf{x}_{*1})] = \mathbf{C}_2(\mathbf{x}_{*2}).$$



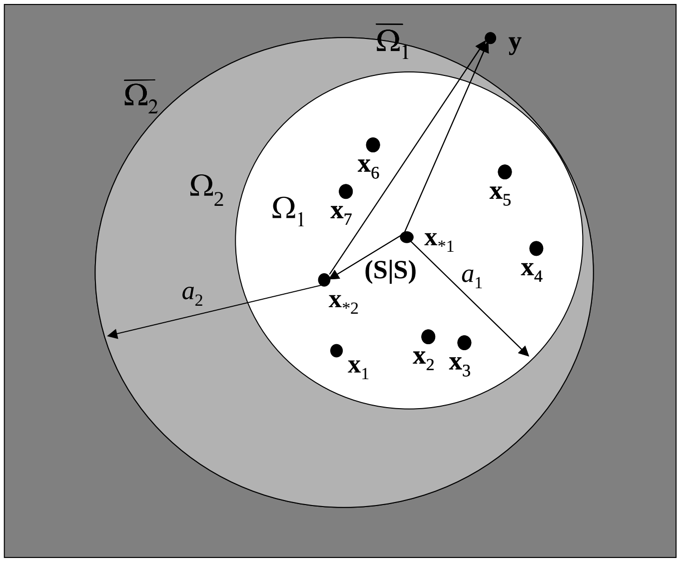
Translations: Multipole-to-local

$$(S|R)(\mathbf{x}_{*2} - \mathbf{x}_{*1})[\mathbf{C}_1(\mathbf{x}_{*1})] = \mathbf{C}_2(\mathbf{x}_{*2}).$$



Translations: Multipole-to-multipole

$$(S|S)(\mathbf{x}_{*2} - \mathbf{x}_{*1})[\mathbf{C}_1(\mathbf{x}_{*1})] = \mathbf{C}_2(\mathbf{x}_{*2}).$$



SLFMM in Terms of Representing Vectors:

- Subdivide the computational domain into N_b boxes.
- For each source $\mathbf{x}_i$ obtain vectors $\mathbf{C}_i\left(\mathbf{x}_{n*}^{(s)}\right)$ of length P_1 corresponding to function $S_{P_1}(\mathbf{y};\mathbf{x}_{n*}^{(s)})$ approximating function $u_i\Phi(\mathbf{y},\mathbf{x}_i)$ in the domain outer to the sphere $\Omega_a\left(\mathbf{x}_{n*}^{(s)}\right)$ (the sphere of radius a includes the box but enclosed into the box neighborhood) and $\mathbf{x}_{n*}^{(s)}$ is the center of the box containing $\mathbf{x}_i$.
- For each source box S_n containing q_n sources $\mathbf{x}_i, i = i_1,\ldots,i_{q_n}$, obtain vector of length $P_2^{(n)}$ (consolidation of all sources inside the source box)

$$\mathbf{C}\left(\mathbf{x}_{n*}^{(s)}\right) = \mathbf{C}_{i_1}\left(\mathbf{x}_{n*}^{(s)}\right)[+]\mathbf{C}_{i_2}\left(\mathbf{x}_{n*}^{(s)}\right)[+]\ldots[+]\mathbf{C}_{i_{q_n}}\left(\mathbf{x}_{n*}^{(s)}\right).$$

  This vector represents potential due to all the sources inside the box in the domain outside the neighborhood of this box S_n^{-}.
- $S|R$ translate each $\mathbf{C}\left(\mathbf{x}_{n*}^{(s)}\right)$ from $\mathbf{x}_{n*}^{(s)}$ to the center $\mathbf{y}_{m*}^{(r)}$ of each receiver box R_m, such that the neighborhood of R_m does not contain S_n

$$(S|R)\left(\mathbf{y}_{m*}^{(r)} - \mathbf{x}_{n*}^{(s)}\right)\left[\mathbf{C}\left(\mathbf{x}_{n*}^{(s)}\right)\right] = \mathbf{D}_n(\mathbf{y}_{m*}^{(r)}), \quad R_m^{+} \cap S_n = \emptyset$$

  where $\mathbf{D}_n$ is the vector of length $P_3^{(n)}$ representing function in the domain inner to the sphere of radius a centered at $\mathbf{y}_{m*}^{(r)}$ due to sources in box S_n.
- For each receiver box R_m obtain vector (consolidation of all sources outside the receiver neighborhood)

$$\mathbf{D}(\mathbf{y}_{m*}^{(r)}) = \mathbf{D}_{n_1}(\mathbf{y}_{m*}^{(r)})[+]\mathbf{D}_{n_2}(\mathbf{y}_{m*}^{(r)})[+]\ldots[+]\mathbf{D}_{n_{pm}}(\mathbf{y}_{m*}^{(r)}), \quad R_m^{+} \cap S_{n_1},\ldots,S_{n_{pm}} = \emptyset.$$

- For each receiver box evaluate the sum

$$\psi(\mathbf{y}_j) = \sum_{\mathbf{x}_i \in R_m^{+}} u_i\Phi(\mathbf{y}_j,\mathbf{x}_i) + R_{P_4}(\mathbf{y}_j;\mathbf{y}_{m*}^{(r)}), \quad \mathbf{y}_j \in R_m,$$

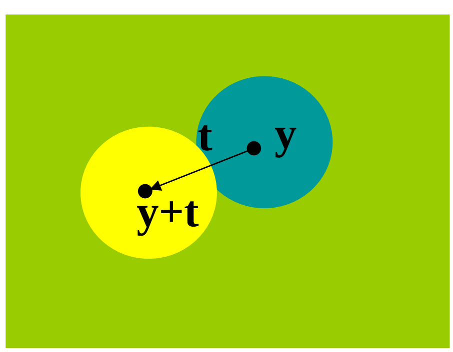
  where $R_{P_4}(\mathbf{y};\mathbf{y}_{m*}^{(r)})$ is the local function represented by $\mathbf{D}(\mathbf{y}_{m*}^{(r)})$ and R_m is the mth receiver box.

Matrix Representations of Translation Operators

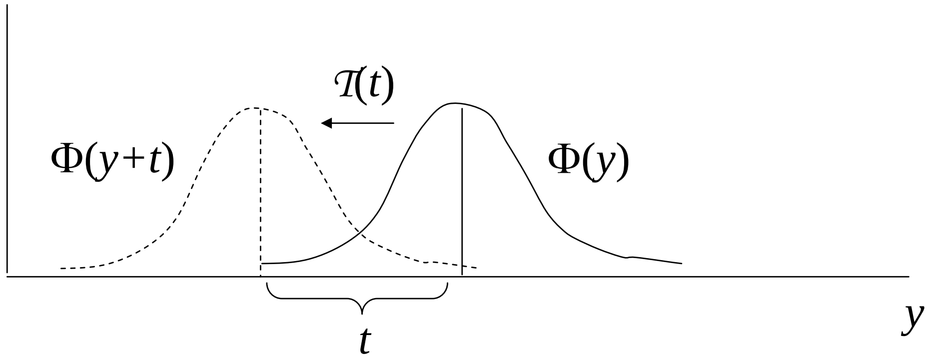
Translation Operator

Operator $\mathcal{T}(\mathbf{t}) : \mathbb{F}(\Omega) \to \mathbb{F}(\Omega'), \Omega' \subset \mathbb{R}^d, \quad \Omega \subset \mathbb{R}^d$ is called *translation* operator corresponding to *translation* vector $\mathbf{t}$, if

$$\mathcal{T}(\mathbf{t})[\Phi(\mathbf{y})] = \Phi(\mathbf{y} + \mathbf{t}), \quad (\mathbf{y} \in \Omega, \quad \mathbf{y} + \mathbf{t} \in \Omega').$$



Example of Translation Operator



R|R-reexpansion

Let $\mathbf{y}-\mathbf{x}_* \in \Omega_r(\mathbf{x}_*) \subset \mathbb{R}^d$, $\Omega_r(\mathbf{x}_*) : |\mathbf{y}-\mathbf{x}_*| < r$, and $\{R_n(\mathbf{y}-\mathbf{x}_*)\}$ be a regular basis in $C(\Omega)$. Let $\mathbf{y}-\mathbf{x}_*+\mathbf{t} \in \Omega_r(\mathbf{x}_*)$ and

$$R_n(\mathbf{y}-\mathbf{x}_*+\mathbf{t}) = \sum_{l=0}^{\infty}(R|R)_{ln}(\mathbf{t})R_l(\mathbf{y}-\mathbf{x}_*).$$

Coefficients $(R|R)_{ln}(\mathbf{t})$ are called $R|R$ – *reexpansion coefficients* (regular-to-regular) , and infinite matrix

$$(\mathbf{R}|\mathbf{R})(\mathbf{t}) = \begin{pmatrix} (R|R)_{00} & (R|R)_{01} & \ldots \\ (R|R)_{10} & (R|R)_{11} & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

is called $R|R$ – *reexpansion matrix*.

Example of R|R-reexpansion

$$R_m(x) = x^m,$$

$$R_m(x+t) = (x+t)^m = x^m + \binom{m}{1} x^{m-1} t + \ldots + \binom{m}{m-1} x t^{m-1} + t^m$$

$$= \sum_{l=0}^{m} \binom{m}{l} t^l x^{m-l} = \sum_{l=0}^{m} \binom{m}{l} t^{m-l} x^l = \sum_{l=0}^{m} \binom{m}{l} t^{m-l} R_l(x),$$

$$(R|R)_{lm}(t) = \begin{cases} \binom{m}{l} t^{m-l}, & l \leqslant m, \\ 0, & l > m. \end{cases}$$

R|R-translation operator

Translation operator $\mathcal{T}(\mathbf{t})$ which is represented in regular basis $\{R_n(\mathbf{y} - \mathbf{x}_*)\}$ by the $R|R$ – *reexpansion matrix* is called $\mathcal{R}|\mathcal{R}$-translation operator.

$$\mathcal{T}(\mathbf{t})[\Phi(\mathbf{y})] = \Phi(\mathbf{y} + \mathbf{t})$$

$$(\mathcal{R}|\mathcal{R})(\mathbf{t}) = \mathcal{T}(\mathbf{t}).$$

Why the same operator named differently?

$$\mathcal{T}(\mathbf{t})[\Phi(\mathbf{y})] = \Phi(\mathbf{y}+\mathbf{t})$$

The first letter shows the basis for $\Phi(\mathbf{y})$

The second letter shows the basis for $\Phi(\mathbf{y}+\mathbf{t})$

$$\mathcal{T}(\mathbf{t}) = \begin{cases} (\mathcal{R}|\mathcal{R})(\mathbf{t}) \\ (\mathcal{S}|\mathcal{S})(\mathbf{t}) \\ (\mathcal{S}|\mathcal{R})(\mathbf{t}) \\ (\mathcal{R}|\mathcal{S})(\mathbf{t}) \end{cases}$$

Needed only to show the expansion basis (for operator representation)

Matrix representation of R|R-translation operator

Consider $\Phi(\mathbf{y}) = \sum_{n=0}^{\infty} A_n(\mathbf{x}_*) R_n(\mathbf{y}-\mathbf{x}_*)$.

$$\Phi(\mathbf{y}+\mathbf{t}) = (\mathcal{R}|\mathcal{R})(\mathbf{t})[\Phi(\mathbf{y})] = \sum_{n=0}^{\infty} A_n(\mathbf{x}_*)(\mathcal{R}|\mathcal{R})(\mathbf{t})[R_n(\mathbf{y}-\mathbf{x}_*)]$$

$$= \sum_{n=0}^{\infty} A_n(\mathbf{x}_*) R_n(\mathbf{y}-\mathbf{x}_*+\mathbf{t})$$

$$= \sum_{n=0}^{\infty} A_n(\mathbf{x}_*) \sum_{l=0}^{\infty} (R|R)_{ln}(\mathbf{t}) R_l(\mathbf{y}-\mathbf{x}_*)$$

$$= \sum_{l=0}^{\infty} \left[\sum_{n=0}^{\infty} (R|R)_{ln}(\mathbf{t}) A_n(\mathbf{x}_*) \right] R_l(\mathbf{y}-\mathbf{x}_*)$$

$$= \sum_{l=0}^{\infty} \widetilde{A}_l(\mathbf{x}_*,\mathbf{t}) R_l(\mathbf{y}-\mathbf{x}_*),$$

Coefficients of shifted function

Coefficients of original function

$$\widetilde{A}_l(\mathbf{x}_*,\mathbf{t}) = \sum_{n=0}^{\infty} (R|R)_{ln}(\mathbf{t}) A_n(\mathbf{x}_*), \quad \widetilde{\mathbf{A}}(\mathbf{x}_*,\mathbf{t}) = (\mathbf{R}|\mathbf{R})(\mathbf{t})\mathbf{A}(\mathbf{x}_*).$$

Reexpansion of the same function over shifted basis

Compact notation:

$$\Phi(\mathbf{y}) = \sum_{n=0}^{\infty} A_n(\mathbf{x}_*) R_n(\mathbf{y}-\mathbf{x}_*) = \mathbf{A}(\mathbf{x}_*) \circ \mathbf{R}(\mathbf{y}-\mathbf{x}_*),$$

$$\Phi(\mathbf{y}+\mathbf{t}) = \sum_{l=0}^{\infty} \widetilde{A}_l(\mathbf{x}_*,\mathbf{t}) R_l(\mathbf{y}-\mathbf{x}_*) = \widetilde{\mathbf{A}}(\mathbf{x}_*,\mathbf{t}) \circ \mathbf{R}(\mathbf{y}-\mathbf{x}_*)$$

We have:

$$\Phi(\mathbf{y}) = \Phi((\mathbf{y}-\mathbf{t})+\mathbf{t}) = \widetilde{\mathbf{A}}(\mathbf{x}_*,\mathbf{t}) \circ \mathbf{R}((\mathbf{y}-\mathbf{t})-\mathbf{x}_*)$$
$$= \widetilde{\mathbf{A}}(\mathbf{x}_*,\mathbf{t}) \circ \mathbf{R}(\mathbf{y}-\mathbf{x}_*-\mathbf{t}).$$

Also

$$\Phi(\mathbf{y}) = \mathbf{A}(\mathbf{x}_*) \circ \mathbf{R}(\mathbf{y}-\mathbf{x}_*) = \mathbf{A}(\mathbf{x}_*+\mathbf{t}) \circ \mathbf{R}(\mathbf{y}-\mathbf{x}_*-\mathbf{t}),$$

so

$$\mathbf{A}(\mathbf{x}_*+\mathbf{t}) = \widetilde{\mathbf{A}}(\mathbf{x}_*,\mathbf{t}) = (\mathbf{R}|\mathbf{R})(\mathbf{t})\mathbf{A}(\mathbf{x}_*).$$

Example

$$\Phi(y,x_i) = \frac{1}{y-x_i}.$$

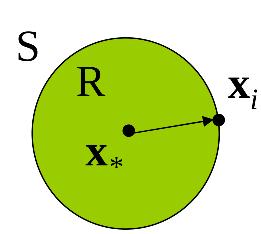
$|y-x_*| < |x_i-x_*|$:

R-expansion

$$\Phi(y,x_i) = \sum_{m=0}^{\infty} a_m(x_i,x_*) R_m(y-x_*),$$

$$a_m(x_i,x_*) = -(x_i-x_*)^{-m-1}, \quad m = 0,1,...,$$

$$R_m(y-x_*) = (y-x_*)^m, \quad m = 0,1,...$$



$|y-x_*| > |x_i-x_*|$:

S-expansion

$$\Phi(y,x_i) = \sum_{m=0}^{\infty} b_m(x_i,x_*) S_m(y-x_*),$$

$$b_m(x_i,x_*) = (x_i-x_*)^m, \quad m = 0,1,...,$$

$$S_m(y-x_*) = (y-x_*)^{-m-1}, \quad m = 0,1,...$$

R|R-operator

$$R_n(y+t) = (y+t)^n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} t^{n-m} y^m = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} t^{n-m} R_m(y).$$

$$(R|R)_{mn}(t) = \begin{cases} 0, & m > n \\ \frac{n!}{m!(n-m)!} t^{n-m}, & m \leqslant n \end{cases}.$$

$$(\mathbf{R}|\mathbf{R})(t) = (R|R)_{mn}(t) = \begin{pmatrix} 1 & t & t^2 & t^3 & \ldots \\ 0 & 1 & 2t & 3t^2 & \ldots \\ 0 & 0 & 1 & 3t & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

S|S-operator

$$S_n(y+t) = (y+t)^{-n-1} = y^{-n-1}\left(1 + \frac{t}{y}\right)^{-n-1} = \sum_{m=n}^{\infty} \frac{(-1)^{m-n} m!}{n!(m-n)!} t^{m-n} S_m(y),$$

$$(S|S)_{mn}(t) = \begin{cases} 0, & m < n \\ \frac{(-1)^{m-n} m!}{n!(m-n)!} t^{m-n}, & m \geqslant n. \end{cases}.$$

$$(\mathbf{S}|\mathbf{S})(t) = (S|S)_{mn}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ -t & 1 & 0 & 0 & \ldots \\ t^2 & -2t & 1 & 0 & \ldots \\ -t^3 & 3t^2 & -3t & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

S|R-operator

$$S_n(y+t) = (t+y)^{-n-1} = t^{-n-1}\left(1+\frac{y}{t}\right)^{-n-1} = \sum_{m=0}^{\infty} \frac{(-1)^m (m+n)!}{m!n!} t^{-n-m-1} y^m$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (m+n)!}{m!n!} t^{-n-m-1} R_m(y).$$

$$(S|R)_{mn}(t) = \frac{(-1)^m (m+n)!}{m!n!t^{n+m+1}},$$

$$(\mathbf{S}|\mathbf{R})(t) = \begin{pmatrix} t^{-1} & t^{-2} & t^{-3} & \ldots \\ -t^{-2} & -2t^{-3} & -3t^{-4} & \ldots \\ t^{-3} & 3t^{-4} & 6t^{-5} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

Renormalized R-functions

$$\widetilde{R}_n(y) = \frac{y^n}{n!}.$$

Then

$$\widetilde{R}_n(y+t) = \frac{1}{n!}(y+t)^n = \frac{1}{n!}\sum_{m=0}^{n} \frac{n!}{m!(n-m)!} t^{n-m} y^m = \sum_{m=0}^{n} \widetilde{R}_{n-m}(t)\widetilde{R}_m(y).$$

$$\left(\widetilde{R}|\widetilde{R}\right)_{mn}(t) = \begin{cases} 0, & m > n \\ \frac{1}{(n-m)!} t^{n-m} = \widetilde{R}_{n-m}(t), & m \leqslant n \end{cases}.$$

Translation Matrix:

$$\left(\widetilde{\mathbf{R}}|\widetilde{\mathbf{R}}\right)(t) = \left(\widetilde{R}|\widetilde{R}\right)_{mn}(t) = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} & \ldots \\ 0 & 1 & t & \frac{t^2}{2} & \ldots \\ 0 & 0 & 1 & t & \ldots \\ 0 & 0 & 0 & 1 & \ldots \end{pmatrix}.$$

Toeplitz

Renormalized S-functions

$$\widetilde{S}_n(y) = \frac{(-1)^n n!}{y^{n+1}}.$$

$$\widetilde{S}_n(y+t) = (-1)^n n!(y+t)^{-n-1} = (-1)^n n! \sum_{m=n}^{\infty} \frac{(-1)^{m-n} m!}{n!(m-n)!} t^{m-n} y^{-m-1} = \sum_{m=n}^{\infty} \widetilde{R}_{m-n}(t)\widetilde{S}_m(y).$$

$$\left(\widetilde{S}|\widetilde{S}\right)_{mn}(t) = \begin{cases} 0, & m < n \\ \frac{1}{(m-n)!} t^{m-n} = \widetilde{R}_{m-n}(t), & m \geq n \end{cases}.$$

Translation Matrix:

Toeplitz

$$\left(\widetilde{\mathbf{S}}|\widetilde{\mathbf{S}}\right)(t) = \left(\widetilde{S}|\widetilde{S}\right)_{mn}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ t & 1 & 0 & 0 & \ldots \\ \frac{t^2}{2} & t & 1 & 0 & \ldots \\ \frac{t^3}{6} & \frac{t^2}{2} & t & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} = \left(\widetilde{\mathbf{R}}|\widetilde{\mathbf{R}}\right)^T(t).$$

Renormalized S-functions

$$\widetilde{S}_n(y+t) = \sum_{m=n}^{\infty} \widetilde{R}_{m-n}(y)\widetilde{S}_m(t) = \sum_{m=0}^{\infty} \widetilde{S}_{m+n}(t)\widetilde{R}_m(y).$$

$$\left(\widetilde{S}|\widetilde{R}\right)_{mn}(t) = \widetilde{S}_{m+n}(t).$$

Hankel

Translation Matrix:

$$\left(\widetilde{\mathbf{S}}|\widetilde{\mathbf{R}}\right)(t) = \left(\widetilde{S}|\widetilde{R}\right)_{mn}(t) = \begin{pmatrix} t^{-1} & -t^{-2} & 2t^{-3} & -6t^{-4} & \ldots \\ -t^{-2} & 2t^{-3} & -6t^{-4} & 24t^{-5} & \ldots \\ 2t^{-3} & -6t^{-4} & 24t^{-5} & -120t^{-6} & \ldots \\ -6t^{-4} & 24t^{-5} & -120t^{-6} & 720t^{-7} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

Integral Representations and Diagonal Forms of Translation Operators

With such renormalized functions all translations can be performed with complexity O($p$log$p$).

But we look for something faster.

Theoretical limit for translation of vector of length p is $O(p)$.

ONLY SPARSE TRANSLATION MATRIX CAN PROVIDE SUCH COMPLEXITY

Representations Based on Signature Functions

Definition *Let*

$$\Phi(y) = \sum_{m=0}^{\infty} C_m \widetilde{R}_m(y),$$

then the Signature Function of $\Phi(y)$ *is a* 2π*-periodic funcion*

$$\Phi^*(s) = \sum_{m=0}^{\infty} C_m e^{ims}.$$

.

Definition *Let*

$$\Phi(y) = \sum_{m=0}^{\infty} C_m \widetilde{S}_m(y),$$

then the Signature Function of $\Phi(y)$ *is a* 2π*-periodic funcion*

$$\Phi^*(s) = \sum_{m=0}^{\infty} C_m e^{-ims}.$$

We assume that series for SF converge. This is always true for finite series, $C_m = 0$, $m > p-1$.

Integral Representation of Regular Functions

$$C_m = \frac{1}{2\pi}\int_0^{2\pi} \Phi^*(s) e^{-ims} ds.$$

← Property of Fourier coefficients

We have then the following representation of $\Phi(y)$:

$$\Phi(y) = \sum_{m=0}^{\infty} \widetilde{R}_m(y) \frac{1}{2\pi}\int_0^{2\pi} \Phi^*(s) e^{-ims} ds = \frac{1}{2\pi}\int_0^{2\pi} \Phi^*(s) \sum_{m=0}^{\infty} \widetilde{R}_m(y) e^{-ims} ds$$

Consider

$$\sum_{m=0}^{\infty} \widetilde{R}_m(y) e^{-ims} = \sum_{m=0}^{\infty} e^{-ims}\frac{y^m}{m!} = \sum_{m=0}^{\infty} \frac{\left(ye^{-is}\right)^m}{m!} = e^{ye^{-is}}.$$

So

$$\Phi(y) = \frac{1}{2\pi}\int_0^{2\pi} e^{ye^{-is}} \Phi^*(s) ds = \frac{1}{2\pi}\int_0^{2\pi} \Lambda_r(y;s) \Phi^*(s) ds,$$

where

$$\Lambda_r(y;s) = e^{ye^{-is}}.$$

← Regular kernel

Integral Representation of Regular Basis Functions

For $\Phi(y) = \widetilde{R}_m(y)$ we have

$$\Phi(y) = \widetilde{R}_m(y) = \sum_{m'=0}^{\infty} C_{m'}\widetilde{R}_{m'}(y), \qquad C_{m'} = \delta_{mm'}$$

Therefore the SF for this function is

$$\Phi^*(s) = \sum_{m'=0}^{\infty} C_{m'}e^{im's} = \sum_{m'=0}^{\infty} \delta_{mm'}e^{im's} = e^{ims}.$$

Then

$$\widetilde{R}_m(y) = \Phi(y) = \frac{1}{2\pi}\int_0^{2\pi} e^{ye^{-is}}\Phi^*(s)ds = \frac{1}{2\pi}\int_0^{2\pi} e^{ye^{-is}}e^{ims}ds = \frac{1}{2\pi}\int_0^{2\pi} \Lambda_r(y;s)e^{ims}ds.$$

Integral Representation of Singular Functions

$$\Phi^{(p)}(y) = \sum_{m=0}^{p-1} C_m\widetilde{S}_m(y), \qquad \Phi^{(p)*}(s) = \sum_{m=0}^{p-1} C_m e^{-ims}.$$

$$C_m = \frac{1}{2\pi}\int_0^{2\pi} \Phi^{(p)*}(s)e^{ims}ds.$$ ← Property of Fourier coefficients

We have then the following representation of $\Phi(y)$:

$$\Phi^{(p)}(y) = \sum_{m=0}^{p-1} \widetilde{S}_m(y)\frac{1}{2\pi}\int_0^{2\pi} \Phi^{(p)*}(s)e^{ims}ds = \frac{1}{2\pi}\int_0^{2\pi} \Phi^{(p)*}(s)\sum_{m=0}^{p-1}\widetilde{S}_m(y)e^{ims}ds$$

Then

$$\boxed{\Phi^{(p)}(y) = \frac{1}{2\pi}\int_0^{2\pi} \Lambda_s^{(p)}(y;s)\Phi^{(p)*}(s)ds,}$$

$$\Lambda_s^{(p)}(y;s) = \sum_{m=0}^{p-1}\widetilde{S}_m(y)e^{ims} = \sum_{m=0}^{p-1} e^{ims}\frac{(-1)^m m!}{y^{m+1}}.$$

Singular kernel

Integral Representation of Singular Basis Functions

For $\Phi(y) = \widetilde{S}_m(y)$ we have

$$\Phi^{(p)}(y) = \widetilde{S}_m(y) = \sum_{m'=0}^{p-1} C_{m'} \widetilde{S}_{m'}(y), \qquad C_{m'} = \delta_{mm'}, \qquad p > m.$$

Therefore the SF for this function is

$$\Phi^{(p)*}(s) = \sum_{m'=0}^{\infty} C_{m'} e^{-im's} = \sum_{m'=0}^{\infty} \delta_{mm'} e^{-im's} = e^{-ims}, \qquad p > m.$$

Then

$$\widetilde{S}_m(y) = \Phi^{(p)}(y) = \frac{1}{2\pi}\int_0^{2\pi} \Lambda_s^{(p)}(y,s)\Phi^{(p)*}(s)ds = \frac{1}{2\pi}\int_0^{2\pi} \Lambda_s^{(p)}(y,s)e^{-ims}ds,$$

$$m < p.$$

R|R-translation of the Signature Function

$$\mathcal{T}(t)[\Phi(y)] = \Phi(y+t) = \frac{1}{2\pi}\int_0^{2\pi} e^{(y+t)e^{-is}}\Phi^*(s)ds = \frac{1}{2\pi}\int_0^{2\pi} e^{ye^{-is}} e^{te^{-is}}\Phi^*(s)ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \Lambda_r(y,s)\Lambda_r(t,s)\Phi^*(s)ds = \frac{1}{2\pi}\int_0^{2\pi} \Lambda_r(y,s)\widehat{\Phi}^*(s,t)ds.$$

$$(\mathcal{R}|\mathcal{R})(t)[\Phi^*(s)] = \widehat{\Phi}^*(s,t) = \Lambda_r(t,s)\Phi^*(s).$$

So the R|R translation of the SF means simply multiplication of the SF by the regular kernel !

S|S-translation of the Signature Function

$$\Phi^{(p)}(y+t) = \sum_{m=0}^{p-1} \widehat{C}_m \widetilde{S}_m(y) = \sum_{m=0}^{p-1}\sum_{n=0}^{\infty} \left(\widetilde{S}|\widetilde{S}\right)_{mn}(t) C_n \widetilde{S}_m(y)$$

$$= \sum_{m=0}^{p-1}\sum_{n=0}^{p-1} \widetilde{R}_{m-n}(t) C_n \widetilde{S}_m(y) = \sum_{m=0}^{p-1}\sum_{n=0}^{p-1} \underbrace{\frac{1}{2\pi}\int_0^{2\pi} e^{te^{-is}} e^{i(m-n)s} ds}\, C_n \widetilde{S}_m(y)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{te^{-is}} \sum_{n=0}^{p-1} C_n e^{-ins} \sum_{m=0}^{p-1} \widetilde{S}_m(y) e^{ims} ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \Lambda_s^{(p)}(y,s) e^{te^{-is}} \Phi^{(p)*}(s) ds, \qquad |t| < |y|.$$

Representation of the regular basis function

So

$$(\mathcal{S}|\mathcal{S})(t)[\Phi^{(p)*}(s)] = \widehat{\Phi}^{(p)*}(s;t) = e^{te^{-is}}\Phi^{(p)*}(s) = \Lambda_r(t,s)\Phi^{(p)*}(s).$$

So the S|S translation of the SF means multiplication of the SF by the regular kernel.

S|R-translation of the Signature Function

In case $|t| > |y|$ we have

$$\Phi^{(p)}(y+t) = \frac{1}{2\pi}\int_0^{2\pi} e^{ye^{-is}} \Lambda_s^{(p)}(t,s)\Phi^{(p)*}(s)ds, \qquad |t| > |y|.$$

This is a representation of the regular function. Therefore,

$$(\mathcal{S}|\mathcal{R})(t)[\Phi^{(p)*}(s)] = \widehat{\Phi}^{(p)*}(s;t) = \Lambda_s^{(p)}(t,s)\Phi^{(p)*}(s).$$

So the S|S translation of the SF means multiplication of the SF by the singular kernel.

Evaluation of Function
based on its Signature Function

Use Gaussian Type Quadrature

function bandwidth

$$\Phi^{(p)}(y) = \frac{1}{2\pi}\int_0^{2\pi} \Lambda_r(y;s)\Phi^{(p)*}(s)ds$$

$$= \sum_{k=0}^{q-1} w_k \Lambda_r(y;s_k)\Phi^{(p)*}(s_k) + error(p,q).$$

quadrature weights

quadrature nodes

quadrature order