Fractional Diffusions:

$$\widehat{(-\Delta u)} = |\xi|^2 \hat{u}$$

and.
$$[(-\Delta)^\alpha u]^\wedge = |\xi|^{2\alpha} \hat{u}.$$

or. $(\alpha < 1)$.

$$(-\Delta)^\alpha u = C_\alpha \int \frac{(u(y) - u(x))}{|x-y|^{n+2\alpha}} \, dy.$$

and.

$$u(x) = \int \frac{f(y)}{|x-y|^{n-2\alpha}} \, dy$$

$(f = (-\Delta)^\alpha u)$.

<u>Boundary operator</u>: Case $\alpha = 1/2$.

Given $u(x)$ in R^n, extend it harmonically to $[R^{n+1}]^-$ by convolving with the Poisson kern.

$$u^*(x,y) = P_y(x) * u.$$

Then $\Delta_{x,y} u^* = 0$. and

$$D_y u^*(x_0, 0) = (-\Delta)^{1/2} u(x_0).$$

<u>Heuristic</u>:

a) $D_y(D_y u^*) = -\Delta_x u.$

or

b) $$D_y u^*(x_0) = \lim \frac{1}{h} \int P_h(x_0 - x)[u(x) - u(x_0)]$$
$$= \int \frac{1}{|x_0 - x|^{n+1}} [u(x) - u(x_0)]$$

or

c) $\widehat{u^*}(\xi, y) = \hat{u}(\xi) e^{+|\xi| y}$. $(y<0)$

$D_y \widehat{u^*} = |\xi| \hat{u}(\xi)$ at $y=0$.

Variational interpretation:

(The trace of H^1 is $H^{1/2}$).

$$\int u u_y^* = \int (\hat{u})^2 |\xi| = \|u\|_{H^{1/2}}$$

but.

$$\int u u_y^* = \int (\nabla u^*)^2 \doteq \inf \int (\nabla v)^2$$

among all extensions v of u to $(\mathbb{R}^{n+1})^-$

also

$$\int \varphi (-\Delta)^{1/2} u = \int \hat{\varphi}\, |\xi|\, \hat{u} \,.$$

that is $(-\Delta^{1/2} u) = \ldots$ is the Euler Lagrange equation of $\|u\|^2_{H^{1/2}}$

<u>Boundary operator</u>: (all α's).

Three well known extensions.

a) Harmonic

b) Cylindrically symmetric harmonic:

Consider $u^*(x,y)$, y the radius in cylindrical variables: i.e.

$$x_{n+1} = y\cos\theta\ ,\quad x_{n+2} = y\sin\theta\ ,$$

u^* harmonic in $\mathbb{R}^{n+2}$. i.e:

$$\frac{1}{y}\,\mathrm{div}_{x,y}\; y\,\nabla u^* = 0.$$

(not really an extension, since u^* is harmonic, and thus real analytic across $y=0$).

c) (In 2-d)

$$y \, \mathrm{div}_{x,y} \, y^{-1} \nabla u^*.$$

Then u^* is the stream function associated to the extension b) above.

<u>Theorem</u>: (C-Silvestre- Arxiv).

Extend $u(x)$ by u^*, solution of $\frac{1}{y^s} \mathrm{div} \, y^s \nabla u^* = 0$

(i.e $\Delta_x u^* + u^*_{yy} + \frac{s}{y} u^*_y$, with $-1 < s < 1$).

(or changing variables: $z = \left(\frac{y}{1-s}\right)^{1-s}$

$$\Delta_x u^* + z^s u_{zz} = 0.\ )$$

<u>Then</u>: $(-\Delta)^{\alpha} u = u_z = y^s u_y$.

for $\alpha = \frac{1-s}{2}$.

Two important remarks:

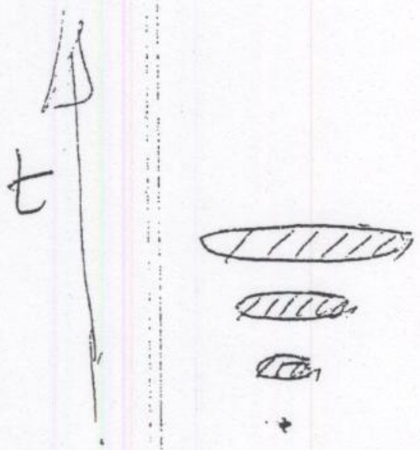
a) The properties of the extension u^* were studied by several authors in the early '80.

In particular, (for $-1<s<1$) they fall into a general class studied by Fabes, Jerison, Kenig and Serapioni that covers basic classical theory (Poincare-Sobolev-Harnack-Boundary Harnack etc)

b) If we think of this extension as a cylindrically symmetric function in $n+1+s$ dimentions, whenever u^* is "harmonic across", i.e: $(-\Delta^s u)=0$, this guides us in formulating the correct "mean value properties", "Almgreen monotonicity formula" etc.

Probabilistic Interpretation.

The heat equation:



$$\left.\begin{array}{l} \varphi_{3h} = \varphi_{2h} * \varphi_h . \\ \varphi_{2h} = \varphi_h * \varphi_h \\ \varphi_h = \frac{1}{|B_\delta|} \chi_{B_\delta} . \\ \text{Dirac's } \delta . \end{array}\right\} \text{probabilit densities.}$$

Every interval of time h, a particle $x(t)$ is kicked randomly inside $B_\delta(x(t$

The probability density satisfies

$$\varphi(x,t) - \varphi(x,t-h) = \fint \varphi(y,t-h) - \varphi(x,t-h)$$

If we divide the left by h $\sim \varphi_t$, and the right by δ^2 ; $\sim \Delta\varphi$.

If we choose $\delta^2 = h$,

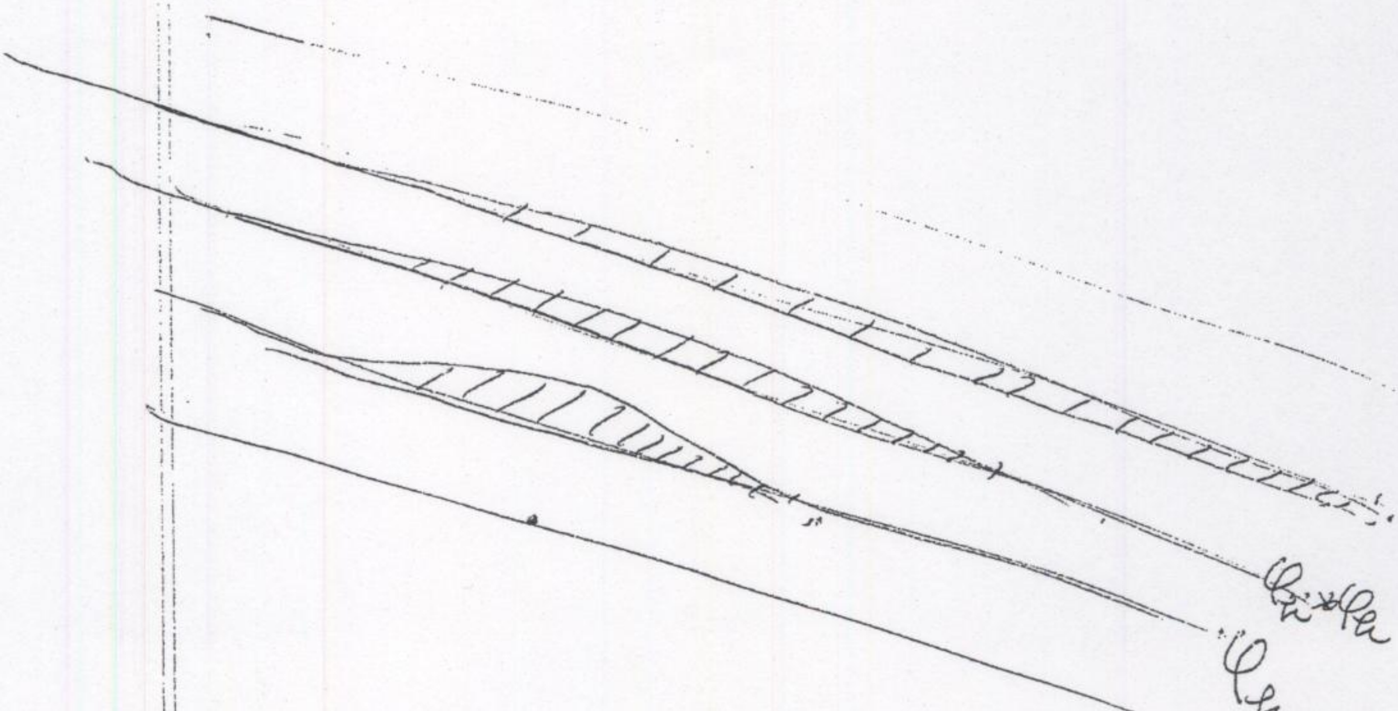
$$\Delta\varphi - \varphi_t = 0 .$$

If $\delta^2 \gg h$, as $h \to 0$, particles drift to ∞, if $\delta^2 \ll h$ particles "stay at x_0".

The central limit theorem assumes that the walk is very organized: both frequency and length of the walk are limited to a precise range (kicks are not short: $\sim (\Delta t)^{1/2}$!!). but variance (second moment) $\sim \Delta t$.

Black box approach:

Two time scales: a very small one on which a random walk takes place and the scale h at which we observe a probability density φ_h -.



For instance we may choose

$$\varphi_h = P_\delta = \frac{\delta}{(|x|^2+\delta^2)^{\frac{n+1}{2}}} .$$

Then, if we choose $\delta = h$ –

$\varphi(x,t) = P(x,t)$, the harmonic extension of Dirac's δ and φ satisfies, at least for $t = kh$., $\varphi_t = -(-\Delta)^{1/2}\varphi$ –.

Note that the "average step length" $\delta \sim \Delta t$ ... but the underlying walk is much more disorganized, resulting in an infinite second moment.

(turbulence, composite materials)

Whatever probability density we observe after an interval ~~h~~, $\varphi(x,h)$, it must satisfy the compatibility condition

$$\varphi(x,h) = \underset{k}{*}\ \varphi(x, h/k).$$

It is an <u>infinitely divisible distribution</u>.

If further we assume that φ is selfsimilar and symmetric

$$\varphi(x,h) = \frac{1}{h^{\alpha n}} f\left(\frac{x}{h^{\alpha}}\right).$$

Then φ is the fundamental solution of a fractional Laplacian, $\hat{\varphi}(\xi,t) = e^{-|\xi|^{1/\alpha} t}$

- Free boundary (phase transition) stationary problems
- Non linear problems of evolution.
  - The quasi geostrophic equation
  - Stefan like problems.
- Other directions..

F.B: · Obstacle like
  · Flame propagation.

Evolution problems:

Several cases of varying complexity:

i) Surface diffusion:

$$u_t = f(u_\nu) + \cdots$$

- Incompressible, irrotational flow
- 2 different diffusion scales

ii) Interior and boundary diffusion:

Integral boundary operators:
fractional dif + memory

Regularity of solutions to drift-diffusion equations. (with A. Vasseur)

- Motivated by quasi geostr. equation.

- Energy based method.

(De Giorgi approach)

- Lipschitz or evolving domains
- Homogenization

(no regularity of "the coefficients")

i) Let θ solve for (x,t) in $(\mathbb{R}^{n+1})^+$

$$\theta_t + v\nabla\theta + (-\Delta)^\alpha(\theta) = 0$$

$$\theta(x,0) \in L^2 \quad , \quad \operatorname{div} v = 0$$

Then: $\sup_x (\theta(x,t)) \leq \dfrac{C\,\|\theta_0\|_{L^2}}{t^{C(\alpha,n)}}$.

ii) θ bounded , v in BMO.

on $B_1 \times [0,1]$.

Then, for $\alpha = 1/2$,

$$\theta\Big|_{B_{1/2} \times [1/2,1]} \text{ is } C^\alpha.$$

Step 2: v in BMO + energy inequality with cut off.

$\Rightarrow \theta \in C^{\alpha}(x,t)$, $\Rightarrow v \in C^{\alpha}(x,t)$

$\Rightarrow \quad \theta \in C^{1,\sigma}(x,t)$ for any $\sigma < 1$

(classical solution

(Note: Kiselev-Nazarov-Volberg, in Arxiv: In 2-D, smooth periodic initial data $\Rightarrow$ smoothness for all time. Yudovich approach for regularity of Euler).

Application to quasi-geostrophic:

Step 1°:

$$v = \nabla^{\perp}\psi \qquad \text{(incompress).}$$

vorticity $\theta = (-\Delta)^{1/2}\psi$:

$$\begin{cases} \theta_t + \operatorname{div} v\theta + (-\Delta)^{1/2}\theta = 0 \\ v = R_j\theta, \quad \operatorname{div} v = 0 \end{cases}$$

Energy inequality: $\theta_\lambda = (\theta - \lambda)^+$.

$$\int (\theta_\lambda)^2(x,t_2)\,dx + \iint_{t_1}^{t_2} [D_{1/2}(\theta_\lambda)]^2\,dx\,dt$$

$$\leq \int \theta_\lambda^2(x,t_1)\,dx.$$

$\Rightarrow$ θ bounded $\Rightarrow$ $v \in BMO$.

Some basic ideas in the proof.

1) $L^2 \to L^\infty$.

- Based on the interplay between energy inequalities (function controls derivatives) and Sobolev (derivatives control function)

  With different homogeneities

- Valid for any power of the Laplacian:

$$\theta_t + v \nabla \theta + (-\Delta)^\alpha \theta = 0$$

$(0 < \alpha < 1)$.

- Sequence of truncations in t and u:

$$A_k = \int_{t_k}^{\infty} \int_{R^n} (u_k)^2 \, dX \, dt$$

with $t_k = 1 - 2^{-k}$

$u_k = (u - \lambda_k)^{+}$ and $\lambda_k = 1 - 2^{-k}$

- Iterative formula.

$$A_{k+1} \leq C^k A_k^{(1+\delta)}$$

The $1+\delta$ comes from the different homogeneities in the "function-derivative" interaction

- A_0 small $\Rightarrow A_\infty \equiv 0 \Rightarrow u < 1$ for $t > 1$

Part ii: L^∞ to C^α to $C^{1,\alpha}$.

- Local theorem, need for cut-off's
  - $u(x,t)$. in $\mathbb{R}^n \times [0,\infty]$
  - $u^*(x,y,t)$ its harmonic.

  extension in y.

  φ a cut-off (in x,y or x)

- Local energy inequality:

$$\int (\varphi u_\lambda)^2 \, dx \Big|_{t_2} + \int_{t_1}^{t_2} \iint (\nabla(\varphi u^*_\lambda))^2 \, dx\,dy\,dt$$

$$\leq \sup|\nabla\varphi| \left[\int u_\lambda^2 \, dx \Big|_{t_1} + \iint (u^*_\lambda)^2 \, dx\,dy\,dt \right.$$

- Oscilation decay lemma:

In the unit cylinder in x, y, t,

$\Gamma_1 = \{ |x| < 1,\ y < 1,\ 0 < t < 1 \}$,

we assume $|u^*| < 1$.

<u>Case 1</u>: $\|u^+\|_{L^2}$ very tiny.

(i.e $u < 0$ most of the time)

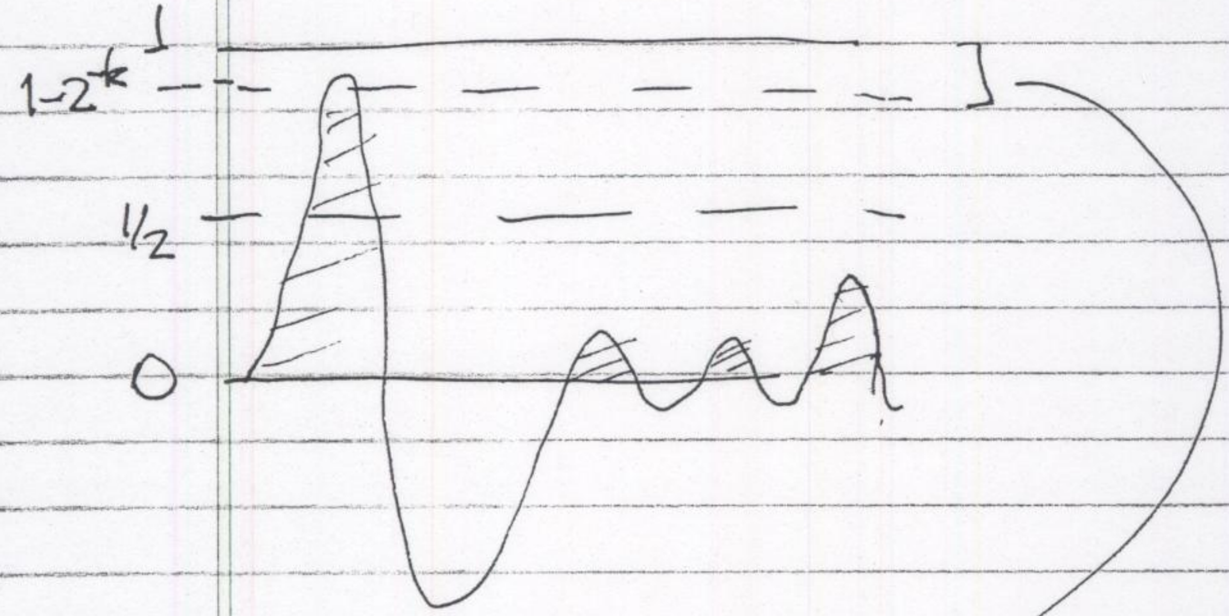
Then $u\big|_{\Gamma_{1/2}} \leq 1/2$.

(Variation of the $L^2 \to L^\infty$ argument)

<u>Case 2</u>: $\|u^+\|_{L^2}$ not so tiny.

(but at least $u < 0$ "half of the time"

Cut and renormalize



$$v_k = \frac{\left[u - (1-2^{-k}\right]^+}{2^{-k}}$$

The set where $v_k > 0$ decreases a fixed amount until v_k falls into Case i after a fixed number of steps.

De Giorgi isoperimetric inequality

$w \in H^1(B_1)$, let

$A = \{w \leq 0\}$, $B = \{w \geq 1\}$.

$D = \{0 < w < 1\}$.

then

$$|A||B| \leq \left[\int_D (\nabla w)^2\right]^{1/2} |D|^{1/2}$$

Some remarks:

- We can always renormalise

$$\int_{B_1} v(x,t) = 0 \quad \text{for every } t$$

in $[0,1]$, by substracting $x - F(t)$, with

$$\dot{F}(t) = \int v(x,t)$$

This is a small renormalization since $v \in BMO$.

- This also is used to show that, for the Q-G equation, $C^{0,\alpha}$ for some $\alpha \Rightarrow C^{1,\beta}$ $\forall \beta < 1$.

Indeed, the term

$$(v\theta)$$

has a zero of twice the order of the other terms, allowing for standard potential theory estimates.

- The oscil. normalized lemma holds for $(-\Delta)^{\alpha}$, for any α.

Ongoing projects:

- Surface flame propagation
- Optimal control and extremal operat.
- Evolution problems:
  - Flow in porous media type equations.
  - Phase transition.
  - Coupled problems (domain evolution)
- Movement by "fractional curvatures".