GAUSSIAN BEAMS

I. Initial Value Problems

The idea underlying Gaussian beams is simply to build asymptotic solutions concentrated on a single curve. This means that, given a curve γ, parametrized by $x = x(s)$, one makes the **Ansatz**

$$u(x,k) = e^{ik\phi(x)}(a_0(x) + \frac{1}{k}a_1(x) + \cdots + \frac{1}{k^N}a_N(x)) = e^{ik\phi(x)}a(x,k), \qquad (1)$$

where $\phi(x(s))$ is real, and $\mathrm{Im}\{\phi(x)\} > 0$ for $x \neq x(s)$. To see what one can do with this Ansatz suppose that $P(x,D)$ is a differential operator of order m with a real principal symbol $p(x,\xi)$, and we wish to build asymptotic solutions to $P(x,D)v = 0$, i.e. we want $P(x,D)u = O(k^{-M})$. Substituting from (1),

$$P(x,D)u = k^m p(x,\phi_x(x))u + O(k^{m-1})$$

Thus we want $p(x,\phi_x(x)) = 0$. However, it will suffice to have $p(x,\phi_x(x))$ vanish to high order on γ. We are going to choose ϕ so that $\mathrm{Im}\{\phi\} \geq cd(x,\gamma)^2$. Note that

$$d(x,\gamma)^r e^{-ckd(x,\gamma)^2} = O(k^{-r/2}).$$

So we need to have $p(x,\phi_x(x)) = O(d(x,\gamma)^{2M})$. This leads to a sequence of equations for the derivatives of ϕ on γ. Using the summation convention, the first three sets of equations, corresponding to vanishing of orders zero, one and two are as follows

$$p = 0 \qquad (2)$$

$$p_{x_i} + p_{\xi_j}\phi_{x_j x_i} = 0 \text{ for } 1 \leq i \leq n \qquad (3)$$

$$p_{x_i x_k} + p_{x_i \xi_j}\phi_{x_j x_k} + p_{x_k \xi_j}\phi_{x_j x_i} + p_{\xi_j \xi_l}\phi_{x_j x_i}\phi_{x_l x_k} + p_{\xi_j}\phi_{x_j x_i x_k} = 0 \text{ for } 1 \leq i,k \leq n \qquad (4)$$

Setting $\phi_x(x(s)) = \xi(s)$, we can rephrase the requirement that (2)-(4) hold on γ as (2)-(4) hold on the curve in (x,ξ)-space given by $(x,\xi) = (x(s),\xi(s))$.

We can also differentiate with respect to the curve parameter s. Letting $\dot{f}$ denote differentiation with respect to s, I am going to assume that

$$\dot{x}(s) = p_\xi(x(s),\xi(s)). \qquad (5)$$

Differentiating $\phi_x(x(s)) = \xi(s)$ with respect to s gives

$$\phi_{x_i x_j}\dot{x}_j = \dot{\xi}_i \qquad (6)$$

and, combining (3),(5) and (6) shows that $(x(s),\xi(s))$ is a solution of

$$\dot{x} = p_\xi(x,\xi) \qquad \dot{\xi} = -p_x(x,\xi), \qquad (7)$$

i.e $(x(s),\xi(s))$ is a bicharacteristic for $P(x,D)$, and (2) says that it is a null bicharacteristic. General results on the behavior of solutions of hyperbolic equations with

highly oscillatory initial data would show that (7) is necessary for this construction, but I will not discuss that here. The **only** additional assumption required for the construction of asymptotic solutions to $P(x,D)v = 0$ using the Ansatz (1) is that $\dot{x}(s)$ never vanishes. Well, almost: if $x(s)$ returns to the same point you will have to use a sum of functions of the form (1) near that point.

The heart of the matter is equation (4). To study it I will introduce the matrices

$$M(s) = \left(\phi_{x_i x_j}(x(s))\right)\,,\ A(s) = \left(p_{x_i x_j}(x(s),\xi(s))\right)\,,\ B(s) = \left(p_{x_i \xi_j}(x(s),\xi(s))\right)$$

$$\text{and } C(s) = \left(p_{\xi_i \xi_j}(x(s),\xi(s))\right)\,. \tag{8}$$

Then (4) becomes

$$A + BM + MB^t + MCM + \dot{M} = 0 \tag{9}$$

which is a matrix Riccati equation, and can be solved in the following well-known (in ODE circles) method: choose matrix solutions to the linear system

$$\dot{Y} = CN + B^t Y \qquad\qquad \dot{N} = -BN - AY \tag{10}$$

such that Y is invertible. Then NY^{-1} will be a solution of (9) (just substitute it into (9) and simplify!). Of course, when you do this everything blows up when you reach a point where Y is not invertible. The good property of Gaussian beams is that you can choose the initial data $(Y(0), N(0))$ so that $Y(s)$ is invertible for all s. To see how to do that we need to use the following property of solutions of (9): writing $\psi_1 = (y^1(s), \eta^1(s))$ and $\psi_2 = (y^2(s), \eta^2(s))$ for a pair of vector solutions of (10), the bilinear form

$$\sigma(\psi_1, \psi_2) = y^2 \cdot \eta^1 - y^1 \cdot \eta^2 \tag{11}$$

is **constant** in s. Since $\overline{\psi}_2$, where $\bar{\cdot}$ denotes complex conjugate, is also a solution of (10)

$$\sigma(\psi_1, \overline{\psi}_2) = \overline{y}^2 \cdot \eta^1 - y^1 \cdot \overline{\eta}^2, \tag{12}$$

is **also** constant in s. You can check (11) and (12) by differentiating them with respect to s, substituting from (9), and noting that the entries in A, B and C are real and A and C are symmetric. If you introduce the symplectic structure on (x,ξ)-space, you can make all this seem less magical, but you will also make the exposition more long-winded. So I will not do that here, but it is helpful to observe that (10) is just the system satisfied by variations of solutions of (7).

Now we are ready to start the construction. I will assume that we have selected the curve γ – and hence the bicharacteristic $(x(s), \xi(s))$ – that we wish to follow. That determines $\phi_x(x(s))$, since it is equal to $\xi(s)$. We choose $\left(\phi_{x_i x_j}(x(s))\right) = M(s)$ as follows: we must have $M(0) = M(0)^t$ and (6) implies $M(0)\dot{x}(0) = \dot{\xi}(0)$. Thus $\mathrm{Im}\{M(0)\}\dot{x}(0) = 0$, and we choose $\mathrm{Im}\{M(0)\}$ so that it is positive definite on the orthogonal complement of $\dot{x}(0)$. Note that we are already using $\dot{x}(s) \neq 0$ here. Next we solve (10) with the initial data $(Y(0), N(0)) = (I, M(0))$. This gives $(Y(s), N(s))$, defined for all s since it is the solution of a system of linear differential equations with (presumably) bounded coefficients. Differentiating (7) with respect to s shows that $(\dot{x}, \dot{\xi})$ is a vector solution of (10), and then uniqueness for the initial value problem implies $(\dot{x}(s), \dot{\xi}(s)) = (Y(s)\dot{x}(0), N(s)\dot{x}(0))$. Next consider the vector

solution of (10) given by $(y(s),\eta(s)) = (Y(s)c, N(s)c)$ for a general vector $c \in \mathbb{C}^n$. If $Y(s_0)c = 0$ for some s_0, then the constancy of (12) with $\psi_1 = \psi_2 = (y(s),\eta(s))$ gives

$$0 = \sigma(\psi(s_0),\overline{\psi}(s_0)) = \sigma(\psi(0),\overline{\psi}(0)) = \overline{y}(0)\cdot\eta(0) - y(0)\cdot\overline{\eta}(0) = 2i\overline{c}\cdot \mathrm{Im}\{M(0)\}c.$$

Thus, by the construction of $M(0)$, $c = \alpha\dot{x}(0)$, and $Y(s)c = \alpha\dot{x}(s)$. Thus, taking $s = s_0$, since $\dot{x}(s) \neq 0$ by assumption, we see that $\alpha = 0$. Thus $c = 0$ and $Y(s_0)$ is invertible. s. For all s the matrix $M(s)$ has the properties of $M(0)$:

$$i)\ M(s)\dot{x}(s) = \dot{\xi}(s) \text{ i.e. it satisfies (6)}$$

$$ii) M(s)^t = M(s) - \text{this follows from the constancy of (11), and}$$

$iii)$ $\mathrm{Im}\{M(s)\}$ is positive definite on the orthogonal complement of $\dot{x}(s)$ – this follows from the constancy of (12).

This completes the crucial part of the construction of the phase ϕ in (1). Of course, one may need to have $p(x,\phi_x)$ vanish to a higher order than two on γ, but the equations corresponding to (2)-(4) for vanishing of order $r \geq 3$ have the form

$$\sum_{j=1}^{n} \partial_{\xi_j}(p)\partial_{x_j}(\partial_x^\alpha\phi) + \sum_{|\beta|=r} c_{\alpha\beta}\partial_x^\beta\phi + d_\alpha = 0$$

for all multi-indices α of length r. Since $\partial_\xi p\cdot\partial_x(\partial_x^\alpha\phi) = (d/ds)\partial_x^\alpha\phi$ on γ, we can solve these equations as linear systems of ODEs in s for the r-th order partial derivatives of ϕ on γ. One does this recursively, since the coefficients $c_{\alpha\beta}$ and d_α depend on all the partials up to order $r-1$, but, since the equations are linear, the solutions exist for all s. One must choose the partial derivatives of order r at $x(0)$ to be compatible with those of lower order, but then they remain compatible along γ. Thus we may assume that $p(x,\phi_x)$ vanishes on γ to any prescribed order.

The complete expansion of $P(x,D)u$ is

$$P(x,D)u = k^m p(x,\phi_x(x))u + e^{ik\phi(x)}\sum_{j=1}^{m} k^{m-j}L_j(x,D)a, \tag{13}$$

where the L_j's are linear differential operators with coefficients depending on ϕ. Recall that

$$a = a_0 + \frac{1}{k}a_1 + \cdots + \frac{1}{k^N}a_N.$$

Thus to make $P(x,D)u = O(k^{-M})$, we now only need to make the terms in (13) which are multiplied by k^{m-j}. $j = 1,..,m+N$ vanish on γ to sufficiently high order. Just as in geometric optics, the terms in (13) multiplied by k^{m-l-1} have the form $L_1a_l + g_l$ where $g_0 = 0$ and g_l only depends on $a_0,..,a_{l-1}$ for $l \geq 1$. Thus we can solve the equations $L_1a_l + g_l = 0$ recursively starting with $l = 0$. Using q for the symbol of the terms of order $m-1$ in $P(x,D)$, we have for $l = 0,1,..,N$

$$0 = L_1a_l + g_l$$

$$= \frac{1}{i}\sum_{j=0}^{n} \partial_{\xi_j} p(x,\phi_x)\partial_{x_j} a_l + \frac{1}{2i}\sum_{0\le j,k\le n} \partial^2_{\xi_j\xi_k} p(x,\phi_x)\partial^2_{x_j x_k}\phi a_l + q(x,\phi_x)a_l + g_l. \quad (14)$$

Since the right hand side of the second equality in (14) is linear in P and involves no x derivatives of the symbols, it suffices to verify it for P of the form $P = D^\alpha$, and this can be done easily by induction on $|\alpha|$. Introducing $v = \partial_\xi p(x,\phi_x)$ and $\mathrm{sub}(P) = q - (2i)^{-1}\partial_{x_j}\partial_{\xi_j} p$, we can rewrite (14) as

$$0 = v_\cdot \cdot \nabla a_l + \frac{1}{2}(\nabla \cdot v)a_l + i\,\mathrm{sub}(P)(x,\phi_x)a_l + i g_l \quad (15)$$

Formula (15) has useful consequences in geometric optics, but here we will just observe that, since $v(x(s)) = \dot{x}$, we can solve it to arbitrarily high order on γ, by solving the linear ODE systems for the partial derivatives of a_l on γ that one gets by differentiating (15) – just as we did for the derivatives of ϕ of order greater than two.

This completes the construction of our asymptotic solutions. To apply this to initial value problems we need to take P to be a hyperbolic differential operator, for example $\partial_{tt} - \Delta$. In the coordinates $(x_0, x_1, x_2) = (t, x, y)$ with γ given by $(t(s), x(s), y(s)) = (s, 0, s)$, for any positive constants a and b the phase

$$\phi(t,x,y) = \frac{y-t}{2} + \frac{a^2x^2t}{1+4a^2t^2} + i\left(\left(\frac{a}{1+4a^2t^2}\right)\frac{x^2}{2} + b\frac{(y-t)^2}{2}\right),$$

and the amplitude

$$a_0 = (1+2iat)^{-1/2}$$

give $u = e^{ik\phi}a_0$ satisfying $(\partial_{tt} - \Delta)u = O(k^{1/2})$ in k. This is the simplest possible example, but it already has one interesting feature: note that the amplitude is complex-valued, and this corresponds to a constant retardation of the phase as t goes from $-\infty$ to $+\infty$ with the total phase shift being $\pi/2$. In ordinary geometric optics there are sharp jumps of multiples of $\pi/2$ of the phase across caustics.

A simple extension of the preceding is to replace $P(x,D)$ by a "semi-classical" operator

$$P(x,D;h) = P_0(x,hD) + hP(x,hD) + h^2P(x,hD) + \cdots$$

Then one can construct asymptotic solutions to $P(x,hD)u = 0$ as $h \to 0$ by taking $k = 1/h$ in the construction above. This can used, for example, for the Schrödinger equation $ihu_t + h^2\Delta u - V(x)u = 0$ and the Helmholtz equation $c^2(x)\Delta u + k^2u = 0$.

If one wants to build asymptotic solutions to $(\partial_{tt} - \Delta)u = 0$ with caustics, one approach is to take superpositions of Gaussian beams. For instance to construct a time-harmonic asymptotic solutions, $u = (exp(ikt)w(x)$, one can construct beams $w(x)$ for the equation $(\Delta + k^2)w = 0$ concentrated on the rays that produce the caustic. For example, suppose that the caustic is a plane curve C, and the tangents to a portion of C are the lines $y = m(s)x + b(s)$. Let $w^s(x,y;k)$ be the beam concentrated at $(0,b(s))$ when $x = 0$ propagating in the direction $(1,m(s))$ denote the analogous beam concentrated at p at time $t = 0$ propagating in the The constructions are uniform in s so superpositions of the beams w^s are valid asymptotic

solutions. Nicolay Tanushev has implemented this construction numerically for fold and cusp caustics. For satisfactory results one needs to balance the asymptotic parameter k and the spacing in the superposition parameter s carefully.

II. Quasimodes

When γ is a stable periodic orbit for the bicharacteristic flow, one can use Gaussian beams to build sequences of functions u_l such that $P(x,D)u_l - \lambda_l u_l = O(\lambda_l^{-M})$. The most common terminology is to refer to the sequence $\{u_l\}$ as a "quasimode". This was the first use of Gaussian beams (dating back to the late sixties – see Bibliography). In this setting we will assume that $p(x,\phi_x) = 1$ on γ, and build beams which satisfy

$$P(x,D)u = (k^m + c_1 k^{m-1} + \dots)u.$$

As in §I, we will use the Ansatz

$$u(x,k) = e^{ik\phi(x)}(a_0(x) + \frac{1}{k}a_1(x) + \cdots + \frac{1}{k^N}a_N(x)) = e^{ik\phi(x)}a, \tag{1}$$

and assume that $\xi(s) = \phi_x(x(s))$. Likewise we will require

$$\dot{x} = p_\xi(x,\xi) \qquad \dot{\xi} = -p_x(x,\xi). \tag{2}$$

Since γ is periodic, the system (I.10)

$$\dot{y} = C\eta + B^t y \qquad \dot{\eta} = -B\eta - Ay \tag{3}$$

has periodic coefficients. Assuming that γ has period S, i.e. S is the smallest positive number such that $x(s+S) = x(s)$, it is natural to consider the Floquet map on vector solutions of (3). This is given by

$$\Phi : (y(0),\eta(0)) \to (y(S),\eta(S)).$$

This map always has 1 as an eigenvalue of algebraic multiplicity at least two:

$$\Phi : \psi_0 = (\dot{x}(0),\dot{\xi}(0)) \to (\dot{x}(S),\dot{\xi}(S)) = \psi_0 \tag{4}$$

$$\Phi : \psi_{00} = (0,\xi(0)) \to (0,\xi(0)) + (m-1)S(\dot{x}(0),\dot{\xi}(0)) = \psi_{00} + (m-1)S\psi_0. \tag{5}$$

The second statement is a consequence of the positive homogeneity (of degree m) of $p(x,\xi)$ which implies that $(x(a^{m-1}s), a\xi(a^{m-1}s))$ is a solution of (2) for $a > 0$. Evaluating the derivative of this solution with respect to a at $a = 1$ gives (5). Note that the vectors in (4) and (5) are independent since $\xi(0)\cdot\dot{x}(0) = mp(x(0),\xi(0)) = m \neq 0$.

The linear mapping Φ preserves the bilinear form σ from (I.11) and it maps real vectors to real vectors – in the language of linear groups Φ is real and symplectic. One can show easily that if λ is an eigenvalue of Φ then both $\overline{\lambda}$ and $1/\lambda$ are also eigenvalues. Thus γ is linearly stable only if all the eigenvalues of Φ lie on the unit circle in the complex plane. I am going to make the strong stability hypothesis that the eigenvalues are $\{1,1,\lambda_1,\overline{\lambda}_1,\dots,\lambda_{n-1},\overline{\lambda}_{n-1}\}$, and that these eigenvalues are

simple – except for 1 which is double. This makes the structure of Φ very rigid. If $\Phi\psi_j = \lambda_j\psi_j$, then $\Phi\overline{\psi}_j = \overline{\lambda}_j\overline{\psi}_j$. We also have

$$\sigma(\psi_j, \psi_k) = 0,\ 1 \le j,k \le n-1,\ \text{since } \lambda_j\lambda_k \ne 1 \tag{6}$$

$$\sigma(\psi_j, \overline{\psi}_k) = 0 \text{ for } j \ne k,\ 1 \le j,k \le n-1,\ \text{since } \lambda_j\overline{\lambda}_k \ne 1$$

$$\sigma(\psi_j, \psi_0) = \sigma(\psi_j, \psi_{00}) = 0,\ 1 \le n-1,\ \text{since } \lambda_j \ne 1 \tag{7}$$

Since σ is nondegenerate, this means that we can choose the ψ_j's so that

$$\sigma(\psi_j, \overline{\psi}_k) = i\delta_{jk},\ 1 \le j,k \le n-1. \tag{8}$$

Now that we have a good basis of eigenfunctions for Φ we can resume the construction. We require $M(0)\dot{x}(0) = \dot{\xi}(0)$ as before, but the rest of $M(0)$ is determined by the following conditions. Writing $\psi_j = (y^j, \eta^j)$, we require

$$M(0)y^j = \eta^j,\ j = 1, ..., n-1. \tag{9}$$

Note that (8) implies that $\{y^1, .., y^{n-1}\}$ is linearly independent: if $c_1y^1 + \cdots + c_{n-1}y^{n-1} = 0$, then $\psi = c_1\psi_1 + \cdots + c_{n=1}\psi_{n-1}$ satisfies

$$0 = \sigma(\psi, \overline{\psi}) = i(|c_1|^2 + \cdots + |c_{n-1}|^2).$$

Likewise, $\{\dot{x}(0), y^1, ...y^{n-1}\}$ is linearly independent: if $\dot{x}(0) = c_1y^1 + \cdots + c_{n-1}y^{n-1}$ and ψ is as before, then $\sigma(\psi_{00}, \psi_0) = \sigma(\psi_{00}, \psi) = 0$ by (7), contradicting $\sigma(\psi_{00}, \psi_0) = m$. Thus (9) determines $M(0)$. We only need to check that $\mathrm{Im}\{M(0)\}$ is positive definite on the orthogonal complement of $\dot{x}(0)$ – which follows from (8) – and that $M(0)$ is symmetric – which follows from (6) and (7). Thus we have $M(s)$ for $0 \le s < S$ with the same properties as in §1. However, since the ψ_j's are eigenvectors for Φ, we also have $M(S) = M(0)$ which makes $\left(\phi_{x_jx_k}(x(s))\right) = M(s)$ well-defined on γ.

The equations for the higher order derivatives of ϕ on γ are inhomogeneous linear, and one has to appeal to the Fredholm alternative to solve them. One can show that there are no solutions to the corresponding homogeneous problem when

$$\lambda_1^{\alpha_1} \cdots \lambda_{n-1}^{\alpha_{n-1}} \ne 1 \tag{10}$$

for all multi-indices α. Writing $\lambda_j = \exp(i\theta_j),\ 0 < \theta_j < 2\pi$, (10) just says that the θ_j's and π are rationally independent. Assuming (10), we can make $p(x, \phi_x)$ vanish to any given order on γ.

The construction of the amplitudes is quite interesting, but I will not try to present it here (see the book of Babich and Buldreyev or Ralston (1976)). It turns out that when sub(P) is real (which happens when P is self-adjoint) for each multi-index α one can find an amplitude a such that $a(x(S)) = \exp(-i\beta(\alpha))a(x(0))$. In the case that sub(P)=0

$$\beta(\alpha) = (\alpha_1 + \frac{1}{2})\theta_1 + \cdots + (\alpha_{n-1} + \frac{1}{2})\theta_{n-1} \mathrm{mod}\ \pi. \tag{11}$$

Introducing the "action" around γ, i.e.

$$\phi(x(S)) - \phi(x(0)) = \int_0^S \dot{x}(s) \cdot \phi_x(x(s))ds = \int_0^S \dot{x}(s) \cdot \xi(s)ds = mS,$$

the condition that determines the sequence of approximate eigenvalues λ_l is just

$$kmS - \beta = 2\pi l \qquad \text{or} \qquad k_l = \frac{2\pi l + \beta(\alpha)}{mS} \tag{12}$$

To leading order $\lambda_l = k_l^m$. Formula (12) says that for l large quite a few eigenvalues of P are associated with γ – assuming that the coefficients of P grow at infinity so that the spectrum of P consists only of eigenvalues. Actually more is true: a positive fraction of *all* the large eigenvalues of P are associated with γ – see Popov (2000).

A Gaussian Beam Bibliography

V.M. Babich and V.S. Buldreyev, Asymptotic Methods in Short Wave Diffraction Problems, Vol. 1 (Russian) Moscow, Nauka 1972

J. A. Arnaud, Beam and Fiber Optics, Academic Press 1976

These are the only book length treatments of Gaussian beams that I know of. The book of Babich and Buldreyev has been translated, and it contains work that Babich and his colleagues published in the Proceedings of the Steklov Institute (Leningrad) as early as 1968.

L. Hörmander, On the existence and regularity of solutions of linear pseudo-differential equations, Enseignement Math. **17**(1971), 99-163.

This is where I first learned the ideas in §1. It may not be immediately apparent that they are here.

J. Ralston, On the construction of of quasi-modes associated with stable periodic orbits, Commun. Math. Phys. **51**(1976), 219-242, Erratum, Commun. Math. Phys. **67**(1979),91.

J. Ralston , Approximate eigenfunctions of the Laplacian, J. Diff. Geometry **12**(1977),87-100, A Correction, J.Diff.Geometry **14**(1979),487.

§2 is based on these papers. However, a good bit of what they contain was already in the work of Babich et al.

J. Ralston, Gaussian beams and the propagation of singularities, Studies in Partial Differential Equations, MAA Studies in Mathematics, **23**, Walter Littman, ed. 1983.

§1 is based on this. It contains some details on the construction. A large part of it is devoted to deriving propagation of singularities results using Gaussian beams instead of pseudo-differential operators.

G. Hagedorn, Semi-classical Quantum Mechanics, I, Commun. Math. Phys. **71** (1980), 77-93, II, Ann. Inst. Henri Poincare **42**(1985), 363-374.

L Klimeš, Expansion of a high-frequency time-harmonic wavefield given on an initial surface into Gaussian beams, Geophys. J. R. astr. Soc.**79**(1984), 105-118.

V.Červený and L. Klimeš, Synthetic body wave seismograms for three dimensional laterally verying media, Geophys. J. R. astr. Soc. **79**(1984), 119-133.

L. Klimeš, The relation between Gaussian beams and Maslov asymptotic theory, Studia geoph. et geod. **28**(1984), 237-247.

L. Klimeš, Gaussian packets in the computation of seismic wavefields, Geophys. J. Int. **99**(1989), 421-433.

M. Combescure and D. Robert, Semiclassical spreading of quantum wave packets and applications near unstable fixed points of the classical flow, Asymptotic Anal. **14**(1997), 377-404.

M. Combescure, J. Ralston and D. Robert, A proof of the Gutzwiller semiclassical trace formula using coherent states decomposition, Commun. Math. Phys **202**(1999),463- 480.

G. Popov, Invariant tori, effective stability and quasimodes with exponentially small error terms, I,II, Ann. Henri Poincaré **1**(2000), 223-248, 249-279.

This last paper does not use Gaussian beams. It is based on KAM theory. However, it does give what may be the definitive result on quasimodes associated with periodic orbits.