Quaternions and particle dynamics in the Euler fluid equations

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Collaborators

JDG & **Darryl Holm** 06: http://arxiv.org/abs/nlin.CD/0607020

JDG, **Holm, Kerr & Roulstone**: Nonlinearity **19**, 1969-83, 2006

JDG, Physica D, **166**, 17-28, 2002.

Galanti, JDG & **Heritage**; Nonlinearity **10**, 1675, 1997.

Galanti, JDG & Kerr, in *Turbulence structure & vortex dynamics*, (pp 23-34, eds: Hunt & Vassilicos, CUP 2000).

Summary of this talk

Question: Do the Euler equations possess some subtle geometric structure that guides the direction of vorticity – see Peter Constantin, *Geometric statistics in turbulence*, SIAM Reviews, **36**, 73–98.

1. Quaternions: what are they?
2. **Lagrangian particle dynamics: We find explicit equations for the Lagrangian derivatives of an ortho-normal co-ordinate system at each point in space. (JDG/Holm 06)**
3. For the $3D$-Euler equations; Ertel's Theorem shows how Euler fits naturally into this framework (JDG, Holm, Kerr & Roulstone 2006).
4. Review of work on the direction of Euler vorticity, particularly that of Constantin, Fefferman & Majda 1996; Deng, Hou & Yu 2005/6 & Chae 2006.
5. A different direction of vorticity result involving the pressure Hessian.

Lord Kelvin (William Thompson) once said:

> **Quaternions came from Hamilton after his best work had been done, & though beautifully ingenious, they have been an unmixed evil to those who have touched them in any way.**

O'Connor, J. J. & Robertson, E. F. 1998 *Sir William Rowan Hamilton*,

http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Hamilton.html

Kelvin was wrong because quaternions are now used in the computer animation, avionics & robotics industries to track objects undergoing sequences of tumbling rotations.

- **Visualizing quaternions**, by Andrew J. Hanson, MK-Elsevier, 2006.
- *Quaternions & rotation Sequences: a Primer with Applications to Orbits, Aerospace & Virtual Reality*, J. B. Kuipers, Princeton University Press, 1999.

What are quaternions? (Hamilton (1843))

Quaternions are constructed from a scalar p & a 3-vector $\boldsymbol{q}$ by forming the tetrad

$$\mathfrak{p} = [p,\, \boldsymbol{q}] = pI - \boldsymbol{q}\cdot\boldsymbol{\sigma}\,, \qquad\qquad \boldsymbol{q}\cdot\boldsymbol{\sigma} = \sum_{i=1}^{3} q_i\,\sigma_i$$

based on the Pauli spin matrices that obey the relations $\sigma_i\sigma_j = -\delta_{ij} - \epsilon_{ijk}\sigma_k$

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Thus quaternions obey the multiplication rule

$$\mathfrak{p}_1 \circledast \mathfrak{p}_2 = \left[p_1p_2 - \boldsymbol{q}_1\cdot\boldsymbol{q}_2\,,\; p_1\boldsymbol{q}_2 + p_2\boldsymbol{q}_1 + \boldsymbol{q}_1\times\boldsymbol{q}_2\right].$$

They are associative but obviously non-commutative.

Quaternions, Rotations and Cayley-Klein parameters

Let $\hat{\mathfrak{p}} = [p,\, \boldsymbol{q}]$ be a unit quaternion with inverse $\hat{\mathfrak{p}}^* = [p,\, -\boldsymbol{q}]$ with $p^2 + q^2 = 1$, which guarantees $\hat{\mathfrak{p}} \circledast \hat{\mathfrak{p}}^* = [1, 0]$. For a pure quaternion $\mathfrak{r} = [0,\, \boldsymbol{r}]$ there exists a transformation $\mathfrak{r} \to \mathfrak{r}'$

$$\mathfrak{r}' = \hat{\mathfrak{p}} \circledast \mathfrak{r} \circledast \hat{\mathfrak{p}}^* = \left[0,\, (p^2 - q^2)\boldsymbol{r} + 2p(\boldsymbol{q} \times \boldsymbol{r}) + 2\boldsymbol{q}(\boldsymbol{r} \cdot \boldsymbol{q})\right].$$

Now choose $p = \pm\cos\frac{1}{2}\theta$ and $\boldsymbol{q} = \pm\,\hat{\boldsymbol{n}}\sin\frac{1}{2}\theta$, where $\hat{\boldsymbol{n}}$ is the unit normal to $\boldsymbol{r}$

$$\mathfrak{r}' = \hat{\mathfrak{p}} \circledast \mathfrak{r} \circledast \hat{\mathfrak{p}}^* = \left[0,\, \boldsymbol{r}\cos\theta + (\hat{\boldsymbol{n}} \times \boldsymbol{r})\sin\theta\right],$$

where

$$\hat{\mathfrak{p}} = \pm\left[\cos\tfrac{1}{2}\theta,\, \hat{\boldsymbol{n}}\sin\tfrac{1}{2}\theta\right].$$

This represents a rotation by an angle θ of the 3-vector $\boldsymbol{r}$ about its normal $\hat{\boldsymbol{n}}$. **The elements of the unit quaternion $\hat{\mathfrak{p}}$ are the Cayley-Klein parameters** from which the Euler angles can be calculated. All terms are quadratic in p and q, and thus allow a double covering ($\pm$) (see Whittaker 1945).

General Lagrangian evolution equations

Consider the general Lagrangian evolution equation for a 3-vector $\boldsymbol{w}$ such that

$$\frac{D\boldsymbol{w}}{Dt} = \boldsymbol{a}(\boldsymbol{x},\, t) \qquad\qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla$$

transported by a velocity field $\boldsymbol{u}$. Define the scalar α_a the 3-vector $\boldsymbol{\chi}_a$ as

$$\alpha_a = |\boldsymbol{w}|^{-1}(\hat{\boldsymbol{w}}\cdot\boldsymbol{a})\,, \qquad\qquad \boldsymbol{\chi}_a = |\boldsymbol{w}|^{-1}(\hat{\boldsymbol{w}}\times\boldsymbol{a})\,.$$

for $|\boldsymbol{w}| \neq 0$. Via the decomposition $\boldsymbol{a} = \alpha_a\boldsymbol{w} + \boldsymbol{\chi}_a \times \boldsymbol{w}$, $|\boldsymbol{w}|$ & $\hat{\boldsymbol{w}}$ satisfy

$$\frac{D|\boldsymbol{w}|}{Dt} = \alpha_a|\boldsymbol{w}|\,, \qquad\qquad \frac{D\hat{\boldsymbol{w}}}{Dt} = \boldsymbol{\chi}_a \times \hat{\boldsymbol{w}}\,.$$

α_a is the growth rate (Constantin 1994) & $\boldsymbol{\chi}_a$ is the 'swing' rate. The 'tetrads'

$$\mathfrak{q}_a = [\alpha_a,\, \boldsymbol{\chi}_a]\,, \qquad\qquad \mathfrak{w} = [0,\, \boldsymbol{w}]\,.$$

allow us to write this as

$$\frac{D\mathfrak{w}}{Dt} = \mathfrak{q}_a \circledast \mathfrak{w}\,.$$

Theorem: (JDG/Holm 06) If $\boldsymbol{a}$ is differentiable in the Lagrangian sense s.t.

$$\frac{D\boldsymbol{a}}{Dt} = \boldsymbol{b}(\boldsymbol{x},\, t)\,,$$

(i) For for $|\boldsymbol{w}| \neq 0$, $\mathfrak{q}_a$ and $\mathfrak{q}_b$ satisfy the Ricatti equation

$$\frac{D\mathfrak{q}_a}{Dt} + \mathfrak{q}_a \circledast \mathfrak{q}_a = \mathfrak{q}_b\,;$$

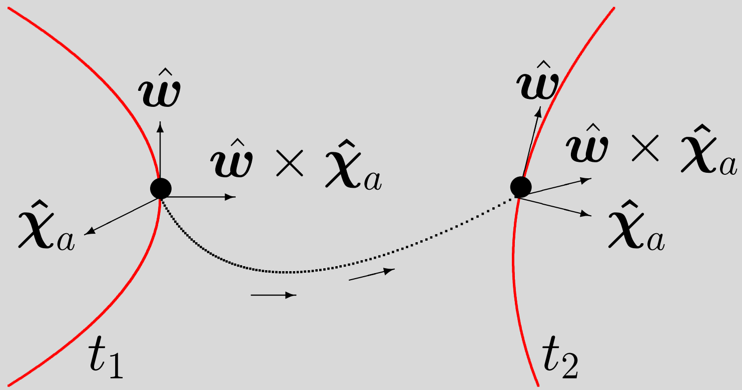
(ii) At each point $\boldsymbol{x}$ there exists an ortho-normal frame $(\hat{\boldsymbol{w}},\, \hat{\boldsymbol{\chi}}_a,\, \hat{\boldsymbol{w}}\times\hat{\boldsymbol{\chi}}_a) \in SO(3)$ whose Lagrangian time derivative is expressed as

$$\begin{aligned} \frac{D\hat{\boldsymbol{w}}}{Dt} &= \boldsymbol{\mathcal{D}}_{ab} \times \hat{\boldsymbol{w}}\,, \\ \frac{D(\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_a)}{Dt} &= \boldsymbol{\mathcal{D}}_{ab} \times (\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_a)\,, \\ \frac{D\hat{\boldsymbol{\chi}}_a}{Dt} &= \boldsymbol{\mathcal{D}}_{ab} \times \hat{\boldsymbol{\chi}}_a\,, \end{aligned}$$

where the Darboux angular velocity vector $\boldsymbol{\mathcal{D}}_{ab}$ is defined as

$$\boldsymbol{\mathcal{D}}_{ab} = \boldsymbol{\chi}_a + \frac{c_1}{\chi_a}\hat{\boldsymbol{w}}\,, \qquad c_1 = \hat{\boldsymbol{w}} \cdot (\hat{\boldsymbol{\chi}}_a \times \boldsymbol{\chi}_b)\,.$$

Lagrangian frame dynamics: tracking a particle



The dotted line represents a particle $(\bullet)$ trajectory moving from $(\boldsymbol{x}_1, t_1)$ to $(\boldsymbol{x}_2, t_2)$. The orientation of the orthonormal unit vectors

$$\{\hat{\boldsymbol{w}}, \quad \hat{\boldsymbol{\chi}}_a, \quad (\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_a)\}$$

is driven by the Darboux vector $\boldsymbol{\mathcal{D}}_{ab} = \boldsymbol{\chi}_a + \frac{c_1}{\chi_a}\hat{\boldsymbol{w}}$ where $c_1 = \hat{\boldsymbol{w}} \cdot (\hat{\boldsymbol{\chi}}_a \times \boldsymbol{\chi}_b)$.
Thus we need the 'quartet' of vectors to make this process work

$$\{\boldsymbol{u}, \, \boldsymbol{w}, \, \boldsymbol{a}, \, \boldsymbol{b}\} \, .$$

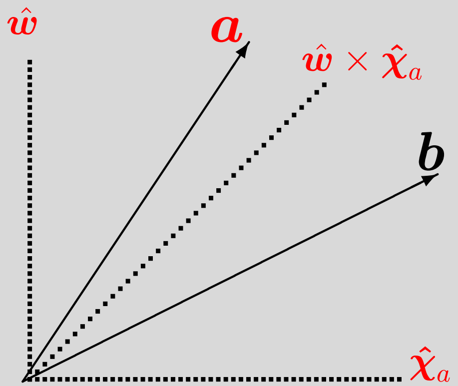
Proof: (i) It is clear that (with $\mathfrak{q}_b = [\alpha_b,\, \boldsymbol{\chi}_b]$)

$$\frac{D^2\mathfrak{w}}{Dt^2} = [0,\, \boldsymbol{b}] = \mathfrak{q}_b \circledast \mathfrak{w}\,.$$

Compatibility between this and the $\mathfrak{q}$-equation means that

$$\left(\frac{D\mathfrak{q}_a}{Dt} + \mathfrak{q}_a \circledast \mathfrak{q}_a - \mathfrak{q}_b\right) \circledast \mathfrak{w} = 0\,,$$

(ii) Now consider the ortho-normal frame $(\hat{\boldsymbol{w}},\, \hat{\boldsymbol{\chi}}_a,\, \hat{\boldsymbol{w}}\times\hat{\boldsymbol{\chi}}_a)$ as in the Figure below.



The evolution of $\boldsymbol{\chi}_a$ comes from

$$\frac{D\mathfrak{q}_a}{Dt} + \mathfrak{q}_a \circledast \mathfrak{q}_a = \mathfrak{q}_b\,,$$

and gives

$$\frac{D\boldsymbol{\chi}_a}{Dt} = -2\alpha_a\boldsymbol{\chi}_a + \boldsymbol{\chi}_b\,.$$

$\boldsymbol{b}$ can be expressed in this ortho-normal frame as the linear combination

$$\boldsymbol{b} = |\boldsymbol{w}|\left[\alpha_b\,\hat{\boldsymbol{w}} + c_1\hat{\boldsymbol{\chi}}_a + c_2(\hat{\boldsymbol{w}}\times\hat{\boldsymbol{\chi}}_a)\right],$$
$$\boldsymbol{\chi}_b = c_1(\hat{\boldsymbol{w}}\times\hat{\boldsymbol{\chi}}_a) - c_2\hat{\boldsymbol{\chi}}_a\,,$$

where $c_1 = \hat{\boldsymbol{w}}\cdot(\hat{\boldsymbol{\chi}}_a\times\boldsymbol{\chi}_b)$ and $c_2 = -(\hat{\boldsymbol{\chi}}_a\cdot\boldsymbol{\chi}_b)$. From the Ricatti equation for the tetrad $\mathfrak{q}_a = [\alpha_a,\,\boldsymbol{\chi}_a]$ (where $\chi_a = |\boldsymbol{\chi}_a|$)

$$\frac{D\boldsymbol{\chi}_a}{Dt} = -2\alpha_a\boldsymbol{\chi}_a + \boldsymbol{\chi}_b\,, \qquad \Rightarrow \qquad \frac{D\chi_a}{Dt} = -2\alpha_a\chi_a - c_2\,,$$

There follows

$$\frac{D\hat{\boldsymbol{\chi}}_a}{Dt} = c_1\chi_a^{-1}(\hat{\boldsymbol{w}}\times\hat{\boldsymbol{\chi}}_a)\,, \qquad \frac{D(\hat{\boldsymbol{w}}\times\hat{\boldsymbol{\chi}}_a)}{Dt} = \chi_a\,\hat{\boldsymbol{w}} - c_1\chi_a^{-1}\hat{\boldsymbol{\chi}}_a\,,$$

which, together with

$$\frac{D\hat{\boldsymbol{w}}_a}{Dt} = \boldsymbol{\chi}_a\times\hat{\boldsymbol{w}}\,,$$

can be re-expressed in terms of the Darboux vector $\boldsymbol{\mathcal{D}}_a = \boldsymbol{\chi}_a + \frac{c_1}{\chi_a}\hat{\boldsymbol{w}}$. ■

Ertel's Theorem & the $3D$ Euler equations

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\omega} \qquad \text{Euler in vorticity format}$$

Theorem: (Ertel 1942) If $\boldsymbol{\omega}$ satisfies the $3D$ incompressible Euler equations then any arbitrary differentiable μ satisfies

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla\mu) = \boldsymbol{\omega}\cdot\nabla\left(\frac{D\mu}{Dt}\right) \qquad \Longrightarrow \qquad \left[\frac{D}{Dt}, \boldsymbol{\omega}\cdot\nabla\right] = 0\,.$$

Proof: Consider $\boldsymbol{\omega}\cdot\nabla\mu \equiv \omega_i\,\mu_{,i}$

$$\begin{aligned}\frac{D}{Dt}(\omega_i\,\mu_{,i}) &= \frac{D\omega_i}{Dt}\mu_{,i} + \omega_i\left\{\frac{\partial}{\partial x_i}\left(\frac{D\mu}{Dt}\right) - u_{k,i}\mu_{,k}\right\}\\ &= \underbrace{\{\omega_j u_{i,j}\,\mu_{,i} - \omega_i u_{k,i}\,\mu_{,k}\}}_{zero\ under\ summation} + \omega_i\frac{\partial}{\partial x_i}\left(\frac{D\mu}{Dt}\right)\end{aligned}$$

In characteristic (Lie-derivative) form, $\boldsymbol{\omega}\cdot\frac{\partial}{\partial\boldsymbol{x}}(t) = \boldsymbol{\omega}\cdot\frac{\partial}{\partial\boldsymbol{x}}(0)$ is a Lagrangian invariant (Cauchy 1859) and is “frozen in”.

Various references

- Ertel; *Ein Neuer Hydrodynamischer Wirbelsatz*, Met. Z. **59**, 271-281, (1942).
- Truesdell & Toupin, Classical Field Theories, *Encyclopaedia of Physics III/1*, ed. S. Flugge, Springer (1960).
- Ohkitani; Phys. Fluids, **A5**, 2576, (1993).
- Kuznetsov & Zakharov; *Hamiltonian formalism for nonlinear waves*, Physics Uspekhi, **40** (11), 1087– 1116 (1997).
- Bauer's thesis 2000 (ETH-Berlin); *Gradient entropy vorticity, potential vorticity and its history*.
- Viudez; *On the relation between Beltrami's material vorticity and Rossby-Ertel's Potential*, J. Atmos. Sci. (2001).

Ohkitani's result & the pressure Hessian

Define the Hessian matrix of the pressure

$$P = \{p_{,ij}\} = \left\{\frac{\partial^2 p}{\partial x_i\,\partial x_j}\right\}$$

then Ohkitani took $\mu = u_i$ (Phys. Fluids, **A5**, 2576, 1993).

Result: The vortex stretching vector $\boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\omega}$ obeys

$$\frac{D(\boldsymbol{\omega}\cdot\nabla\boldsymbol{u})}{Dt} = \frac{D(S\boldsymbol{\omega})}{Dt} = \boldsymbol{\omega}\cdot\nabla\left(\frac{D\boldsymbol{u}}{Dt}\right) = -P\,\boldsymbol{\omega}$$

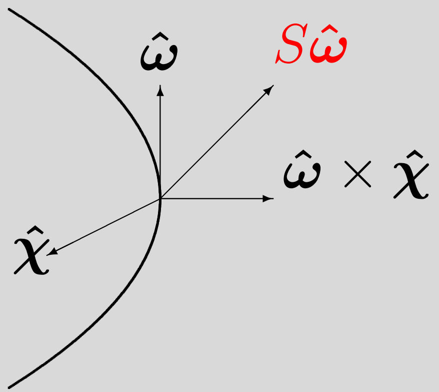
Thus for Euler, via Ertel's Theorem, we have the identification:

$$\boldsymbol{w} \equiv \boldsymbol{\omega} \qquad \boldsymbol{a} \equiv \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\omega} \qquad \boldsymbol{b} \equiv -P\boldsymbol{\omega}$$

with a quartet

$$(\boldsymbol{u},\,\boldsymbol{w},\,\boldsymbol{a},\,\boldsymbol{b}) \equiv (\boldsymbol{u},\,\boldsymbol{\omega},\,S\boldsymbol{\omega},\,-P\boldsymbol{\omega})\,.$$

Euler: the variables $\alpha(\boldsymbol{x}, t)$ and $\boldsymbol{\chi}(\boldsymbol{x}, t)$



$$S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi} \times \hat{\boldsymbol{\omega}}$$

See JDG, Holm, Kerr & Roulstone 2006.

$$(\alpha_a) \qquad \alpha = \hat{\boldsymbol{\omega}} \cdot S\hat{\boldsymbol{\omega}} \qquad\qquad \boldsymbol{\chi} = \hat{\boldsymbol{\omega}} \times S\hat{\boldsymbol{\omega}} \qquad (\boldsymbol{\chi}_a)$$

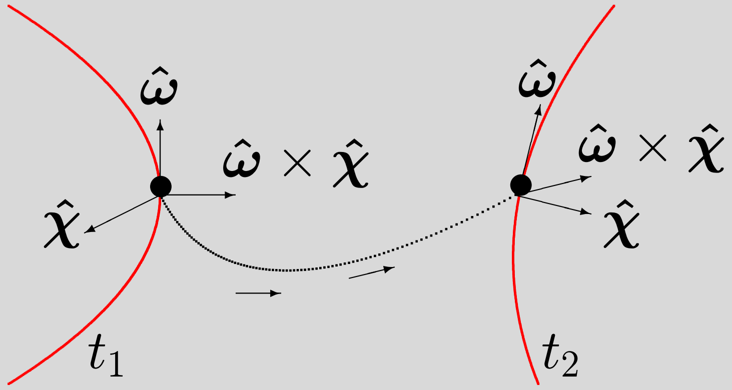
$$(-\alpha_b) \qquad \alpha_p = \hat{\boldsymbol{\omega}} \cdot P\hat{\boldsymbol{\omega}} \qquad\qquad \boldsymbol{\chi}_p = \hat{\boldsymbol{\omega}} \times P\hat{\boldsymbol{\omega}} \qquad (-\boldsymbol{\chi}_b)$$

$$\mathfrak{q} = [\alpha,\, \boldsymbol{\chi}] \qquad\qquad \mathfrak{q}_b = -\mathfrak{q}_p = -[\alpha_p,\, \boldsymbol{\chi}_p]$$

$$\boxed{\frac{D\mathfrak{q}}{Dt} + \mathfrak{q} \circledast \mathfrak{q} + \mathfrak{q}_p = 0},$$

constrained by $TrP = \Delta p = -u_{i,j}u_{j,i} = \frac{1}{2}\omega^2 - TrS^2$.

Lagrangian frame dynamics: tracking an Euler fluid particle



The dotted line represents the fluid packet $(\bullet)$ trajectory moving from $(\boldsymbol{x}_1, t_1)$ to $(\boldsymbol{x}_2, t_2)$. The orientation of the orthonormal unit vectors

$$\{\hat{\boldsymbol{\omega}},\;\; \hat{\boldsymbol{\chi}},\;\; (\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{\chi}})\}$$

is driven by the Darboux vector

$$\boldsymbol{\mathcal{D}} = \boldsymbol{\chi} + \frac{c_1}{\chi}\hat{\boldsymbol{\omega}}\,, \qquad\qquad c_1 = -\hat{\boldsymbol{\omega}} \cdot (\hat{\boldsymbol{\chi}} \times \boldsymbol{\chi}_p)\,.$$

Thus the pressure Hessian within c_1 drives the Darboux vector $\boldsymbol{\mathcal{D}}$.

The α and $\boldsymbol{\chi}$ equations

In terms of α and $\boldsymbol{\chi}$, the Ricatti equation for $\mathfrak{q}$

$$\frac{D\mathfrak{q}_a}{Dt} + \mathfrak{q}_a \circledast \mathfrak{q}_a = \mathfrak{q}_b\,;$$

becomes

$$\frac{D\alpha}{Dt} = \chi^2 - \alpha^2 - \alpha_p\,, \qquad\qquad \frac{D\boldsymbol{\chi}}{Dt} = -2\alpha\boldsymbol{\chi} - \boldsymbol{\chi}_p\,.$$

(Galanti, JDG & Heritage; Nonlinearity **10**, 1675, 1997). Stationary values are

$$\alpha = \gamma_0\,, \qquad \boldsymbol{\chi} = \mathbf{o}\,, \qquad \alpha_p = -\gamma_0^2$$

which correspond to **Burgers'-like vortices**.

When tubes & sheets bend & tangle then $\boldsymbol{\chi} \neq 0$ and $\mathfrak{q}$ becomes a full tetrad driven by $\mathfrak{q}_p$ which is coupled back through the elliptic pressure condition.

Note: Off-diagonal elements of P change rapidly near intense vortical regions across which χ_p and α_p change rapidly.

Phase plane

On Lagrangian trajectories, the $\alpha - \boldsymbol{\chi}$ equations become

$$\frac{\partial \alpha}{\partial t} = \chi^2 - \alpha^2 - \alpha_p\,, \qquad \frac{\partial \chi}{\partial t} = -2\alpha\chi + C_p\,.$$

where $C_p = -\hat{\boldsymbol{\chi}} \cdot \boldsymbol{\chi}_p$.

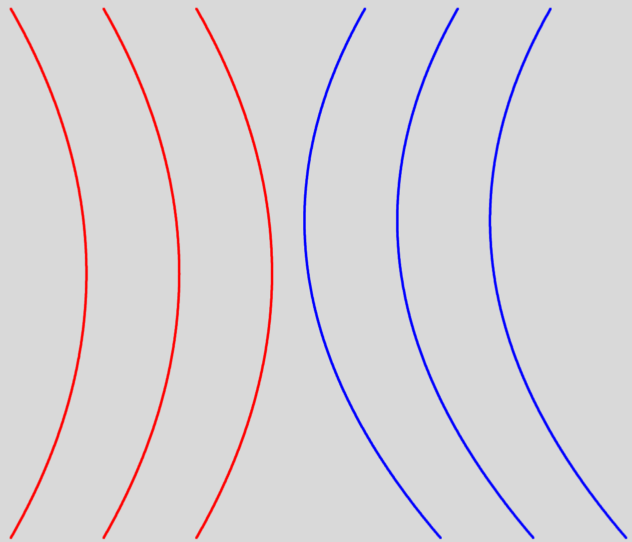
In regions of the $\alpha - \chi$ phase plane where $\alpha_p = const, \quad C_p = const$ there are 2 critical points:

$$(\alpha,\, \chi) = (\pm\alpha_0,\, \chi_0) \qquad 2\alpha_0^2 = \alpha_p + [\alpha_p^2 + C_p^2]^{1/2}$$

- The critical point in the LH-half-plane $(-\alpha_0,\, \chi_0)$ is an unstable spiral;
- The critical point in the RH-half-plane is $(\alpha_0,\, \chi_0)$ is a stable spiral.

The next few slides: remarks on the "direction of vorticity" in Euler

1. The BKM theorem



2. • The work of Constantin, Fefferman & Majda 1996 and Constantin 1994
   • The work of Deng, Hou & Yu 2005/6
   • Can our quaternionic Ricatti equation give anything in terms of P?

The Beale-Kato-Majda Theorem (CMP **94**, 61-6, 1984)

Theorem: *There exists a global solution of the Euler equations* $\boldsymbol{u} \in C([0,\,\infty]; H^s) \cap C^1([0,\,\infty]; H^{s-1})$ *for* $s \geq 3$ *if, for every* $t^* > 0$,

$$\int_0^{t^*} \|\boldsymbol{\omega}(\tau)\|_{L^\infty(\Omega)}\, d\tau < \infty\,.$$

The proof is based on $\|\nabla \boldsymbol{u}\|_\infty \leq c\,\|\boldsymbol{\omega}\|_\infty\,[1 + \log H_3]$.

Thus one needs to numerically monitor only $\int_0^{t^*} \|\boldsymbol{\omega}(\tau)\|_\infty\, d\tau$.

Corollary: If a singularity is observed in a numerical experiment of the form $\|\boldsymbol{\omega}\|_\infty \sim (t^* - t)^{-\beta}$ then β must lie in the range $\beta \geq 1$ for the singularity to be genuine & not an artefact of the numerical calculation.

Constantin, Fefferman & Majda; Comm PDEs, **21**, 559-571, 1996

The image $\boldsymbol{W}_t$ of a set $\boldsymbol{W}_0$ is given by $\boldsymbol{W}_t = \boldsymbol{X}(t,\, \boldsymbol{W}_0)$. $\boldsymbol{W}_0$ is said to be *smoothly directed* if there exists a length $\rho > 0$ and a ball $0 < r < \frac{1}{2}\rho$ such that:

1. $\hat{\boldsymbol{\omega}}(\cdot,\, t)$ has a Lipschitz extn to the ball of radius 4ρ centred at $\boldsymbol{X}(\boldsymbol{q},\, t)$ &
$$M = \lim_{t\to T}\, \sup_{\boldsymbol{q}\in\boldsymbol{W}_0^*} \int_0^t \|\nabla\hat{\boldsymbol{\omega}}(\cdot,\, t)\|^2_{L^\infty(B_{4\rho})}\, dt < \infty\,.$$
i.e. the direction of vorticity is well-behaved in the nbhd of a set of trajectories.
2. The condition $\sup_{B_{3r}(\boldsymbol{W}_t)} |\boldsymbol{\omega}(\boldsymbol{x},\, t)| \leq m \sup_{B_r(\boldsymbol{W}_t)} |\boldsymbol{\omega}(\boldsymbol{x},\, t)|$ holds for all $t \in [0,\, T)$ with $m =$ const > 0; i.e. this nbhd captures large & growing vorticity but not so that it overlaps with another similar region & $\sup_{B_{4r}(\boldsymbol{W}_t)} |\boldsymbol{u}(\boldsymbol{x},\, t)| \leq U(t) := \sup_{\boldsymbol{x}} |\boldsymbol{u}(\boldsymbol{x},\, t)| < \infty$ (Cordoba & Fefferman 2001; for tubes).

Theorem: (CFM 1996) *Assume that $\boldsymbol{W}_0$ is smoothly directed as in (i)–(ii). Then $\exists$ a time $\tau > 0$ & a constant Γ s.t. for any $0 \leq t_0 < T$ and $0 \leq t - t_0 \leq \tau$*
$$\sup_{B_r(\boldsymbol{W}_t)} |\boldsymbol{\omega}(\boldsymbol{x},\, t)| \leq \Gamma \sup_{B_\rho(\boldsymbol{W}_t)} |\boldsymbol{\omega}(\boldsymbol{x},\, t_0)|\,.$$

The work of Deng, Hou & Yu; Comm PDEs, **31**, 293–306, 2006

Consider a family of vortex line segments L_t in a region of max-vorticity. Denote by $L(t)$ the arc length of L_t, $\hat{\boldsymbol{n}}$ the unit normal & κ the curvature. DHY define

$$U_{\hat{\omega}}(t) \equiv \max_{\boldsymbol{x},\boldsymbol{y}\in L_t} \left|(\boldsymbol{u}\cdot\hat{\boldsymbol{\omega}})(\boldsymbol{x},t) - (\boldsymbol{u}\cdot\hat{\boldsymbol{\omega}})(\boldsymbol{y},t)\right|,$$

$U_n(t) \equiv \max_{L_t} |\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|$, and $M(t) \equiv \max\left(\|\nabla\cdot\hat{\boldsymbol{\omega}}\|_{L^\infty(L_t)}, \|\kappa\|_{L^\infty(L_t)}\right)$.

Theorem: *(Deng, Hou & Yu 06)***:** *Let $A, B \in (0,1)$ with $B = 1 - A$, and C_0 be a positive constant. If*

1. $U_{\hat{\omega}}(t) + U_n(t) \lesssim (T-t)^{-A}$,
2. $M(t)L(t) \leq C_0$,
3. $L(t) \gtrsim (T-t)^B$,

then there will be no blow-up up to time T.

Also J. Deng, T. Y. Hou & X. Yu; Comm. PDEs, **30**, 225-243, 2005.

Using the pressure Hessian

(see also Chae: $\int_0^T \|S\hat{\boldsymbol{\omega}}\cdot P\hat{\boldsymbol{\omega}}\|_\infty\,d\tau < \infty$; *Comm. P&A-M.*, **109**, 1–21, 2006).

Theorem: (JDG, Holm, Kerr & Roulstone 06) **:** $\exists$ *a global solution of the Euler equations,* $\boldsymbol{u}\in C([0,\,\infty];H^s)\cap C^1([0,\,\infty];H^{s-1})$ *for* $s\geq 3$ *if*

$$\int_0^T \|\boldsymbol{\chi}_p\|_{L^\infty(\mathbb{D})}\,d\tau < \infty\,,$$

with the exception of when $\hat{\boldsymbol{\omega}}$ *becomes collinear with an e-vec of* P *at* $t=T$*.*

Proof: With $|S\hat{\boldsymbol{w}}|^2 = \alpha^2 + \boldsymbol{\chi}^2$,

$$\frac{D|S\hat{\boldsymbol{w}}|}{Dt} \leq -\alpha|S\hat{\boldsymbol{w}}| + \frac{|\alpha||\alpha_p| + |\boldsymbol{\chi}||\boldsymbol{\chi}_p|}{(\alpha^2+\boldsymbol{\chi}^2)^{1/2}}\,.$$

Because $D|\boldsymbol{\omega}|/Dt = \alpha|\boldsymbol{\omega}|$, our concern is with $\alpha\geq 0$

$$\frac{D|S\hat{\boldsymbol{w}}|}{Dt} \leq |\alpha_p| + |\boldsymbol{\chi}_p|\,.$$

Possible that $|P\hat{\boldsymbol{\omega}}|$ blows up simultaneously as the angle between $\hat{\boldsymbol{\omega}}$ and $P\hat{\boldsymbol{\omega}}\to 0$ thus keeping $\boldsymbol{\chi}_p$ finite; i.e. $\int_0^t \|\boldsymbol{\chi}_p\|_{L^\infty(\mathbb{D})}d\tau < \infty$ but $\int_0^t \|\alpha_p\|_{L^\infty(\mathbb{D})}d\tau \to \infty$.

Frame dynamics & the Frenet-Serret equations

With $\hat{\boldsymbol{w}}$ as the unit tangent vector, $\hat{\boldsymbol{\chi}}$ as the unit bi-normal and $\hat{\boldsymbol{w}}\times\hat{\boldsymbol{\chi}}$ as the unit principal normal, the matrix N can be formed

$$N = \left(\hat{\boldsymbol{w}}^T,\, (\hat{\boldsymbol{w}}\times\hat{\boldsymbol{\chi}})^T,\, \hat{\boldsymbol{\chi}}^T\right)\,,$$

with

$$\frac{DN}{Dt} = GN\,, \qquad\qquad G = \begin{pmatrix} 0 & -\chi_a & 0 \\ \chi_a & 0 & -c_1\chi_a^{-1} \\ 0 & c_1\chi_a^{-1} & 0 \end{pmatrix}.$$

The Frenet-Serret equations for a space-curve are

$$\frac{dN}{ds} = FN \qquad \text{where} \qquad F = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix},$$

where κ is the curvature and τ is the torsion.

The arc-length derivative d/ds is defined by

$$\frac{d}{ds} = \hat{\boldsymbol{\omega}} \cdot \nabla \,.$$

The evolution of the curvature κ and torsion τ may be obtained from Ertel's theorem expressed as the commutation of operators $\left[\frac{D}{Dt},\, \boldsymbol{\omega}\cdot\nabla\right] = 0$

$$\alpha_a \frac{d}{ds} + \left[\frac{D}{Dt},\, \frac{d}{ds}\right] = 0\,.$$

This commutation relation immediately gives

$$\alpha_a F + \frac{DF}{Dt} = \frac{dG}{ds} + [G,\, F]\,.$$

Thus Ertel's Theorem gives explicit evolution equations for the curvature κ and torsion τ that lie within the matrix F and relates them to c_1, χ_a and α_a.

Mixing

Consider a passive vector line-element $\boldsymbol{\delta\ell}$ in a flow transported by an independent velocity field $\boldsymbol{u}$. For *small* $\boldsymbol{\delta\ell}$ we have the same equations as Euler for $\boldsymbol{\omega}$

$$\frac{D\boldsymbol{\delta\ell}}{Dt} = \boldsymbol{\delta\ell}\cdot\nabla\boldsymbol{u} \qquad\qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla$$

Following the analogy with Euler, Ertel's Theorem holds so there is a $\boldsymbol{b}$-field:

$$\frac{D(\boldsymbol{\delta\ell}\cdot\nabla\boldsymbol{u})}{Dt} = \boldsymbol{\delta\ell}\cdot\nabla\left(\frac{D\boldsymbol{u}}{Dt}\right)$$

$D\boldsymbol{u}/Dt$ represents any dynamics one wishes to impose on the problem. Thus all the conditions hold for Theorem 1:

1. $\boldsymbol{w} = \boldsymbol{\delta\ell}$
2. $\boldsymbol{a} = \boldsymbol{\delta\ell}\cdot\nabla\boldsymbol{u}$
3. $\boldsymbol{b} = \boldsymbol{\delta\ell}\cdot\nabla\left(\frac{D\boldsymbol{u}}{Dt}\right) \longrightarrow$ a Ricatti equation plus an ortho-normal frame ...

Ideal MHD

Consider a magnetic field $\boldsymbol{B}$ coupled to a fluid ($\text{div}\,\boldsymbol{u} = 0 = \text{div}\,\boldsymbol{B}$)

$$\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \nabla p \qquad\qquad \frac{D\boldsymbol{B}}{Dt} = \boldsymbol{B}\cdot\nabla\boldsymbol{u}$$

Defining Elsasser variables with $\pm$-material derivatives (two time-clocks)

$$\boldsymbol{v}^{\pm} = \boldsymbol{u} \pm \boldsymbol{B}\,; \qquad\qquad \frac{D^{\pm}}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v}^{\pm}\cdot\nabla$$

the magnetic field $\boldsymbol{B}$ and $\boldsymbol{v}^{\pm}$ satisfy with $\text{div}\,\boldsymbol{v}^{\pm} = 0$

$$\frac{D^{\pm}\boldsymbol{v}^{\mp}}{Dt} = -\nabla p\,; \qquad\qquad \frac{D^{\pm}\boldsymbol{B}}{Dt} = \boldsymbol{B}\cdot\nabla\boldsymbol{v}^{\pm}$$

Moffatt (1985) suggested that $\boldsymbol{B}$ takes the place of $\boldsymbol{\omega}$ in ideal MHD.

Ertel's Theorem (proof omitted) for this system is

$$\frac{D^{\mp}(\boldsymbol{B}\cdot\nabla\boldsymbol{v}^{\pm})}{Dt} = -P\boldsymbol{B}\,.$$

With two time-clocks, we have the correspondence

$$\boldsymbol{w} \equiv \boldsymbol{B} \qquad \boldsymbol{a}^{\pm} \equiv \boldsymbol{B}\cdot\nabla\boldsymbol{v}^{\pm} \qquad \boldsymbol{b} \equiv -P\boldsymbol{B}$$

$$\alpha_{pb} = \hat{\boldsymbol{B}}\cdot P\hat{\boldsymbol{B}} \qquad \boldsymbol{\chi}_{pb} = \hat{\boldsymbol{B}}\times P\hat{\boldsymbol{B}}$$

Define tetrads $\mathfrak{q}^{\pm}$ and $\mathfrak{q}_{pb}$ as follows

$$\mathfrak{q}^{\pm} = \left[\alpha^{\pm},\,\boldsymbol{\chi}^{\pm}\right] \qquad \mathfrak{q}_{pb} = \left[\alpha_{pb},\,\boldsymbol{\chi}_{pb}\right].$$

The tetrads $\mathfrak{q}^{\pm}$ satisfy the compatibility relation

$$\boxed{\frac{D^{\mp}\mathfrak{q}^{\pm}}{Dt} + \mathfrak{q}^{\pm}\circledast\mathfrak{q}^{\mp} + \mathfrak{q}_{pb} = 0}$$

MHD-Lagrangian frame dynamics

We have 2 sets of orthonormal vectors $\hat{\boldsymbol{B}}$, $(\hat{\boldsymbol{B}} \times \hat{\boldsymbol{\chi}}^{\pm})$, $\hat{\boldsymbol{\chi}}^{\pm}$ acted on by their opposite Lagrangian time derivatives.

$$\begin{aligned}\frac{D^{\mp}\hat{\boldsymbol{B}}}{Dt} &= \boldsymbol{\mathcal{D}}^{\mp} \times \hat{\boldsymbol{B}}\,,\\ \frac{D^{\mp}(\hat{\boldsymbol{B}} \times \hat{\boldsymbol{\chi}}^{\pm})}{Dt} &= \boldsymbol{\mathcal{D}}^{\mp} \times (\hat{\boldsymbol{B}} \times \hat{\boldsymbol{\chi}}^{\pm})\,,\\ \frac{D^{\mp}\hat{\boldsymbol{\chi}}^{\pm}}{Dt} &= \boldsymbol{\mathcal{D}}^{\mp} \times \hat{\boldsymbol{\chi}}^{\pm}\end{aligned}$$

where the pair of Darboux vectors $\boldsymbol{\mathcal{D}}^{\mp}$ are defined as

$$\boldsymbol{\mathcal{D}}^{\mp} = \boldsymbol{\chi}^{\mp} - \frac{c_1^{\mp}}{\chi^{\mp}}\hat{\boldsymbol{B}}\,, \qquad c_1^{\mp} = \hat{\boldsymbol{B}} \cdot \left[\hat{\boldsymbol{\chi}}^{\pm} \times (\boldsymbol{\chi}_{pb} + \alpha^{\pm}\boldsymbol{\chi}^{\mp})\right].$$