Global existence in nonlinear elastodynamics

Thomas C. Sideris

University of California, Santa Barbara

http://www.math.ucsb.edu/∼sideris

Abstract

Discuss the equations of motion for homogeneous, isotropic elastic bodies, in the compressible and incompressible case.

Present results on global existence of solutions to the initial value problem, under small deformations and appropriate structural conditions.

Road Map

Viscoelasticity	$\longrightarrow$	Incompressible Viscoelasticity
$\downarrow$		$\downarrow$
Elasticity	$\longrightarrow$	Incompressible Elasticity

Road Map

Viscoelasticity	$\longrightarrow$	Incompressible Viscoelasticity
$\downarrow$		$\downarrow$
Elasticity	$\longrightarrow$	Incompressible Elasticity

$$\left(\begin{array}{ccc} \text{Compressible Navier-Stokes} & \longrightarrow & \text{Navier-Stokes} \\ \downarrow & & \downarrow \\ \text{Compressible Euler} & \longrightarrow & \text{Incompressible Euler} \end{array} \right)$$

Cast of Characters

- Deformation – basic unknown

$$\varphi : \mathbb{R}^+ \times \Omega \to \mathbb{R}^3$$

Orientation-preserving diffeomorphism carrying material points to their spatial position at a given time.

$$(t, X) \mapsto x = \varphi(t, X)$$

Cast of Characters

- Deformation – basic unknown

$$\varphi : \mathbb{R}^+ \times \Omega \to \mathbb{R}^3$$

Orientation-preserving diffeomorphism carrying material points to their spatial position at a given time.

$$(t, X) \mapsto x = \varphi(t, X)$$

- Reference configuration – assume $\Omega = \mathbb{R}^3$. No boundaries.

- Deformation gradient

$$F(t,X) = D_X\varphi(t,X), \quad \det F > 0$$

$$F^i_\ell = D_\ell \varphi^i$$

- Deformation gradient

$$F(t,X) = D_X\varphi(t,X), \quad \det F > 0$$

$$F^i_\ell = D_\ell \varphi^i$$

- Strain energy function (homegeneous)

$$W : GL(3,\mathbb{R}) \to \mathbb{R}^+$$

$$F \mapsto W(F)$$

Equations of motion

Lagrangian

$$\mathcal{L}[\varphi] = \iint [\tfrac{1}{2}\bar{\rho}|D_t\varphi|^2 - W(D_X\varphi)]dX\,dt$$

($\bar{\rho}$ is the constant reference density)

Equations of motion

Lagrangian

$$\mathcal{L}[\varphi] = \iint [\tfrac{1}{2}\bar{\rho}|D_t\varphi|^2 - W(D_X\varphi)]\,dX\,dt$$

($\bar{\rho}$ is the constant reference density)

Principle of stationary action: find formal critical points

$$\delta\mathcal{L}[\varphi] = 0$$

Equations of motion

Lagrangian

$$\mathcal{L}[\varphi] = \iint [\tfrac{1}{2}\bar{\rho}|D_t\varphi|^2 - W(D_X\varphi)]dX\,dt$$

($\bar{\rho}$ is the constant reference density)

Principle of stationary action: find formal critical points

$$\delta\mathcal{L}[\varphi] = 0$$

PDEs

$$D_t^2\varphi^i - D_\ell\left[\frac{\partial W}{\partial F_\ell^i}(D\varphi)\right] = 0$$

Equations of motion

Lagrangian

$$\mathcal{L}[\varphi] = \iint [\tfrac{1}{2}\bar{\rho}|D_t\varphi|^2 - W(D_X\varphi)]\,dX\,dt$$

($\bar{\rho}$ is the constant reference density)

Principle of stationary action: find formal critical points

$$\delta\mathcal{L}[\varphi] = 0$$

PDEs

$$D_t^2\varphi^i - D_\ell\left[\frac{\partial W}{\partial F_\ell^i}(D\varphi)\right] = 0$$

(Summation convention.)

Small displacements

We will only consider small displacements from the equilibrium reference configuration

$$|\varphi(t, X) - X| \ll 1,$$

which guarantees the invertibility of $\varphi(t, \cdot)$.

Small displacements

We will only consider small displacements from the equilibrium reference configuration

$$|\varphi(t, X) - X| \ll 1,$$

which guarantees the invertibility of $\varphi(t, \cdot)$.

(More generally, it is possible to perturb from a simple 'pre-stressed' state σX, $\sigma > 0$.)

PDEs for the displacement

Write

$$\varphi(t, X) = X + u(t, X), \quad F = I + G$$

PDEs for the displacement

Write

$$\varphi(t, X) = X + u(t, X), \quad F = I + G$$

The PDEs may for u be written as

$$D_t^2 u^i - A^{ij}_{\ell m} D_\ell D_m u^j = B^{ijk}_{\ell mn}(Du) D_\ell (D_m u^j D_n u^k)$$

PDEs for the displacement

Write

$$\varphi(t,X) = X + u(t,X), \qquad F = I + G$$

The PDEs may for u be written as

$$D_t^2 u^i - A^{ij}_{\ell m} D_\ell D_m u^j = B^{ijk}_{\ell m n}(Du) D_\ell (D_m u^j D_n u^k)$$

where

$$A^{ij}_{\ell m} = \frac{\partial^2 W}{\partial F^i_\ell \partial F^j_m}(I)$$

and

$$B^{ijk}_{\ell m n}(G) = \frac{1}{2}\frac{\partial^3 W}{\partial F^i_\ell \partial F^j_m \partial F^k_n}(I+G)$$

Assumptions

- Isotropic motion: $W(F)$ depends on $F \in GL(3, \mathbb{R})$ only through the principal invariants of the strain matrix FF^T.

Assumptions

- Isotropic motion: $W(F)$ depends on $F \in GL(3,\mathbb{R})$ only through the principal invariants of the strain matrix FF^T.
- Legendre-Hadamard condition: makes linear problem hyperbolic

$$A^{ij}_{\ell m} D_\ell D_m u^j = c_2^2 \Delta u^i + (c_1^2 - c_2^2) D_i D_j u^j, \quad c_1 > c_2 > 0$$

Assumptions

- Isotropic motion: $W(F)$ depends on $F \in GL(3,\mathbb{R})$ only through the principal invariants of the strain matrix FF^T.
- Legendre-Hadamard condition: makes linear problem hyperbolic

$$A^{ij}_{\ell m} D_\ell D_m u^j = c_2^2 \Delta u^i + (c_1^2 - c_2^2) D_i D_j u^j, \quad c_1 > c_2 > 0$$

c_1 and c_2 are propagation speeds for pressure and shear waves, resp.

Assumptions

- Isotropic motion: $W(F)$ depends on $F \in GL(3, \mathbb{R})$ only through the principal invariants of the strain matrix FF^T.
- Legendre-Hadamard condition: makes linear problem hyperbolic

$$A^{ij}_{\ell m} D_\ell D_m u^j = c_2^2 \Delta u^i + (c_1^2 - c_2^2) D_i D_j u^j, \quad c_1 > c_2 > 0$$

c_1 and c_2 are propagation speeds for pressure and shear waves, resp.

The hydrodynamical case, $W(F) = \hat{W}(\det FF^T)$, is ruled out because in this case $c_2 = 0$.

- Null condition / linear degeneracy condition: restricts the self-interaction of individual wave families.

- Null condition / linear degeneracy condition: restricts the self-interaction of individual wave families.

Shear waves are automatically linearly degenerate in the isotropic case.

- Null condition / linear degeneracy condition: restricts the self-interaction of individual wave families.

Shear waves are automatically linearly degenerate in the isotropic case.

Pressure waves

$$B^{ijk}_{\ell mn}(0)x_i x_j x_k x_\ell x_m x_n \equiv 0, \quad \text{for all} \quad x \in \mathbb{R}^3$$

Consistent with physically meaningful examples.

Initial value problem

Consider the PDEs

$$D_t^2 u^i - A^{ij}_{\ell m} D_\ell D_m u^j = B^{ijk}_{\ell mn}(Du) D_\ell (D_m u^j D_n u^k)$$

Initial value problem

Consider the PDEs

$$D_t^2 u^i - A^{ij}_{\ell m} D_\ell D_m u^j = B^{ijk}_{\ell m n}(Du) D_\ell (D_m u^j D_n u^k)$$

with an initial displacement and an initial velocity

$$u(0, X) = u_0(X), \quad D_t u(0, X) = u_1(X)$$

which are sufficiently small in an appropriate energy norm.

Global existence - compressible case

Theorem: The IVP has a unique global classical solution of finite energy $\ll 1$.

Global existence - compressible case

Theorem: The IVP has a unique global classical solution of finite energy $\ll 1$.

Moreover, for each ℓ and m,

$$\| \, (|X| - c_1 t) \, \bar{X} \cdot D_\ell D_m u(t, \cdot) \, \|_{L^2(\mathbb{R}^3)} \ll 1$$

and

$$\| \, (|X| - c_2 t) \, \bar{X} \times D_\ell D_m u(t, \cdot) \, \|_{L^2(\mathbb{R}^3)} \ll 1,$$

Global existence - compressible case

Theorem: The IVP has a unique global classical solution of finite energy $\ll 1$.

Moreover, for each ℓ and m,

$$\| \, (|X| - c_1 t) \, \bar{X} \cdot D_\ell D_m u(t, \cdot) \, \|_{L^2(\mathbb{R}^3)} \ll 1$$

and

$$\| \, (|X| - c_2 t) \, \bar{X} \times D_\ell D_m u(t, \cdot) \, \|_{L^2(\mathbb{R}^3)} \ll 1,$$

where $\bar{X} = X/|X|$.

Remarks

- More detailed decomposition and asymptotic behavior available.

Remarks

- More detailed decomposition and asymptotic behavior available.
- Without the null condition, small solutions exist almost globally. Initial data of size ε give local solutions with a lifespan of order $\exp(\varepsilon^{-1})$. John, Klainerman-S

Remarks

- More detailed decomposition and asymptotic behavior available.
- Without the null condition, small solutions exist almost globally. Initial data of size ε give local solutions with a lifespan of order $\exp(\varepsilon^{-1})$. John, Klainerman-S
- Without the null condition, there are spherically symmetric examples where singularities form in finite time, for arbitrarily small initial conditions. John

1st order formulation

- Reference map (back-to-labels map)

$$\varphi^{-1} : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3, \qquad (t,x) \mapsto X = \varphi^{-1}(t,x)$$

1st order formulation

- Reference map (back-to-labels map)

$$\varphi^{-1} : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (t, x) \mapsto X = \varphi^{-1}(t, x)$$

- Inverse deformation gradient

$$H(t, x) = \nabla_x \varphi^{-1}(t, x) = F^{-1}(t, X)\Big|_{X=\varphi^{-1}(t,x)}$$

1st order formulation

- Reference map (back-to-labels map)

$$\varphi^{-1} : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3, \qquad (t,x) \mapsto X = \varphi^{-1}(t,x)$$

- Inverse deformation gradient

$$H(t,x) = \nabla_x \varphi^{-1}(t,x) = F^{-1}(t,X)\Big|_{X=\varphi^{-1}(t,x)}$$

- Density $\qquad \rho(t,x) = \bar{\rho} \det H(t,x)$

1st order formulation

- Reference map (back-to-labels map)

$$\varphi^{-1} : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3, \qquad (t,x) \mapsto X = \varphi^{-1}(t,x)$$

- Inverse deformation gradient

$$H(t,x) = \nabla_x \varphi^{-1}(t,x) = F^{-1}(t,X)\Big|_{X=\varphi^{-1}(t,x)}$$

- Density $\qquad \rho(t,x) = \bar{\rho} \det H(t,x)$

- Velocity $\qquad v(t,x) = D_t\varphi(t,X)\Big|_{X=\varphi^{-1}(t,x)}$

1st order PDEs – compressible motion

Balance laws

$$\partial_t \rho + v \cdot \nabla v + \rho \nabla \cdot v = 0$$

$$\rho(\partial_t v + v \cdot \nabla v) - \nabla \cdot T(H) = 0$$

1st order PDEs – compressible motion

Balance laws

$$\partial_t \rho + v \cdot \nabla v + \rho \nabla \cdot v = 0$$

$$\rho(\partial_t v + v \cdot \nabla v) - \nabla \cdot T(H) = 0$$

The stress tensor T is determined from W

$$T = (\det F)\, \frac{\partial W}{\partial F} F^T = -(\det H^{-1})\, H^T \frac{\partial W}{\partial H}.$$

1st order PDEs – compressible motion

Balance laws

$$\partial_t \rho + v \cdot \nabla v + \rho \nabla \cdot v = 0$$

$$\rho(\partial_t v + v \cdot \nabla v) - \nabla \cdot T(H) = 0$$

The stress tensor T is determined from W

$$T = (\det F)\, \frac{\partial W}{\partial F} F^T = -(\det H^{-1})\; H^T \frac{\partial W}{\partial H}.$$

Transport equation $\qquad \partial_t H + v \cdot \nabla H + \nabla v H = 0$

1st order PDEs – compressible motion

Balance laws

$$\partial_t \rho + v \cdot \nabla v + \rho \nabla \cdot v = 0$$

$$\rho(\partial_t v + v \cdot \nabla v) - \nabla \cdot T(H) = 0$$

The stress tensor T is determined from W

$$T = (\det F)\,\frac{\partial W}{\partial F} F^T = -(\det H^{-1})\; H^T \frac{\partial W}{\partial H}.$$

Transport equation $\quad \partial_t H + v \cdot \nabla H + \nabla v H = 0$

Constraints $\quad \partial_\ell H^i_m = \partial_m H^i_\ell$ and $\rho = \bar{\rho} \det H$

1st order PDEs – compressible motion

Balance laws

$$\partial_t \rho + v \cdot \nabla v + \rho \nabla \cdot v = 0$$
$$\rho(\partial_t v + v \cdot \nabla v) - \nabla \cdot T(H) = 0$$

The stress tensor T is determined from W

$$T = (\det F)\, \frac{\partial W}{\partial F} F^T = -(\det H^{-1})\; H^T \frac{\partial W}{\partial H}.$$

Transport equation $\quad \partial_t H + v \cdot \nabla H + \nabla v H = 0$

Constraints $\quad \partial_\ell H^i_m = \partial_m H^i_\ell$ and $\rho = \bar{\rho} \det H$

(In the hypdrodynamical case, $-T = P(\rho) I$.)

Incompressible motion

Here the deformation satisfies the internal constraint

$$\det D\varphi(t, X) \equiv 1.$$

This can be enforced on the level of the variational problem through the addition of a Lagrange multiplier.

PDEs of incompressible elastic motion

- Balance law

$$\partial_t v + v \cdot \nabla v - \nabla \cdot T(H) + \nabla p = 0$$

PDEs of incompressible elastic motion

- Balance law

$$\partial_t v + v \cdot \nabla v - \nabla \cdot T(H) + \nabla p = 0$$

- Transport equation

$$\partial_t H + v \cdot \nabla H + \nabla v H = 0$$

PDEs of incompressible elastic motion

- Balance law

$$\partial_t v + v \cdot \nabla v - \nabla \cdot T(H) + \nabla p = 0$$

- Transport equation

$$\partial_t H + v \cdot \nabla H + \nabla v H = 0$$

- Constraints

$$\nabla \cdot v = 0, \quad \det H = 1, \quad \text{and} \quad \partial_\ell H^i_m = \partial_m H^i_\ell$$

Assumptions

- Isotropic strain energy function, as before.

Assumptions

- Isotropic strain energy function, as before.
- Legendre-Hadamard condition – linearized problem has only one propagation speed because of the incompressibilty constraint.

Assumptions

- Isotropic strain energy function, as before.
- Legendre-Hadamard condition – linearized problem has only one propagation speed because of the incompressibilty constraint.

Shear waves, but no pressure waves.

Assumptions

- Isotropic strain energy function, as before.
- Legendre-Hadamard condition – linearized problem has only one propagation speed because of the incompressibilty constraint.

Shear waves, but no pressure waves.

Null condition not necessary.

Initial conditions

Take initial conditions

$$H(0,x) = H_0(x), \quad v(0,x) = v_0(x),$$

which satisfy the incompressibility constraints

$$\det H_0 = 1, \quad \nabla \cdot v_0 = 0,$$

as well as

$$\partial_\ell (H_0)^i_m = \partial_m (H_0)^i_\ell.$$

Initial conditions

Take initial conditions

$$H(0,x) = H_0(x), \quad v(0,x) = v_0(x),$$

which satisfy the incompressibility constraints

$$\det H_0 = 1, \quad \nabla \cdot v_0 = 0,$$

as well as

$$\partial_\ell (H_0)^i_m = \partial_m (H_0)^i_\ell.$$

Assume that

$$H_0(x) - I, \quad v_0(x),$$

are sufficiently small in an appropriate energy norm.

Global existence - incompressible case

Theorem: (With Becca Thomases) The initial IVP has a unique global classical solution of finite energy $\ll 1$.

Global existence - incompressible case

Theorem: (With Becca Thomases) The initial IVP has a unique global classical solution of finite energy $\ll 1$.

Moreover, for each ℓ, m,

$$\| \, (c_2|x| - t) \; \bar{x} \times D_\ell D_m v(t, \cdot)\|_{L^2(\mathbb{R}^3)} \ll 1$$

$$\| \, (c_2|x| - t) \; \bar{x} \times D_\ell D_m H(t, \cdot)\|_{L^2(\mathbb{R}^3)} \ll 1$$

where $\bar{x} = x/|x|$.

Global existence - incompressible case

Theorem: (With Becca Thomases) The initial IVP has a unique global classical solution of finite energy $\ll 1$.

Moreover, for each ℓ, m,

$$\| \, (c_2|x| - t) \; \bar{x} \times D_\ell D_m v(t,\cdot) \|_{L^2(\mathbb{R}^3)} \ll 1$$

$$\| \, (c_2|x| - t) \; \bar{x} \times D_\ell D_m H(t,\cdot) \|_{L^2(\mathbb{R}^3)} \ll 1$$

where $\bar{x} = x/|x|$.

More detailed asymptotic information available.

Incompressible limit

- Modify the strain energy function so as to penalize pressure waves.

$$\hat{W}(F) = W(F) + \lambda^2 h(\rho),$$

$$h(\bar{\rho}) = h'(\bar{\rho}) = 0, \quad h''(\rho) > 0, \quad \lambda \gg 1.$$

Incompressible limit

- Modify the strain energy function so as to penalize pressure waves.

$$\hat{W}(F) = W(F) + \lambda^2 h(\rho),$$
$$h(\bar{\rho}) = h'(\bar{\rho}) = 0, \quad h''(\rho) > 0, \quad \lambda \gg 1.$$

- Fast propagation speed $\sim \lambda$.

Incompressible limit

- Modify the strain energy function so as to penalize pressure waves.

$$\hat{W}(F) = W(F) + \lambda^2 h(\rho),$$
$$h(\bar{\rho}) = h'(\bar{\rho}) = 0, \quad h''(\rho) > 0, \quad \lambda \gg 1.$$

- Fast propagation speed $\sim \lambda$.
- Penalization term does not satisfy the null condition.

Incompressible limit

- Modify the strain energy function so as to penalize pressure waves.

$$\hat{W}(F) = W(F) + \lambda^2 h(\rho),$$

$$h(\bar{\rho}) = h'(\bar{\rho}) = 0, \quad h''(\rho) > 0, \quad \lambda \gg 1.$$

- Fast propagation speed $\sim \lambda$.
- Penalization term does not satisfy the null condition.
- Consider the compressible system with data close to equilibrium, satisfying the incompressibility constraints. (This can be relaxed.)

Long time local existence

Theorem: (With Becca Thomases) The penalized initial value problem parameterized by λ has a classical small energy solution on the time interval $0 \le t \le \lambda$, satisfying the uniform bound

$$\lambda^2 \|\rho^\lambda(t,\cdot) - \bar{\rho}\|_{L^2(\mathbb{R}^3)} \ll 1, \quad 0 \le t \le \lambda.$$

Long time local existence

Theorem: (With Becca Thomases) The penalized initial value problem parameterized by λ has a classical small energy solution on the time interval $0 \le t \le \lambda$, satisfying the uniform bound

$$\lambda^2 \|\rho^\lambda(t,\cdot) - \bar{\rho}\|_{L^2(\mathbb{R}^3)} \ll 1, \quad 0 \le t \le \lambda.$$

As $\lambda \to \infty$ the solution family converges locally uniformly in $\mathbb{R}^+ \times \mathbb{R}^3$ to a global solution of the corresponding incompressible problem.

Long time local existence

Theorem: (With Becca Thomases) The penalized initial value problem parameterized by λ has a classical small energy solution on the time interval $0 \le t \le \lambda$, satisfying the uniform bound

$$\lambda^2 \|\rho^\lambda(t,\cdot) - \bar{\rho}\|_{L^2(\mathbb{R}^3)} \ll 1, \quad 0 \le t \le \lambda.$$

As $\lambda \to \infty$ the solution family converges locally uniformly in $\mathbb{R}^+ \times \mathbb{R}^3$ to a global solution of the corresponding incompressible problem.

Improves a result of Schochet, which established the convergence on a fixed time interval. (See also Klainerman-Majda, Ukai in the hydrodynamical case.)

Method of proof

- Generalized energy method using vector fields based on rotational and scaling invariance.

Method of proof

- Generalized energy method using vector fields based on rotational and scaling invariance.
- Strong dispersive estimates, thanks to the form of the linearized equations.

Method of proof

- Generalized energy method using vector fields based on rotational and scaling invariance.
- Strong dispersive estimates, thanks to the form of the linearized equations.
- Localization of individual wave families near their respective characteristic cones. Controls the nonlinear interaction of distinct wave families.

Method of proof

- Generalized energy method using vector fields based on rotational and scaling invariance.
- Strong dispersive estimates, thanks to the form of the linearized equations.
- Localization of individual wave families near their respective characteristic cones. Controls the nonlinear interaction of distinct wave families.
- Null structure to control nonlinear interactions of waves of the same family (pressure waves).

Viscoelastic materials

- Stress depends on strain and strain rate:
  $T = T(H, \nabla H)$. No field theoretic description.

Viscoelastic materials

- Stress depends on strain and strain rate: $T = T(H, \nabla H)$. No field theoretic description.
- Example: Oldroyd model at infinite Weisberg number – incompressible case

$$\bar{\rho}(\partial_t v + v \cdot \nabla v) - \nabla \cdot T(H) + \nabla p = \nu \Delta v$$

$$\partial_t H + v \cdot \nabla H + \nabla v H = 0$$

$$\nabla \cdot v = 0, \quad \partial_\ell H^i_m = \partial_m H^i_\ell, \quad \det H = 1$$

Viscoelastic materials

- Stress depends on strain and strain rate: $T = T(H, \nabla H)$. No field theoretic description.
- Example: Oldroyd model at infinite Weisberg number – incompressible case

$$\bar{\rho}(\partial_t v + v \cdot \nabla v) - \nabla \cdot T(H) + \nabla p = \nu \Delta v$$

$$\partial_t H + v \cdot \nabla H + \nabla v H = 0$$

$$\nabla \cdot v = 0, \quad \partial_\ell H^i_m = \partial_m H^i_\ell, \quad \det H = 1$$

- Global existence for initial conditions sufficiently small w.r.t. the Reynolds number. Liu, Lin, Zhang and Lei, Liu, Zhou.

Current project

- Implement a modified version of the hyperbolic strategy in the viscoelastic case.

Current project

- Implement a modified version of the hyperbolic strategy in the viscoelastic case.
- Relaxation time $\sim \nu^{-1}$ when diffusive effects begin to dominate.

Current project

- Implement a modified version of the hyperbolic strategy in the viscoelastic case.
- Relaxation time $\sim \nu^{-1}$ when diffusive effects begin to dominate.
- Construct global solutions with a smallness condition that is independent of the size of the Reynolds number.

Current project

- Implement a modified version of the hyperbolic strategy in the viscoelastic case.
- Relaxation time $\sim \nu^{-1}$ when diffusive effects begin to dominate.
- Construct global solutions with a smallness condition that is independent of the size of the Reynolds number.
- Joint work with Paul Kessenich.