Well-posedness of the generalized Proudman-Johnson equation without viscosity

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Generalized Proudman-Johnson equation

- Proposed in 2000 by Zhu and O. in order to measure the balance of the convection and stretching terms.

$$f_{txx} + ff_{xxx} - af_x f_{xx} = \nu f_{xxxx}$$

convection stretching viscosity

0 < x < 1, 0 < t. a is a parameter.

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2 D Navier-Stokes + $\mathbf{u} = (f(t,x), -yf_x(t,x)) \Rightarrow$

Proudman-Johnson eq. ($a = 1$) ('62)

Riabouchinski ('24)

Generalized Proudman-Johnson equation

- Why this equation is interesting to me?

$$\omega = -f_{xx}, \quad \omega_t + f\omega_x - af_x\omega = \nu\omega_{xx}$$

Cf. 3D vorticity equations.

$$\boldsymbol{\omega} = \mathrm{curl}\,\mathbf{u}, \quad \boldsymbol{\omega}_t + (\mathbf{u} \bullet \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \bullet \nabla)\mathbf{u} = \nu\Delta\boldsymbol{\omega}$$

- 3D Navier-Stokes is formidable to me, but, 1D analogue could be solved, I hoped. However, …

Though simple, it contains some known equations as particular members.

- ❶ a= -(m-3)/(m-1), axisymmetric exact solutions of the Navier-Stokes equations in $\mathbf{R}^m$. (Zhu & O. Taiwanese J. Math. 2000) (a=0 for 3D Euler)

- ❷ a=1 (m=2) Proudman-Johnson equation ('24, '62)
- ❸ a=-2, ν=0. Hunter-Saxton equation ('91)
- ❹ a=-3 Burgers equation ('40)
- ❺ a= ∞

$$u_t = \nu u_{xx} + u^2 - \int_0^1 u(t,x)^2\,dx.$$

The Hunter-Saxton equation is a model appearing in the nematic liquid crystal theory. SIAM J. Appl. Math. (1991)

$$f_{tx} + ff_{xx} + \frac{1}{2}(f_x)^2 = 0.$$

(known to be integrable)

By differentiation

$$f_{txx} + ff_{xxx} + 2f_x f_{xx} = 0$$

The Burgers equation

$$f_t + ff_x = \nu f_{xx}$$

Differentiate

$$f_{tx} + ff_{xx} + (f_x)^2 = \nu f_{xxx}$$

Differentiate once more

$$f_{txx} + ff_{xxx} + 3 f_x f_{xx} = \nu f_{xxxx}$$

My goal: To determine whether blow-up occurs or not, depending on the parameter a and the initial data.

- What is expected is: global existence for small |a| and blow-up for large |a|.

$$\omega = -f_{xx}, \quad \omega_t + f\omega_x - af_x\omega = \nu\omega_{xx}$$

convection stretching viscosity

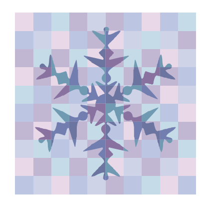
- Stretching is a cause of blow-up, viscosity suppresses blow-up, and convection is neutral. Are these heuristic statements really substantiated? A little surprise: convection term isn't a bystander. It suppresses blow-up: O & Ohkitani , J. Phys. Soc. Japan, '05.
- For the sake of simplicity, we consider in $0<x<1$ with periodic boundary condition.

Summary of results in the case of $\nu > 0$.

- If $-3 \leqq a \leqq 1$, no blow-up occurs. Every solution tends to zero.

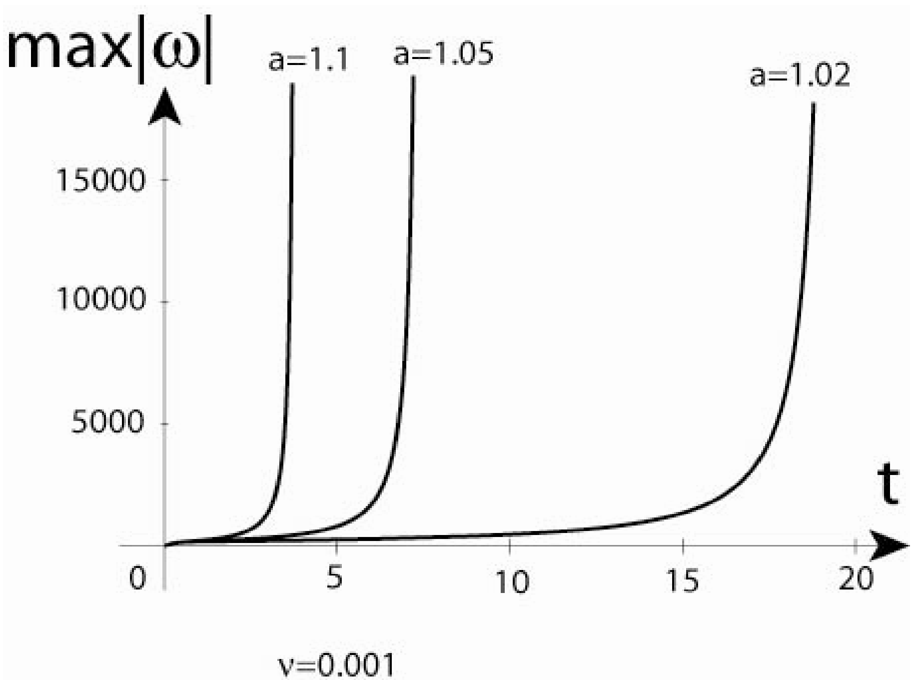
X. Chen & O., Proc. Japan Acad., (2002)

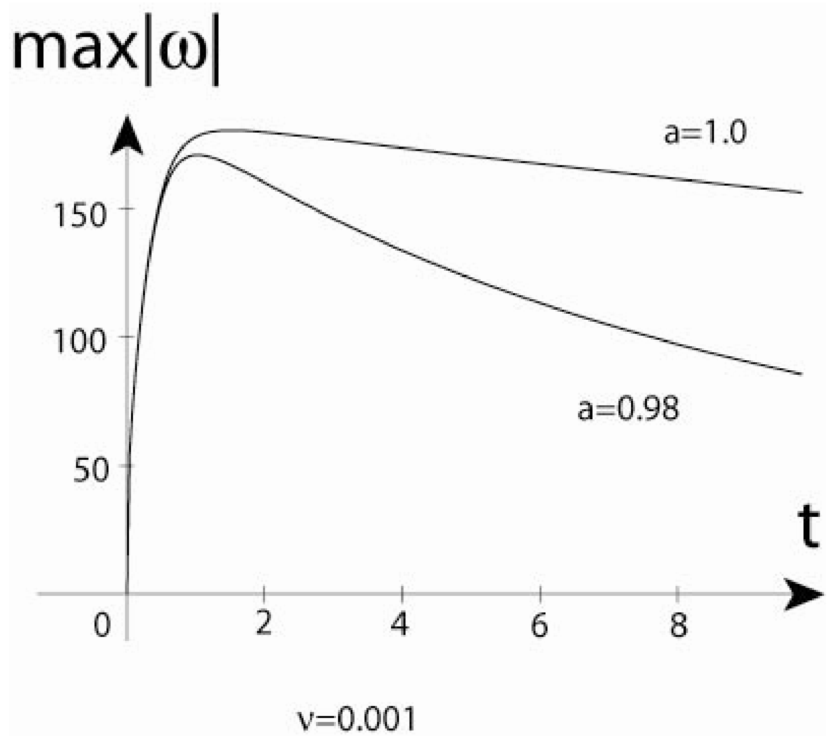
- If $a < -3$ or $1 < a$, numerical experiments strongly suggest that:
  - ✓ large solutions blow up
  - ✓ small solutions decay to zero.

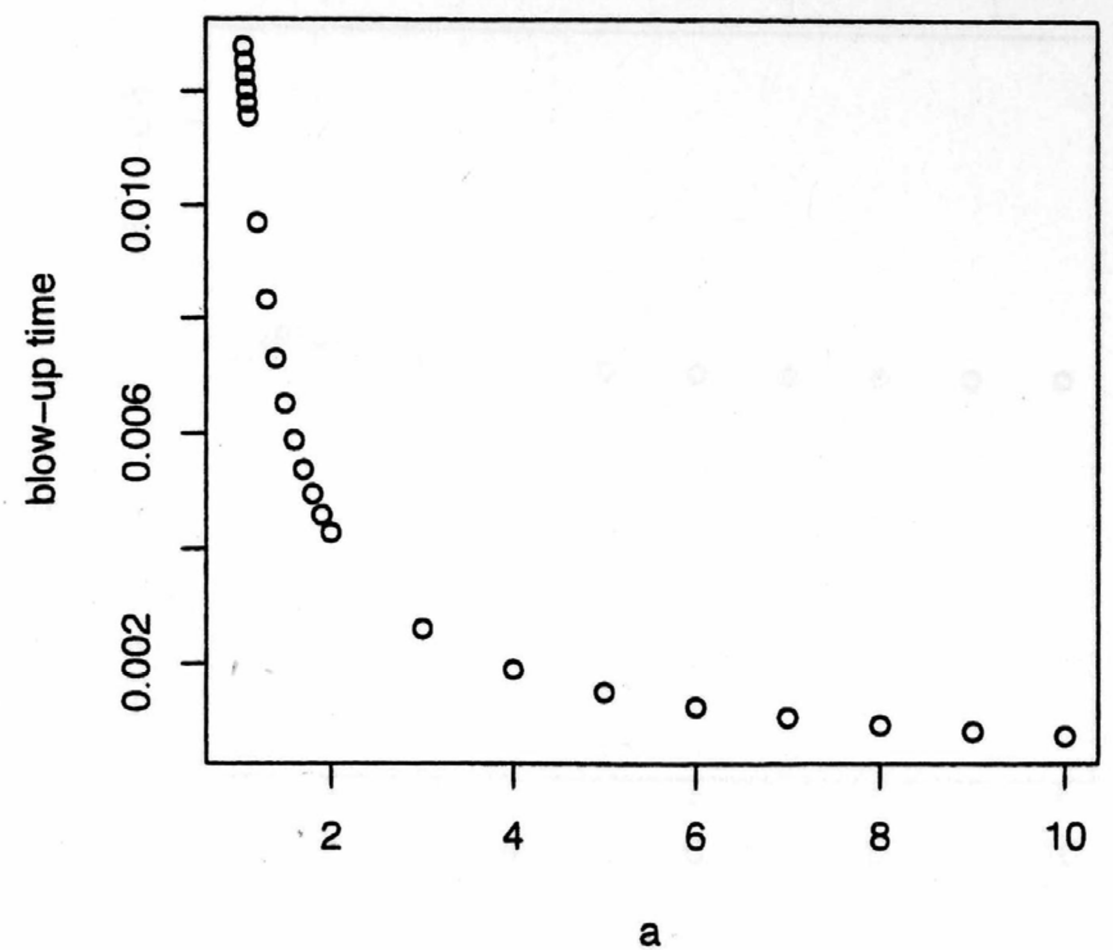


Numerical experiments (Zhu & O. Taiwanese J. Math. 2000)

- a=1 is a threshold.







blow−up time
0.002
0.006
0.010
2
4
6
8
10
a

The limit as $a \to \infty$

$$f_{txx} + ff_{xxx} - af_x f_{xx} = \nu f_{xxxx}$$

- redefine

$$f \to \frac{1}{a} f \qquad \frac{1}{a} f_{txx} + \frac{1}{a^2} ff_{xxx} - \frac{1}{a} f_x f_{xx} = \frac{\nu}{a} f_{xxxx}$$

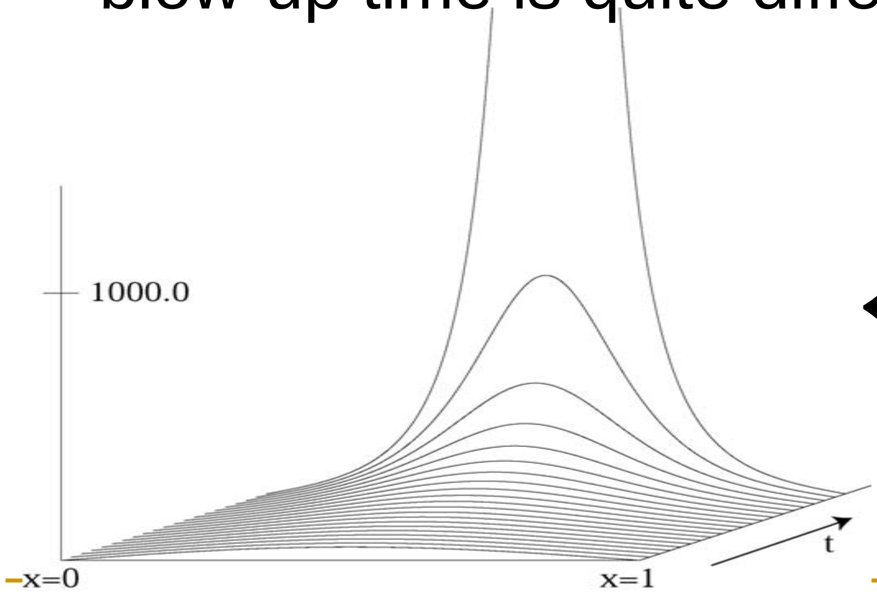
- and let a tend to infinity. $f_{txx} - f_x f_{xx} = \nu f_{xxxx}$

$$f_{tx} - \frac{1}{2}(f_x)^2 = \nu f_{xxx} + \gamma(t).$$

$$u = \frac{1}{2} f_x, \qquad u_t = \nu u_{xx} + u^2 - \int_0^1 u(t,x)^2\, dx.$$

Blow-up occurs in $u_t = \nu u_{xx} + u^2 - \int_0^1 u(t,x)^2 dx.$

- Large solutions blow-up and small solutions exists and decay to zero. Budd et al. ('93, SIAM J. Appl. Math.), O.& Zhu ('00)
- But the asymptotic behavior as t approach the blow-up time is quite different.



$$\leftarrow u_t = \nu u_{xx} + u^2$$

Budd, Dold & Stuart ('93), Zhu &O. ('00)

$$u_t = \nu u_{xx} + u^2 - \int_0^1 u(t,x)^2\,dx.$$

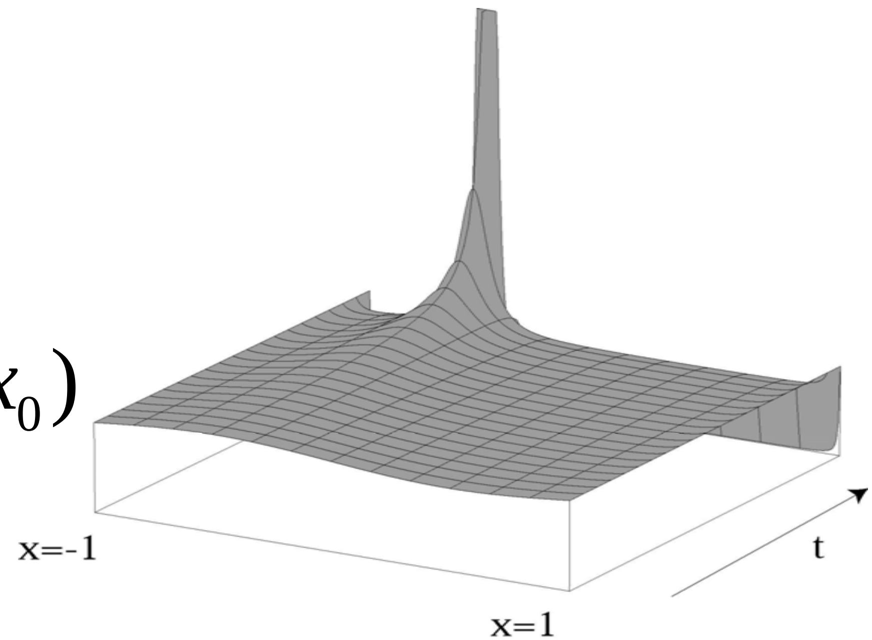
- $\exists\ x_0$

$$\int_0^1 u(t,x)dx = \int_0^1 u(0,x)dx$$

$$\lim_{t\to T} u(t,x_0) = +\infty,$$

$$\lim_{t\to T} u(t,y) = -\infty \quad (y \neq x_0)$$

$$\lim_{t\to T} \frac{u(t,y)}{u(t,x_0)} = 0$$

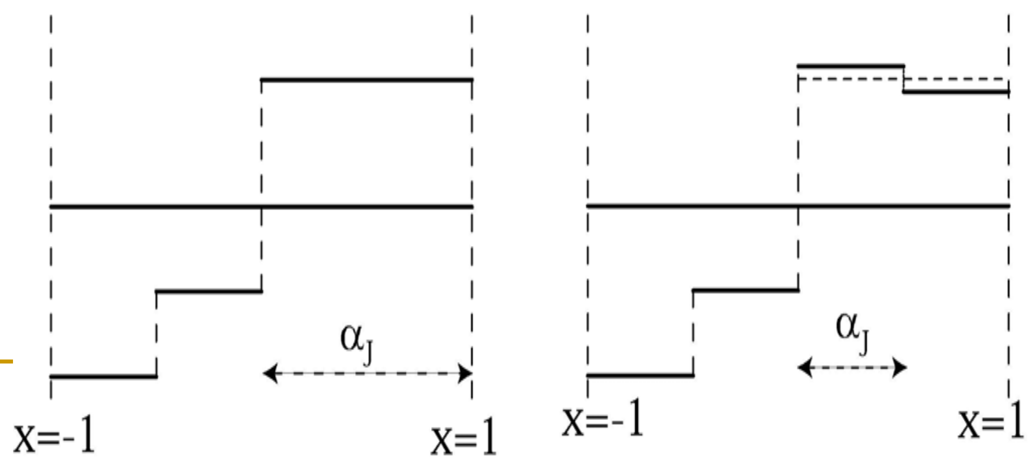


If $\nu = 0$, $$u_t(t,x) = u(t,x)^2 - \int_0^1 u(t,y)^2\,dy$$

$$\int_0^1 u(t,x)\,dx = 0$$

Theorem (X. Chen & O., ‘03 , J. Math. Sci. Univ. Tokyo).

Blow up iff $|\{ x ; u(0,x) = \max u(0,\cdot) \}| < 1/2$



I want to know a proof for blow-up when $\nu>0$, $-\infty< a < -3$, $1 < a < \infty$.

The case of $\nu=0$. We have fragmental knowledge only.

- Blow-ups occur if $a < -2$ (Zhu & O.)
- No blow-up for $a = 0$ (Zhu & O.)
- Blow-ups occur if $a = 1$ (Childress & others)
- Blow-ups occur if $a = -3$ (Burgers, shock wave)
- Blow-ups occur if $a=-2$ (Hunter & Saxton)

My report today

- Blow-up for $-2 < a < -1$. (Remember that the solutions exist globally in this region if $\nu > 0$. Viscosity helps global existence.)

- Global existence for $-1 \leqq a < 1$ & smooth initial data.
- Self-similar, non-smooth blow-up solutions exist for $-1 < a < \infty$.
- So far, I have no conclusion in the case of $1 < a$.

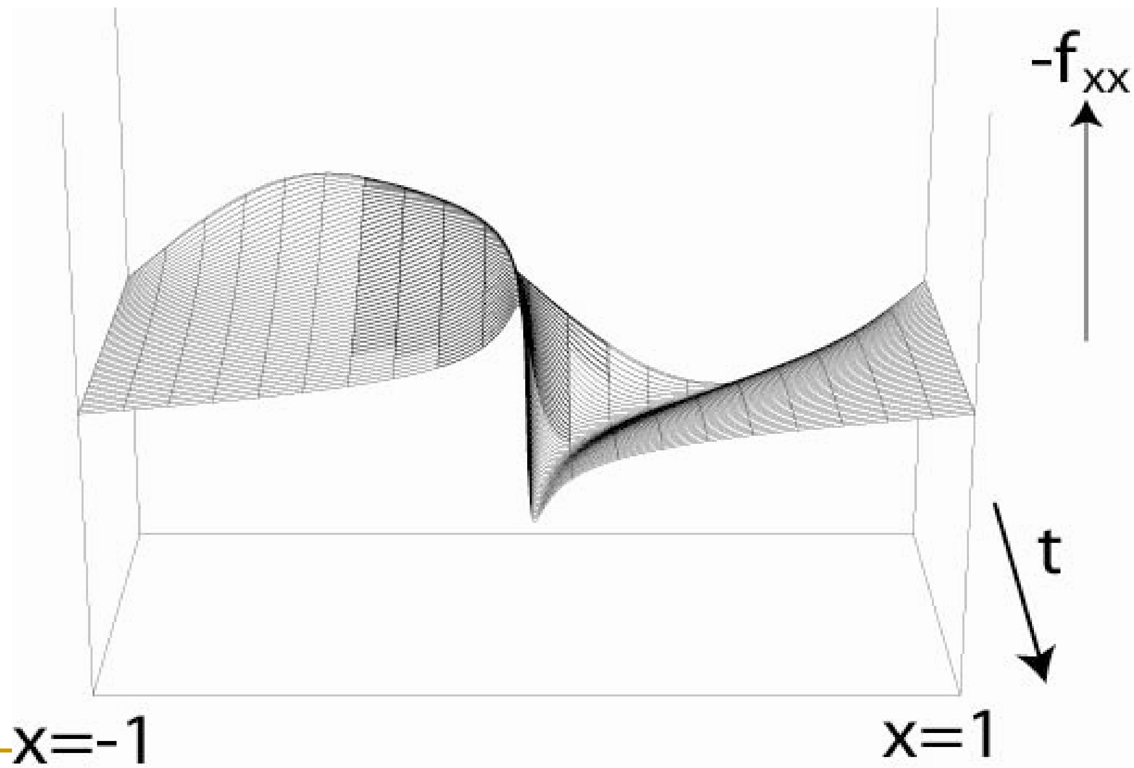
A remark on numerical experiments

- In the case of ν=0, (Euler), numerical experiments are sometimes (but not often) misleading.

$$f_{txx} + ff_{xxx} = 0$$

(a=0, 3D Euler)

Rigorous analysis is necessary



Starting point: local existence theorem

- With a help of Kato-Lai theorem (J. Func. Anal. '84),

$$\omega = -f_{xx},\ f = G(\omega),\ \omega_t + f\omega_x - af_x\omega = \nu\omega_{xx}$$

- Theorem (Zhu & O. '00). For all $\omega(0) \in L^2(0,1)/\mathbf{R}$, there exist *T* and a unique solution in $0 \leqq t < T$.

$$\omega \in C([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-1}(0,1))$$

- A priori bound for $\|\omega(t)\|_2$ is enough for global existence

Analysis for global existence/blow-up proceeds in different ways in different philosophy in

$-\infty < a < -2$, $-2 \leqq a < -1$,

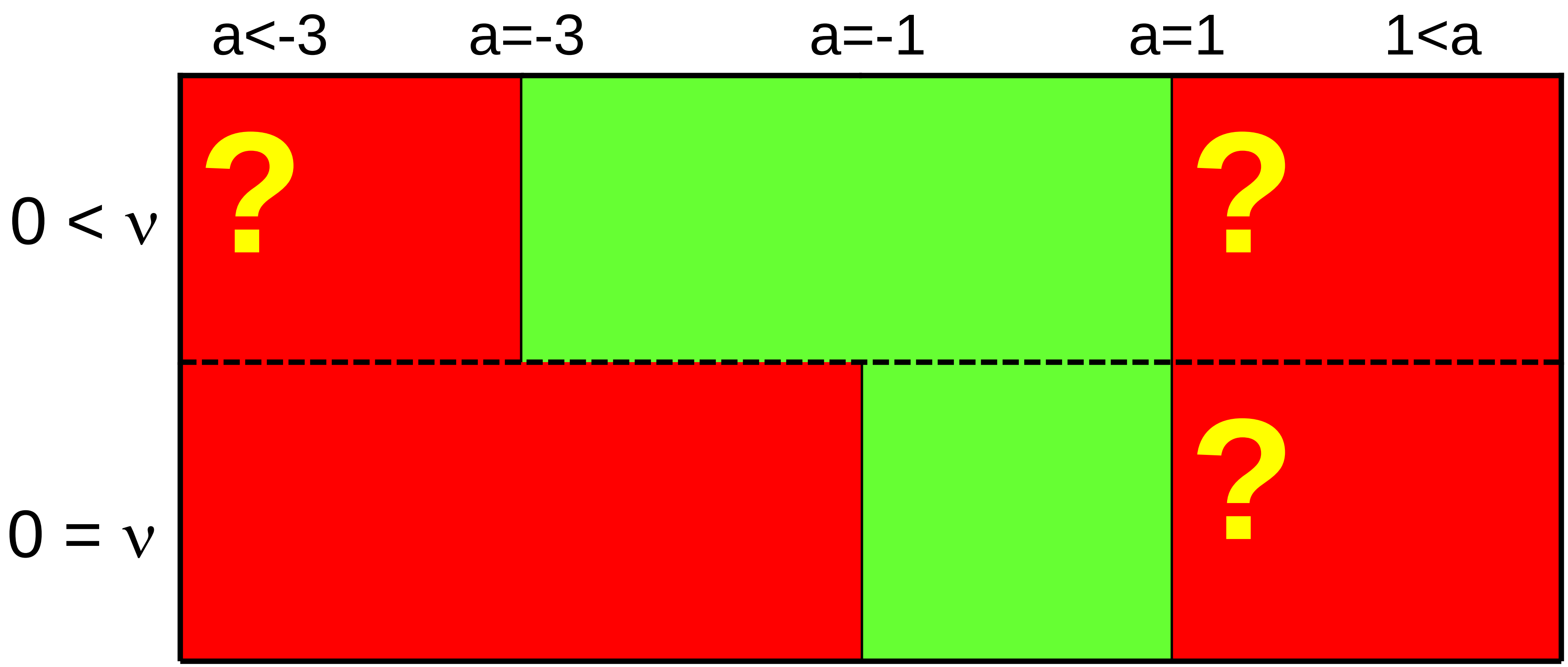
$-1 \leqq a < 0$, $0 \leqq a < 1$

- The case of $-\infty < a < -2$ is settled in Zhu & O., Taiwanese J. Math. (2000).

$$\phi(t) \equiv \int_0^1 f_x(t,x)^2 dx$$

$$\frac{d^2}{dt^2}\phi(t) \geq b\phi(t)^3$$

Summary of the results.



-2 ≦ a < -1. Follows the recipe of Hunter & Saxton ('91)

- Use the Lagrangian coordinates

$$X_t = f(t, X(t,\xi)), \quad X(0,\xi) = \xi, \quad (0 \le \xi \le 1)$$

- Define $V(t,\xi) = X_\xi(t,\xi).$

$$VV_{tt} = (V_t)^2 - I(t)V, \quad I(t) = \int_0^1 \frac{V_t^{\,2}}{V} d\xi$$

- V tends to -∞.
- Global weak solution in the case of a= -2 (Bressan & Constantin '05).

Blow-up occurs both in **-∞< a < -2** and in **-2 ≦ a < -1**, but

- Asymptotic behavior is quite different.
- $\| f_x(t) \|_{L^2}$ blow up. (-∞ < a < -2)
- $\| f_x(t) \|_{L^2}$ is bounded. $\| f_x(t) \|_{L^\infty}$ blows up.

(-2 ≦ a < -1)

-1 ≦ a < 0.   Follows the recipe of Chen &O. Proc. Japan Acad., (2002)

- Define $\Phi(u) = |u|^{-1/a}$

- Invariant

$$\frac{d}{dt}\int_0^1 \Phi(f_{xx}(t,x))dx = \int_0^1 \Phi'(f_{xx})[-ff_{xxx} + af_x f_{xx}]dx$$

$$= \int_0^1 [\Phi(f_{xx}) + af_{xx}\Phi'(f_{xx})] f_x dx = 0.$$

- Boundedness of $\int_0^1 |f_{xx}(t,x)|^{-1/a} dx, \quad \int_0^1 |f_{xx}(t,x)| dx$

-1 ≦ a < 0. Continued.

- $\left\| f_x(t) \right\|_\infty \leq c$
- $f_{txx} + ff_{xxx} - af_x f_{xx} = \nu f_{xxxx}$ gives us

$$\frac{d}{dt}\int_0^1 f_{xx}(t,x)^2 dx = (2a+1)\int_0^1 f_x f_{xx}{}^2 dx$$

$$\frac{d}{dt}\int_0^1 f_{xx}(t,x)^2 dx \leq c(2a+1)\int_0^1 f_{xx}(t,x)^2 dx$$

$0 \leqq a < 1$. Follows the recipe of Chen &O. Proc. Japan Acad., (2002)

- Define

$$\Phi(u) = \begin{cases} |u|^{1/(1-a)} & (u<0) \\ 0 & (0<u) \end{cases}$$

- Then $\dfrac{d}{dt}\int_0^1 \Phi(f_{xxx})dx = a\int_0^1 f_{xx}^{\,2}\Phi'(f_{xxx})dx \leq 0$

- $\int_0^1 \left| f_{xxx}(t,x) \right| dx$ is bounded. $\left\| f_{xx} \right\|_\infty$ is bounded.

Non-smooth, self-similar blow-up solutions when -1 < a < +∞

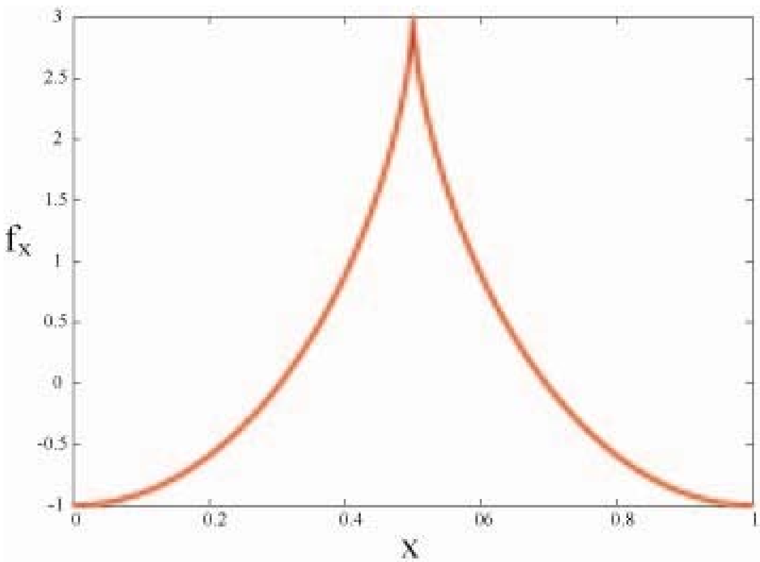
- $$f(t,x) = \frac{F(x)}{T-t}$$

  $$F'' + FF''' - aF'F'' = 0.$$

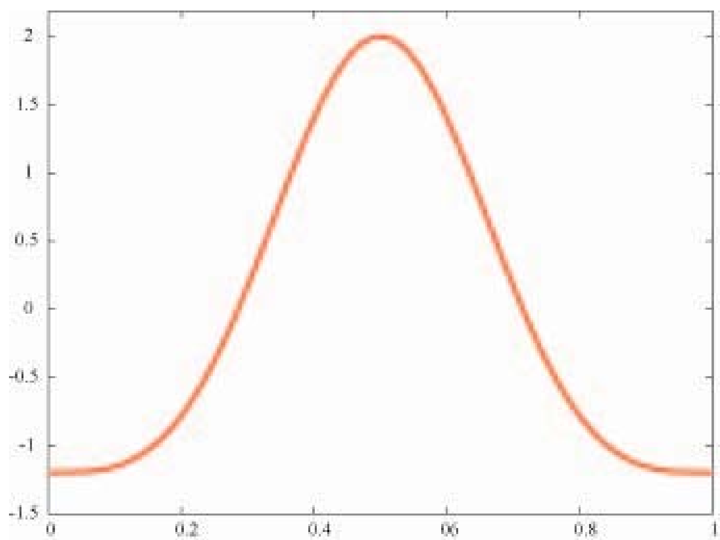
- Nontrivial solution exists for all -1 < a < +∞.

Some profiles

- Periodic, but not smooth.

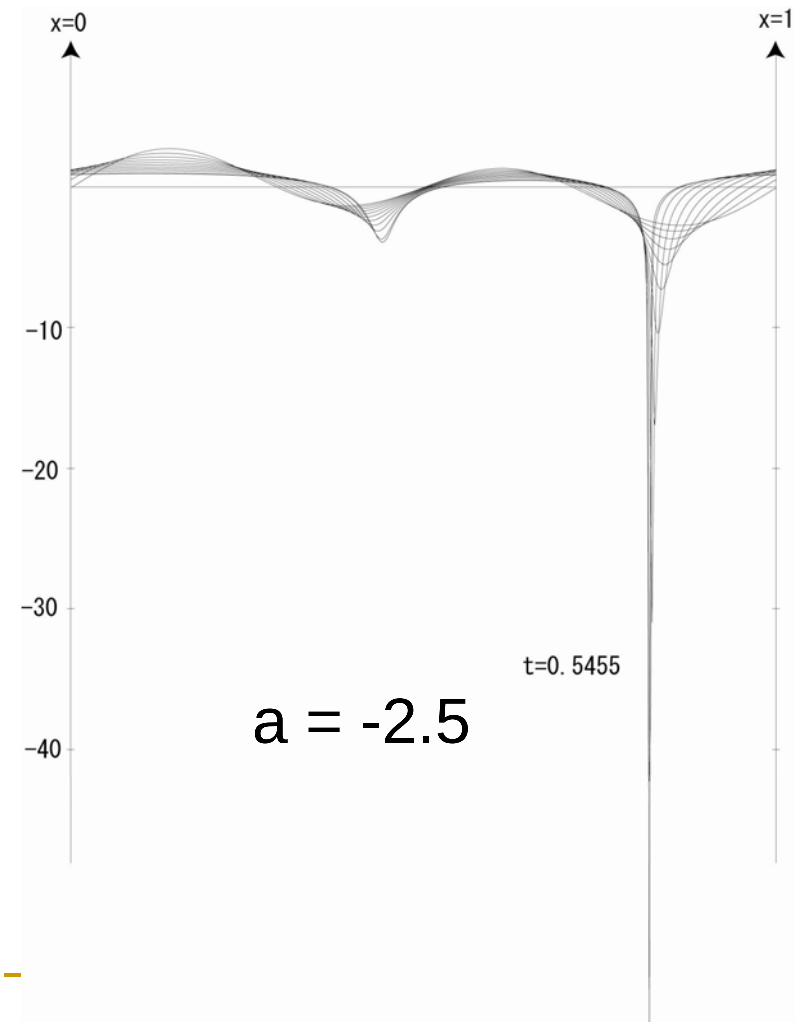


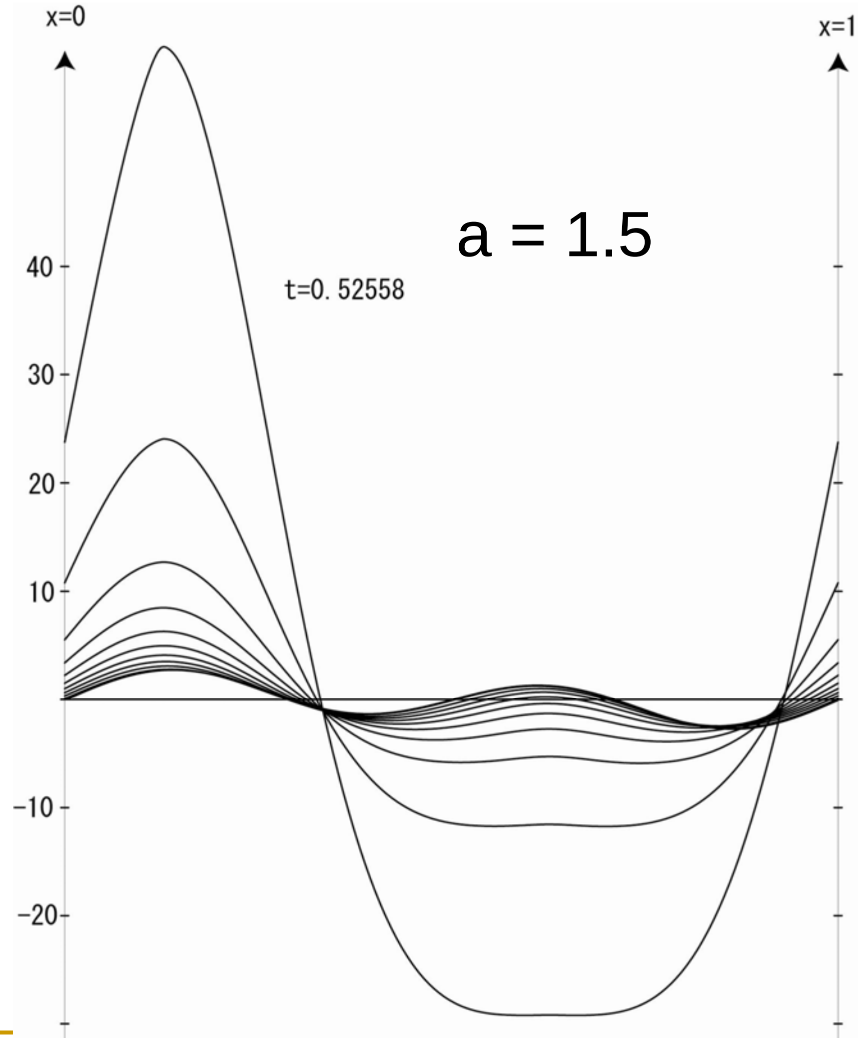
a=0



a=1.5

If 1 < a, we expect blow-up occurs even for smooth initial data.





Conclusion.

- Inviscid generalized Proudman-Johnson equation is analyzed.
- Except for the case of $1 < a < \infty$, global existence/blow-up are determined depending on a.
- Smooth initial data give us global solutions for $-1 < a < 1$. But non-smooth blow-up solutions co-exist.
- For $1 < a$, even smooth initial data are expected to lead to blow-up.

Current Status