On the shrinking obstacle limit in a viscous incompressible flow

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Setting:

Exterior domain $\Omega_\varepsilon = \varepsilon\Omega$, simply-connected obstacle.

Incompressible Navier-Stokes equations:

$$\partial_t u - \nu \triangle u + u \cdot \nabla u = -\nabla p, \quad \operatorname{div} u = 0$$

with Dirichlet boundary conditions.

Limit flow as $\varepsilon \longrightarrow 0$?

Initial data (motivated by inviscid study):

$$u_0 = K_\varepsilon[\omega_0] + \alpha H_\varepsilon$$

where

- initial vorticity ω_0 is fixed (independent of ε), smooth and compactly supported outside 0;
- initial circulation γ of u_0 along $\partial\Omega$ is independent of ε.

K_ε is the $\nabla_x^\perp$ of the Green function and H_ε is a canonical harmonic vector field.

Inviscid case

Circulation $\gamma = \alpha - m, \quad m = \int \omega_0$.

The limit vorticity is

$$\operatorname{curl} u = \omega + \gamma \delta_0$$

with limit equation in vorticity formulation

$$\partial_t \omega + \operatorname{div}\big[\big(v + \gamma H\big)\omega\big] = 0$$

$$v = K[\omega], \qquad H = \frac{x^\perp}{2\pi |x|^2}.$$

Here, K is the usual kernel of the Biot-Savart law in $\mathbb{R}^2$.

The equation of the limit velocity is roughly the Euler equation with an additional term which takes into account the circulation and a (fixed) Dirac mass in 0.

Viscous case

Convergence to the Navier-Stokes equations in the case of small circulation:

Theorem. There exists $\gamma_0 > 0$ independent of ε such that if $|\gamma| \leq \gamma_0$ then u_ε converges to the solution of the incompressible Navier-Stokes equations in $\mathbb{R}^2$ with initial vorticity $\omega_0 + \gamma\delta_0$.

The initial data makes sense. The circulation vanishes instantly.

The limit vorticity at time $t = 0$ has a Dirac mass in 0.

In $\mathbb{R}^2$ the global existence holds (Kato, Cottet, Giga-Miyakawa-Osada) but uniqueness was proved only very recently (Gallagher-Gallay).

The existence in the full plane case uses L^1 estimates on the vorticity; these are unavailable for domains with boundaries.

L^2 a priori estimates?

From the inviscid work we know that the behavior of the initial velocity can be described as follows :

- for $|x| < 1$: $\quad \gamma H$
- for $|x| > M$: $\quad \alpha H$
- plus a remainder bounded in all L^p, $1 < p < \infty$.

Two problems occur:

- initial velocity not square-integrable at ∞;
- initial velocity not square-integrable in 0.

The problem at infinity subsists for $t > 0$ but can be solved because it is independent of ε.

The problem in 0 disappears for $t > 0$, but local estimates are required. These are done with a fixed point argument and demand smallness of circulation.

Once the local estimates done, global L^2 estimates are not difficult.

Local estimates

Weighted in time norms:

$$\|f\|_{p,T} = \sup_{t\in[0,T]} t^{\frac{1}{2}-\frac{1}{p}}\|f(t)\|_{L^p}.$$

$$\widetilde{u} = S(t)(u_0)$$
$$w = u - \widetilde{u}$$

verifies

$$w(t) = \int_0^t S(t-\tau)\mathbb{P}\,\mathrm{div}\big(w\otimes w + w\otimes\widetilde{u} + \widetilde{u}\otimes w + \widetilde{u}\otimes\widetilde{u}\big)(\tau)d\tau.$$

so (via Maremonti-Solonnikov, Dan-Shibata and the change of functions $f_\varepsilon(t,x) \leftrightarrow f(\varepsilon^2 t, \varepsilon x)$)

$$\|w\|_{p,t} \le C\big(\|w\|_{q_1,t}\|w\|_{q_2,t} + \|w\|_{q_1,t}\|\widetilde{u}\|_{q_2,t} + \|\widetilde{u}\|_{q_1,t}\|\widetilde{u}\|_{q_1,t}\big)$$

where

$$\frac{1}{q_1} + \frac{1}{q_2} < \frac{1}{2} + \frac{1}{p}.$$

We need to have that $\|\widetilde{u}\|_{p,t}$ is small. This requires the smallness of the circulation and demands to show that $S(t)H_\varepsilon$ belongs to the weighted in time spaces.

We assume that $\varepsilon = 1$, set $T : \Omega \to B(0,1)^c$ a biholomorphism, $S = T^{-1}$ and prove that $\widetilde{u} = Stokes\big[h(|T|)H_\Omega\big]$ belongs to the weighted in time space on $\mathbb{R}_+$.

Obvious in the circular-symmetric case by the maximum principle. In the general case we reduce the problem to that case by a change of variables:

$$\widetilde{u} = (\nabla T)^t v \circ T$$

$$\partial_t v + \nu \nabla^\perp \Big(\frac{1}{|S'|^2} \operatorname{curl} v \Big) = -\nabla q$$

$$\operatorname{div} v = 0, \quad v(0,y) = \frac{y^\perp}{2\pi |y|^2} h(|y|).$$

Next, $\overline{w} = v - \overline{v}$ ($\overline{v}$ = leading term) verifies

$$\partial_t \overline{w} + \nu \nabla^\perp \Big(\frac{1}{|S'|^2} \operatorname{curl} \overline{w} \Big) = -\nabla q_2 - \nu \nabla^\perp \Big[\operatorname{curl} \overline{v} \big(\alpha - \frac{1}{|S'|^2} \big) \Big]$$

By duality

$$\int \overline{w}(t,x) \cdot \varphi_0(x)\, dx \le C \int_0^t \|\operatorname{curl} \overline{v}(\tau)\|_{L^q} \|\operatorname{curl} \varphi(t-\tau)\|_{L^r}\, d\tau \le C \|\varphi_0\|_{L^{p'}} t^{\frac{1}{p} - \frac{1}{2}}.$$

Global estimates

Energy estimates on w:

$$\partial_t \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \leq \frac{C}{t}\|w\|_{L^2}^2 + \frac{C}{t}.$$

$$\frac{\|w(t_2)\|_{L^2}^2}{t_2^C} + \int_{t_1}^{t_2} \frac{\|\nabla w(s)\|_{L^2}^2}{s^C}\,ds \leq \frac{1}{t_1^C} - \frac{1}{t_2^C} + \frac{\|w(t_1)\|_{L^2}^2}{t_1^C}.$$

We multiply by t_1^{a+C-1} ($a > 0$) and integrate w.r.t. t_1:

$$\int_0^{t_2} s^a \|\nabla w(s)\|_{L^2}^2 \leq \frac{C}{a} t_2^a - \|w(t_2)\|_{L^2}^2 t_2^{a+C} + (a+C)\int_0^{t_2} \|w(s)\|_{L^2}^2 s^{a-1}\,ds.$$

Therefore, w is bounded in

$$L^\infty_{loc}(\mathbb{R}_+; L^2) \cap L^p_{loc}(\mathbb{R}_+; H^1)$$

for all $p \in [1,2)$.

Similar estimates easily hold for $S(t)u_0 = u - w$.

Strong convergence

We need some equicontinuity in time. We extend everything with 0 inside the obstacle. To avoid estimating the pressure, we use the vorticity equation.

$\varphi \in C_0^\infty(\mathbb{R}^2)$ div free test vector field

ψ such that $\nabla^\perp \psi = \varphi$ and $\psi(0) = 0$.

Smooth cut-off functions:

$g_\lambda = g(\cdot/\lambda)$ localizes in $|x| > \lambda$

$h_\lambda = h(\cdot/\lambda)$ localizes in $|x| < \lambda$.

Multiply the vorticity equation by $g_\varepsilon \psi h_R$:

$$\int \big[u(t_2) - u(t_1)\big] \nabla^\perp (g_\varepsilon \psi h_R) = \underbrace{\int_{t_1}^{t_2} \int \triangle\omega \, g_\varepsilon \psi h_R}_{I_1} - \underbrace{\int_{t_1}^{t_2} \int u \cdot \nabla\omega \, g_\varepsilon \psi h_R}_{I_2}$$

and send $R \to \infty$. Then

$$\limsup_{R\to\infty} |I_1| \le C \|\varphi\|_{H^2} \|\omega\|_{L^{\frac{9}{5}}(t_1,t_2;L^2)} |t_1 - t_2|^{\frac{4}{9}}.$$

and

$$\limsup_{R\to\infty} |I_2| \le C \|\varphi\|_{H^2} \|\omega\|_{L^{\frac{9}{5}}(t_1,t_2;L^2)} \|u\|_{L^3(t_1,t_2;L^4)} |t_1 - t_2|^{\frac{1}{9}}.$$

and finally, since $u(t_2) - u(t_1) \in L^2$,

$$\lim_{R\to\infty} \int \big[u(t_2) - u(t_1)\big] \nabla^\perp (g_\varepsilon \psi h_R) \\ = \langle g_\varepsilon u(t_2) - g_\varepsilon u(t_1), \varphi \rangle + o(\varepsilon).$$

By the Ascoli theorem, the strong convergence of u in $L^2_{loc}(\mathbb{R}_+^* \times \mathbb{R}^2)$ follows.

Passing to the limit

We denote by $\overline{u}$ the limit velocity.
$\varphi \in C_0^\infty(\mathbb{R}_+^* \times \mathbb{R}^2)$ div free test vector field
ψ such that $\nabla^\perp \psi = \varphi$ and $\psi(t,0) = 0$.
Multiply the vorticity equation by $g_\eta \psi h_R$, integrate in time and space and pass to the limit $\varepsilon \to 0$ to obtain

$$\underbrace{\Big\langle \partial_t \overline{\omega}, g_\eta \psi h_R \Big\rangle}_{J_1} - \underbrace{\Big\langle \overline{\omega}, \triangle(g_\eta \psi h_R) \Big\rangle}_{J_2} - \underbrace{\Big\langle \overline{u}\,\overline{\omega}, \nabla(g_\eta \psi h_R) \Big\rangle}_{J_3} = 0.$$

We finally take the limits $\eta \to 0$ and $R \to \infty$:

$$\lim_{R\to\infty} \lim_{\eta\to 0} J_3 = -\iint \overline{u}\,\overline{\omega}\,\varphi^\perp = -\iint \overline{u} \cdot \nabla \overline{u} \cdot \varphi$$

$$\lim_{R\to\infty} \lim_{\eta\to 0} J_2 = \iint \overline{\omega}\,\triangle \psi = \langle \triangle \overline{u}, \varphi \rangle.$$

$$\lim_{R\to\infty} \lim_{\eta\to 0} J_1 = \langle \partial_t \overline{u}, \varphi \rangle.$$

$\overline{u}$ verifies the Navier-Stokes equations in the distributional sense.

The initial data follows from the equicontinuity in time and the inviscid result.