On a new scale of

borderline regularity spaces for Euler equations

Eitan TADMOR

Center for Scientific Computation
and Mathematical Modeling (CSCAMM)
Department of Mathematics & IPST
University of Maryland College Park

Euler's Equations

$$\begin{cases} u_t + \nabla_x \cdot (u \otimes u) = -\nabla_x\, p, & u = (u_1, \ldots, u_d) \\ \text{div } u = 0 \\ \text{initial and boundary data} \end{cases}$$

<u>Weak solutions</u>

$\mathcal{P}1$. Finite Energy: L^2_{loc}-energy – $u(x,t) \in L^\infty([0,T]; L^2_{\text{loc}}(\mathbb{R}^d))$.

$\mathcal{P}2$. Balance Law: $\forall \varphi \in C_c^\infty([0,T) \times \mathbb{R}^d; \mathbb{R}^d)$ with div $\varphi = 0$:

$$\int_0^T \int_{\mathbb{R}^d} \varphi_t \cdot u + D\varphi\, (u \otimes u)\, dxdt + \int_{\mathbb{R}^d} \varphi(x,0) \cdot u(x,0)\, dx = 0.$$

$\mathcal{P}3$. Incompressibility: div $u = 0$ in $\mathcal{D}'$.

- <u>Weak regularity in time</u> (Lopes & Schochet) $u \in Lip((0,T); H^{-L}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d))$.

$\odot$ Assume the initial data in H^{-L}, $L > 1$ sense

- <u>Existence</u>. Passing to a limit with a sequence of approximate solutions.

Approximate Solutions

$\mathcal{P}1$. L^2_{loc}-Energy bound: $\{u^\varepsilon\} \hookrightarrow L^\infty([0,T]; L^2_{\text{loc}}(\mathbb{R}^d))$.

$\mathcal{P}2$. Weak Consistency: $\forall \varphi \in C_c^\infty([0,T) \times \mathbb{R}^d)$ with div $\varphi = 0$:

$$\int_0^T \int_{\mathbb{R}^d} \varphi_t \cdot u^\varepsilon + D\varphi \, (u^\varepsilon \otimes u^\varepsilon) \, dxdt + \int_{\mathbb{R}^d} \varphi(x,0) \cdot u^\varepsilon(x,0) \, dx \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

$\mathcal{P}3$. (Approximate) Incompressibility: div $u^\varepsilon = 0$ in $\mathcal{D}'$ ($\to 0$ in H^{-1}_{loc}).

⊙ In practice, H^{-s} consistency: $\varphi \in H^s_c([0,T] \times \mathbb{R}^d)$ ...

⊙ Energy-bound implies $\{u^\varepsilon\} \hookrightarrow Lip((0,T); H^{-L}_{\text{loc}}(\mathbb{R}^d))$, $L(s,n) > 1$.

<u>EXAMPLES</u> • *Mollification of initial data*: $u_0^\varepsilon = K_\varepsilon * \omega_0, \quad K_\varepsilon := \eta_\varepsilon * K$.

• *Navier-Stokes approximate solutions.*

• *Vortex blob approximations*

• *Discrete methods*: High-resolution difference, Spectral and FEM methods

Existence of Weak Solutions

$\odot$ Energy bound $\Longrightarrow u^\varepsilon \rightharpoonup u$ in $L^\infty([0,T], L^2_{\mathrm{loc}}(\mathbb{R}^d))$

$\odot$ Weak regularity in time: $\{u^\varepsilon\} \hookrightarrow Lip((0,T), H^{-L}_{\mathrm{loc}}(\mathbb{R}^d))$

• Main issue: passing to limit in quadratic terms: $u \bigotimes u$ ...

$\odot$ Either $u^\varepsilon \longrightarrow u$ in $L^\infty([0,T], L^2_{\mathrm{loc}}(\mathbb{R}^d)) \Longrightarrow u$ is a weak solution;

$\odot$ Or no strong convergence: $\int_E |u|^2 dxdt < \liminf \int_E |u^\varepsilon|^2 dxdt$

$\Longrightarrow$ Energy concentrates on sets with non-zero reduced defect measure

$$\mu(E) := \limsup_{\varepsilon} \int_{E \subset [0,t] \times \mathbb{R}^d} |u^\varepsilon - u|^2 dxdt > 0$$

• (DiPerna-Majda). The phenomena of concentration-cancelation.

$$u_i^\varepsilon u_j^\varepsilon \rightharpoonup u_i u_j, \quad i \neq j.$$

H^{-1} Stability

- Characterize lack of concentrations (and hence existence)

- Typically, formulated in terms of vorticity $\omega_{ij}^{\varepsilon} = \frac{\partial u_i^{\varepsilon}}{\partial x_j} - \frac{\partial u_j^{\varepsilon}}{\partial x_i} \in \mathbb{A}^d$

Definition [H^{-1}-stability]: *The sequence $\{u^{\varepsilon}\}$ is H^{-1}-stable if $\{\omega^{\varepsilon}\}$ is a precompact in $C((0,T); H_{loc}^{-1}(\mathbb{R}^d; \mathbb{A}^d))$.*

$\odot$ No growth conditions at infinity $\odot$ $u^{\varepsilon} \cdot \widehat{n} = 0$ for bounded domains

Statement of main result *(M. Lopes, H. Lopes-Nussenzveig, T.).*

If $\{u^{\varepsilon}\}$ is H^{-1}–stable, then a subsequence converges strongly to a weak solution u in $L^{\infty}([0,T]; L_{loc}^2(\mathbb{R}^d))$.

- H^{-1}–stability as a criterion which excludes concentrations.

Proof div $u^{\varepsilon} \hookrightarrow C([0,T], H_{loc}^{-1}(\mathbb{R}^d))$ and curl $u^{\varepsilon} \hookrightarrow C([0,T], H_{loc}^{-1}(\mathbb{R}^d))$

div-curl lemma $\Longrightarrow u^{\varepsilon_k} \cdot u^{\varepsilon_k} \rightharpoonup \overline{u} \cdot \overline{u}$, No concentration: $u^{\varepsilon_k} \longrightarrow \overline{u}$, $L^2([0,T], L_x^2)$

- Passing information from ω^ε to u^ε

$$\text{div } u^\varepsilon = 0 \ (\stackrel{\text{comp}}{\hookrightarrow} H^{-1}) \quad \text{curl } u^\varepsilon = \omega^\varepsilon \stackrel{\text{comp}}{\hookrightarrow} \text{“nice space”}$$

1. Biot-Savart Kernel (the 2D case): $u^\varepsilon = K * \omega^\varepsilon, \ K(x) \sim \frac{x^\perp}{|x|^2}$

⊙ CZ + Sobolev imbedding $L^p(\mathbb{R}^2) \longrightarrow W^{1,p}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$.

* Delicate as $p \downarrow 1$.

2. Stream-function formulation: $\Delta\Psi^\varepsilon := \omega^\varepsilon, \quad u^\varepsilon = \nabla^\perp\Psi^\varepsilon$

⊙ Elliptic Regularity (delicate as $p \downarrow 1$).

* For $W^{2,p}$ regularity of Ψ^ε – requires growth control at infinity

3. Our approach – *generalized* Div-Curl Lemma (Tartar-Murat)

* Sharp local condition – simplifies & generalize previous results

- Greatly simplify previous results
- Generalization – unbounded domains, $d > 2$ dimensions

- Crystallize new regularity spaces...

A Retrospect of L^p Scales of Regularity Spaces

- Lebesgue $L^p(\mathbb{R}^d)$: $\left|\int \omega\varphi dx\right| \leq Const.\|\varphi\|_{L^{p'}}, \quad \forall \varphi \in L^{p'}$
- Lorentz - $L^{p\infty}(\mathbb{R}^d)$: $\varphi \longmapsto \chi_E, \quad \forall E's \in \mathbb{R}^d$

$$\int_E |\omega| dx \leq Const.|E|^{1/p'}, \quad \text{arbitrary sets } E's,$$

- Morrey - $M^p(\mathbb{R}^d)$: $\varphi \longmapsto \chi_B, \quad \forall$arbitrary balls $B \in \mathbb{R}^d$

$$\int_{B_R} |\omega| dx \leq Const.|R|^{d/p'},$$

- Logarithmic refinements: $L^p(\log L)^\alpha,\ L^{p\infty}(\log L)^\alpha,\ M^{p,\alpha}, \ldots$

$$L^p(\log L)^\alpha := \{\omega \mid \int |\omega|^p(\log^+|\omega|)^\alpha dx \leq Const.\}$$

$$M^{p,\alpha} := \{\omega \mid R^{-d/p'}|\log R|^\alpha \int_{B_R(x_0)} |\omega| dx \leq Const., \quad R \downarrow 0\}$$

The 2D problem – scalar vorticity transported

- Transport equation $\omega_t + u \cdot \nabla_x \omega = 0, \quad \omega = \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2}$
- $H^{-1}(R^{d=2})$-compactness: Critical $p_{crit} = \frac{2d}{d+2} = 1$

⊙ Lebesgue (Yudovich, DiPerna-Majda)– borderline $\mathcal{BM}_c$ (vortex sheets)

$$\omega_0 \in L^p_c(\mathbb{R}^2),\ p > 1 \Longrightarrow \omega^\varepsilon(\cdot, t) \in L^p_{loc} \hookrightarrow H^{-1}_{loc}(\mathbb{R}^2)$$

⊙ Orlicz (Morgulis, Chae)– propagation of compactness in borderline $L(\log L)^{\frac{1}{2}}$

$$\omega_0 \in L(\log L)^\alpha_c(\mathbb{R}^2), \alpha \geq 1/2 \Longrightarrow \omega^\varepsilon(\cdot, t) \in L(\log L)^\alpha_{loc} \hookrightarrow H^{-1}_{loc}(\mathbb{R}^2)$$

⊙ Lorentz (P. L. Lions)– propagation of compactness in borderline $L^{(12)}$

$$\omega_0 \in L^{(1q)}(\mathbb{R}^2), q \leq 2 \Longrightarrow \omega^\varepsilon(\cdot, t) \in L^{(1q)}_{loc} \hookrightarrow H^{-1}_{loc}(\mathbb{R}^2)$$

- $L^{(12)}$ – largest rearrangement invariant borderline case in $H^{-1}(\mathbb{R}^2)$

⊙ ... beyond rearrangement invariant spaces ...

Beyond Rearrangement Invariant Spaces

- Morrey spaces: $\widetilde{M}^{(p;\alpha)} := \{\omega | \sup_x \int_{B_R(x)} |\omega| \leq Const. R^{\frac{d}{p'}} |\log R)|^{-\alpha}\}$

<u>Assertion</u> *(R. DeVore & T. Tao). $\widetilde{M}^{p;\alpha}(\Omega)$, $\Omega \subset \mathbb{R}^d$, is compactly imbedded in $H^{-1}(\Omega)$ if either: (i) $p > \frac{d}{2}$ or (ii) $p = \frac{d}{2}$ and $\alpha > 1$.*

- <u>Two-dimensional Morrey space</u> (DiPerna-Majda)

⊙ $\widetilde{M}^{(1,\alpha)}(\mathbb{R}^2)$: $\int_{B_R} |\omega^\varepsilon| \leq C|\log R|^{-\alpha}$, $\alpha > 1 \Longrightarrow$ no concentration

$$\omega^\varepsilon(\cdot, t) \in L^\infty([0,T], \widetilde{M}^{(1,\alpha)}(\mathbb{R}^2) \overset{\text{comp}}{\hookrightarrow} H^{-1}_{\text{loc}}(\mathbb{R}^2), \quad \alpha > 1$$

⊙ Positive vorticity (Delort, Majda): $\omega^\varepsilon(\cdot, t) \in \mathcal{BM}_c^+ \Longrightarrow \widetilde{M}_c^{(1;\frac{1}{2})}(\mathbb{R}^2)$

⊙ **Q**. Is $\widetilde{M}^{(1,\frac{1}{2})}(\mathbb{R}^2)$ <u>borderline regularity</u> space for concentration-cancelation?

⊙ **Q**. On the borderline gap $\widetilde{M}^{(1,\alpha)}(\mathbb{R}^2)$, $\frac{1}{2} < \alpha \leq 1$.

* Uniqueness: L^∞ Borderline − Besov $B^s_{2/s,1}$ (Vishik)

No Concentration – the multiD ($d > 2$) case

- No concentration for $\omega^\varepsilon(\cdot, t) \in L^\infty([0,T], X)$

$$\text{Lebesgue}: \quad X = L^p_c(\mathbb{R}^d), \qquad p > \frac{2d}{d+2} \longmapsto L^p \hookrightarrow H^{-1}(\mathbb{R}^d)$$

(since $H^1 \hookrightarrow L^p_c(\mathbb{R}^d),\ p < p^* = \dfrac{2d}{d-2}$)

$$\text{Morrey}: \quad X = M^p(\mathbb{R}^d), \qquad p > \frac{d}{2} \longmapsto M^p \hookrightarrow H^{-1}(\mathbb{R}^d)$$

Q1. On the borderline gap $\frac{6}{5} < p < \frac{3}{2}$ for the $d = 3$-D case?

- The 3D Navier-Stokes - $M^{3/2}$ existence (Giga-Miyakawa)

$$\frac{1}{R}\int_{B_R(x_0)} |\omega| dx \leq Const.$$

⊙ Comparison of $L^{6/5}$ and $M^{3/2}$ - measures of singular support (CKN)

⊙ Identify borderline regularity: $p = \frac{2d}{d+2} = \frac{6}{5}$

* Uniqueness & Energy loss – Brenier, Shnirelman,...

Borderline regularity – the multiD ($d \geq 2$) case

Theorem Assume borderline regularity: $\omega^\varepsilon(\cdot,t) \in L_c^{\frac{2d}{d+2}}(\mathbb{R}^d)$. Then there is no concentration with 'super-critical' energy bound

$$u^\varepsilon(\cdot,t) \in L^\infty([0,T], L^{p>2}(\mathbb{R}^d))$$

Proof (by Murat Lemma). By interpolation of $X_r := W^{-1,r}(\mathbb{R}^d)$

$$\begin{cases} L_c^{\frac{2d}{d+2}}(\mathbb{R}^d) \overset{\text{comp}}{\hookrightarrow} X_q, \ q<2 \\ \{\omega^\varepsilon\} \quad \text{in} \quad X_p, \ p>2 \end{cases} \Longrightarrow \omega^\varepsilon(\cdot,t) \overset{\text{comp}}{\hookrightarrow} X_2 = H^{-1}(\mathbb{R}^d).$$

⊙ Example ($d=2$). The critical regularity $\omega_0 \in \mathcal{BM}_c(\mathbb{R}^2)$:

$u^\varepsilon(\cdot,t) \in L^\infty([0,T], L^{p>2}(\mathbb{R}^2)) \Longrightarrow$ no concentration (DiPerna-Majda)

⊙ Example ($d=3$). The critical regularity $\omega_0 \in L_c^{6/5}(\mathbb{R}^3)$:

$u^\varepsilon(\cdot,t) \in L^\infty([0,T], L^{p>2}(\mathbb{R}^3)) \Longrightarrow$ no concentration.

Q2. What can we say about $L^{6/5}$ as a regularity space for $\omega^\varepsilon(\cdot,t)$?

High Resolution Central Scheme (Levy-T.)

- Solution is realized by cell-averages, $\omega(x,y,t^n) = \sum_{j,k} \bar{\omega}^n_{j,k} 1_{\mathcal{C}_{j,k}}$
- Recovery of the velocity field (u,v) from the discrete vorticity:

Define the discrete vorticity $\bar{\omega}_{j+\frac{1}{2},k+\frac{1}{2}} := \frac{1}{4}(\bar{\omega}_{j+1,k+1} + \bar{\omega}_{j,k+1} + \bar{\omega}_{j,k} + \bar{\omega}_{j+1,k})$

and use a five-point stream-function , ψ, such that $\Delta\psi = -\bar{\omega}$

$$u_{j,k} := \mu_x \nabla_y \psi_{j,k}, \quad v_{j,k} := -\mu_y \nabla_x \psi_{j,k}, \qquad (\nabla_x^2 + \nabla_y^2)\psi = -\bar{\omega}$$

$\odot \Longrightarrow$ Discrete incompressibility: $\mu_y \nabla_x u_{j+\frac{1}{2},k+\frac{1}{2}} + \mu_x \nabla_y v_{j+\frac{1}{2},k+\frac{1}{2}} = 0.$

- Time Evolution: predict the $t^n + \frac{\Delta t}{2}$-midvalues ($\lambda_x := \frac{\Delta t}{\Delta x}$ and $\lambda_y := \frac{\Delta t}{\Delta y}$)

$$\omega^{n+\frac{1}{2}}_{j,k} = \bar{\omega}^n_{j,k} - \frac{\lambda_x}{2}(u\omega)'_{j,k} - \frac{\lambda_y}{2}(v\omega)^{\grave{}}_{j,k},$$

$\odot$ Compute the *staggered* cell-averages at $t^{n+1} = t^n + \Delta t$

$$\begin{aligned}\bar{\omega}^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}} \quad = \quad & \mu_x\mu_y \bar{\omega}^n_{j+\frac{1}{2},k+\frac{1}{2}} + \frac{1}{8}\mu_y \nabla_x \omega'_{j+\frac{1}{2},k+\frac{1}{2}} + \frac{1}{8}\mu_x \nabla_y \omega^{\grave{}}_{j+\frac{1}{2},k+\frac{1}{2}} \\ & - \Delta t \{ \mu_y \nabla_x (u\omega)^{n+\frac{1}{2}}_{j+\frac{1}{2},k+\frac{1}{2}} + \mu_x \nabla_y (v\omega)^{n+\frac{1}{2}}_{j+\frac{1}{2},k+\frac{1}{2}} \}.\end{aligned}$$

High Resolution Central Scheme(cont'd)

- Numerical slopes

$f'_{j,k}$ and $g^{\backprime}_{j,k}$, denote discrete 'numerical slopes' in x- and y-directions.

$\odot$ $f' \equiv g^{\backprime} \equiv 0 \longmapsto$ first-order Lax-Friedrichs scheme:

$$\bar{\omega}^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}} = \frac{\bar{\omega}^n_{j,k} + \bar{\omega}^n_{j+1,k} + \bar{\omega}^n_{j,k+1} + \bar{\omega}^n_{j+1,k+1}}{4} - \Delta t\{\mu_y \nabla_x (u\omega)^n_{j+\frac{1}{2},k+\frac{1}{2}} + \mu_x \nabla_y (v\omega)^n_{j+\frac{1}{2},k+\frac{1}{2}}\}.$$

$\odot$ key observation – discrete incompressibility implies convexity

$$\bar{\omega}^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}} = \sum_{\alpha,\beta} \theta_{\alpha,\beta} \bar{\omega}^n_{\alpha,\beta}, \qquad \sum \theta_{\alpha,\beta} = 1, \ \theta_{\alpha,\beta} \geq 0.$$

- Higher resolution:

$$f'_{j,k} \sim \Delta x \cdot f_x(x_j, y_k, t^n) + \mathcal{O}(\Delta x)^2, \ g^{\backprime}_{j,k} \sim \Delta y \cdot g_y(x_j, y_k, t^n) + \mathcal{O}(\Delta y)^2$$

- Discrete evolution: $\omega(\cdot, 0) \longmapsto \omega(\cdot, t^n)$ maps any Orlicz space into itself

Candidates for regularity spaces

$\odot$ Lebesgue L^p : $\left\{\omega \mid \left|\int_x \omega\varphi dx\right| \leq Const.\|\varphi\|_{L^{p'}}\right\}$

$\odot$ Lorentz wk $- L^{p\infty}$: $\left\{\omega \mid \varphi = \chi_E, \text{ arbitrary } E's\right\}$

$\odot$ Morrey M^p : $\left\{\omega \mid \varphi = \chi_B, \text{ arbitrary } B's \right\}$

$$\|\omega\|_{M^p} = \sup_B \frac{1}{|B|^{1/p'}} \int_B |\omega| dx \leq \infty$$

- A new scale $\vee^{pq}$: $\left\{\omega \mid \varphi = \chi_{\cup B_j}, \text{ arbitrary covering } \mathcal{B}'s\right\}$

$$\vee^{pq} := \sup_{\{B_j\}\subset\mathcal{B}} \left\{\frac{1}{|B_j|^{1/p'}} \int_{B_j} |\omega| dx\right\}_{\ell^q} < \infty, \quad arbitrary\ \{B_j\}'s \subset \mathcal{B}$$

The new scale of regularity spaces

$$\vee^{pq}: \qquad \sum_j \left(\frac{1}{|B_j|^{1/p'}} \int_{B_j} |\omega| dx \right)^q \leq Const.$$

$$\vee^{pp}: \qquad \sum_j \left(\frac{1}{|B_j|} \int_{B_j} |\omega| \right) \chi_{B_j}(x) \in L^p \longmapsto L^p$$

$$\vee^{p\infty}: \qquad \frac{1}{|B|^{1/p'}} \int_B |\omega| dx \leq Const. \longmapsto M^p$$

- $L^{p\infty}$ measures total mass on arbitrary <u>sets</u>

- M^p measures total mass on arbitrary <u>balls</u>

- $\vee^{pq}$ - are intermediate scales of spaces: $\vee^{pq} = (L^p, M^p)_{\theta,q}$ $\theta = \frac{p}{q} \leq 1$

  measuring ℓ^q weighted distribution of L^p mass on <u>arbitrary coverings</u>

A New Scale of Regularity Spaces

- New scale $\vee^{pq}(\Omega)$, $1 \leq p \leq q \leq \infty$: for all coverings $\cup_j B_j$

$$\sup_{\cup B_j = \Omega} \left(\sum_j (R_j^{-d/p'} \int_{B_j} |\omega(x)| dx)^q \right)^{1/q} \leq Const, \quad 1 \leq p \leq q \leq \infty.$$

- Logarithmic refinement: $\vee^{pq,\alpha} = \vee^{pq}(\log \vee)^\alpha$

$$\|\omega\|_{\vee^{pq}(\log \vee)^\alpha(\Omega)} := \sup_{R_j < R_0} \|\{R_j^{d/p} |\log R_j|^\alpha \bar{\omega}_j\}\|_{\ell^q}, \quad q > p.$$

$\odot$ Example: $\|\omega\|_{\vee^{pq}(\log \vee)^\alpha(\Omega)} < \infty$: Covering Ω by a dyadic lattice $\mathcal{C}_{jk}$

$$\sum_j (\int_{\mathcal{C}_{jk}} |\omega(x)| dx)^q \leq 2^{-kNq/p'} |1 + k_+|^{-\alpha q}, \quad \mathcal{C}_{jk}(\cdot) := 2^{-k} \mathcal{C}(\cdot + j).$$

$\odot$ weak-L^p spaces: measure the total mass of over *arbitrary sets*;
$\odot$ Morrey spaces M^p: measure the total mass over *arbitrary balls*.
$\odot$ $\vee$-spaces: measure ℓ_q-weighted mass over *collection of disjoint balls*

- $\vee^{pq}$ bridges the gap: $\vee^{pq} = (L^p, M^p)_{\theta,q}$ $\theta = \frac{p}{q} \leq 1$, $\quad \vee^{pp} = L^p \ \ldots \ \vee^{p\infty} = M^p$

Readers' digest

$$\mathsf{V}^{pq}: \qquad \sum_j \left(\frac{1}{|B_j|^{1/p'}} \int_{B_j} |\omega| dx \right)^q < \infty$$

- Cover $\Omega \subset \mathbb{R}^d$ by dyadic covering of cubes $C_{jk} := 2^{-k} C(\cdot + j)$

$$\mathsf{V}^{pq}(\Omega): \qquad \sum_j \left(\int_{C_{jk}} |\omega| dx \right)^q \le 2^{-kdq/p'}$$

- Comparison of borderline regularity spaces

$$p = \frac{3}{2}, \quad M^{\frac{3}{2}}(\mathbb{R}^3): \quad \frac{1}{R} \int_{B_R(x_0)} |\omega| dx \le Const.$$

$$p = \frac{6}{5}, \quad \mathsf{V}^{\frac{6}{5}2}(\mathbb{R}^3): \quad \sum_j \frac{1}{R_j} \left(\int_{B_j} |\omega| dx \right)^2 \le Const.$$

- difference in Hausdorff dim (sing support ω)

Compact Embeddings of $\mathbb{V}$'s in H^{-1}

Statement of compactness. $\mathbb{V}^{p2,\alpha} \overset{\text{comp}}{\hookrightarrow} H^{-1}_{\text{loc}}(\mathbb{R}^d)$ if

$$\text{(i) } p > \frac{2d}{d+2} \qquad \text{or} \qquad \text{(ii) } p = \frac{2d}{d+2},\ \alpha > 1/2.$$

⊙ The 2D borderline case: $\widetilde{\mathbb{V}}_c^{12,\alpha}(\mathbb{R}^2) \overset{\text{comp}}{\hookrightarrow} H^{-1}_{\text{loc}}(\mathbb{R}^2), \quad \alpha > \frac{1}{2}$

$$\widetilde{\mathbb{V}}^{12}(\log \widetilde{\mathbb{V}})^{1/2}(\Omega) = \{\omega \mid \sup_{\cup B_j = \Omega} \sum_j |\log R_j| \left(\int_{B_j} |\omega|\right)^2 \leq Const.\},\ \Omega \subset \mathbb{R}^2.$$

⊙ The 3D borderline case: $\widetilde{\mathbb{V}}_c^{p2}(\mathbb{R}^3) \overset{\text{comp}}{\hookrightarrow} H^{-1}_{\text{loc}}(\mathbb{R}^3), \quad p > \frac{6}{5}$

$$\|\omega\|^2_{\widetilde{\mathbb{V}}^{\frac{6}{5}2}(\Omega)} = \sup_{\cup B_j = \Omega} \sum_j \frac{1}{R_j} \left(\int_{B_j} |\omega|\right)^2 \leq Const.,\ \Omega \subset \mathbb{R}^3$$

Proof (of $\vee$-compact imbedding).

Measure the H^{-1} size of f^ε in terms of its wavelet expansion

$$f^\varepsilon = \sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}+}\sum_{j\in\mathbb{Z}^d} \widehat{f}^\varepsilon_{jk}\psi_{jk}, \qquad \psi_{jk} := 2^{kd/2}\psi(2^k x - j)$$

Using the $\vee^{p2}(\log\vee)^\alpha$-bounds

$$\begin{aligned}\sum_{j\in Z^d}|\widehat{f}^\varepsilon_{jk}|^2 &\leq 2^{kd}\sum_{j\in Z^d}\Big(\int_{\mathcal{C}_{jk}}|f^\varepsilon(x)|dx\Big)^2 \\ &\leq Const\cdot 2^{kd}\|f^\varepsilon\|^2_{\vee^{p2,\alpha}}\cdot 2^{-2kd/p'}|1+k_+|^{-2\alpha}.\end{aligned}$$

we conclude: if $(p-\frac{2d}{d+2})_+ + (\alpha-1/2)_+ > 0$

$$\begin{aligned}\|\sum_{k>k_0}\sum_{j\in\mathbb{Z}^d}\widehat{f}^\varepsilon_{jk}\psi_{jk}\|^2_{H^{-1}} &= \sum_{\psi\in\Psi}\sum_{(j,k)\in(\mathbb{Z}^d,\mathbb{Z}+)}|\widehat{f}^\varepsilon_{jk}|^2\|\psi_{jk}\|^2_{H^{-1}} \\ &\leq Const.\sum_{k>k_0}2^{k(d-2d/p'-2)}|1+k_+|^{-2\alpha}\to 0.\end{aligned}$$

Concentration-Cancelation in 2D Euler's equations

• 2D pseudo-energy $H(\omega) := -\frac{1}{2\pi}\int\int_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y| d\omega(x)d\omega(y) \leq H_0$

⊙ $\vee$-scale classification of 2D regularity: $X_\alpha = \widetilde{\vee}^{12}(\log\widetilde{\vee})^{\alpha}_{c}(\mathbb{R}^2)$

<u>Theorem</u>. {i} No concentration if $\omega^\varepsilon \in X_\alpha,\ \alpha > 1/2$;

{ii} Concentration-Cancelation if $\omega^\varepsilon \in X_\alpha,\ \alpha \in (0, \frac{1}{2}]$.

⊙ Extension of Delort's result for one-signed measures: $\mathcal{BM}^+(\mathbb{R}^2) \subset X_{1/2}$

Q1. Is $X_\alpha = \widetilde{\vee}^{12}(\log\widetilde{\vee})^{\alpha}_{c}(\mathbb{R}^2)$ an invariant regularity space for 2D Euler?

$$\omega_0 \in X_\alpha \Longrightarrow \ \eta_\varepsilon * \omega_0 \mapsto \omega^\varepsilon(\cdot,t) \in X_\alpha \ ?$$

⊙ Propagation of compactness in borderline regularity: $X = L(\log L)^{1/2},\ L^{(12)}, \ldots$

$$\{\omega_0^\varepsilon\} \subset X_{1/2} \not\hookrightarrow H^{-1}(R^2) \quad \text{but does} \quad \eta_\varepsilon * \omega_0 \mapsto \omega^\varepsilon(\cdot,t) \hookrightarrow H^{-1}(R^2)$$

Q2. ... No concentration phenomena for one-signed measures?

$$\{\omega^\varepsilon\} \subset X_{1/2} \not\hookrightarrow H^{-1}(R^2):\ \omega_0^\varepsilon = \frac{1}{\varepsilon^2\sqrt{|\log\varepsilon|}}\omega\left(\frac{|x|}{\varepsilon}\right), \text{but} \ldots \eta_\varepsilon * \omega_0 \mapsto \omega^\varepsilon(\cdot,t) \hookrightarrow H^{-1}(R^2)\ ?$$

Concentration-Cancelation in 3D Euler's equations

• 3D Coulomb energy $H(\omega(x,t)) := \frac{1}{8\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\langle\omega(x,t),\omega(y,t)\rangle}{|x-y|}dxdy \equiv H_0$

⊙ Split between long-range and short-range $H(\omega) =: H_{ie}(\omega)+H_{si}(\omega)$

⊙ Long-range interaction energy - bounded from below...

$$H_{ie}(\omega^\varepsilon(x,t)) = \frac{1}{8\pi}\sum_{j\neq k}\int\int_{\mathcal{C}_j\times\mathcal{C}_k}\frac{\langle\omega^\varepsilon(x,t),\omega^\varepsilon(y,t)\rangle}{|x-y|}dxdy \geq -Const_{ie}.$$

$\Longrightarrow$ Upper-bound short-range self-induced energy:

$$H_0+Const_{ie} \geq H_{si}(\omega^\varepsilon(\cdot,t)) = \frac{1}{8\pi}\sum_j\int\int_{\mathcal{C}_j\times\mathcal{C}_j}\frac{\langle\omega^\varepsilon(x,t),\omega^\varepsilon(y,t)\rangle}{|x-y|}dxdy \geq \ldots?$$

Q1. Seeking $\vee^{\frac{6}{5}}2$-bound: $\boxed{\langle\omega^\varepsilon(x),\omega^\varepsilon(y)\rangle \geq (1-\theta^2)|\omega^\varepsilon(x)|\cdot|\omega^\varepsilon(y)|}$

$$\sum_j\int\int_{\mathcal{C}_j\times\mathcal{C}_j}\frac{\langle\omega^\varepsilon(x,t),\omega^\varepsilon(y,t)\rangle}{|x-y|}dxdy \geq (1-\theta^2)\sum_j\frac{1}{2R_j}\left(\int_{\mathcal{C}_j}|\omega^\varepsilon(x,t)|dx\right)^2$$

No concentration in 3D Euler's equations

- A weak alignment condition (Constantin-Fefferman-Majda):

$$\left|\frac{\omega^\varepsilon(x,t)}{|\omega^\varepsilon(x,t)|} - \frac{\omega^\varepsilon(y,t)}{|\omega^\varepsilon(y,t)|}\right|_{|x-y|\le\delta} \le \sqrt{2}\theta. \quad \theta < 1.$$

<u>Theorem</u> A weak alignment condition implies

$$\left|\frac{\omega^\varepsilon(x,t)}{|\omega^\varepsilon(x,t)|} - \frac{\omega^\varepsilon(y,t)}{|\omega^\varepsilon(y,t)|}\right|_{|x-y|\le\delta} < \sqrt{2} \qquad \Longrightarrow \|\omega^\varepsilon(\cdot,t)\|_{\vee^{\frac{6}{5}2}(\Omega)} \le Const_T.$$

Q2. Propagation of compactness of borderline regularity $X = \vee^{\frac{6}{5}2}(\mathbb{R}^3)$?!

<u>Theorem</u> Assume a weak alignment condition. Then as long as $u^\varepsilon \in L^\infty([0,T], L^{p>2}(\mathbb{R}^3))$ there is no 3D concentration: $u^\varepsilon \longrightarrow u$

THANK YOU