Dynamic Depletion of Vortex Stretching and Dynamic Stability of the 3-D Incompressible Flow

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Special thanks to Prof. Linbo Zhang
and Institute of Computational Mathematics.

October 23, 2006

3D incompressible Euler equations

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\nabla p \\ \nabla\cdot\mathbf{u} &= 0 \\ \mathbf{u}\,\big|_{t=0} &= \mathbf{u}_0 \end{cases}$$

Define vorticity $\boldsymbol{\omega} = \nabla\times\mathbf{u}$, then $\boldsymbol{\omega}$ is governed by

$$\boldsymbol{\omega}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = \nabla\mathbf{u}\cdot\boldsymbol{\omega},$$
$$\boldsymbol{\omega}\big|_{t=0} = \boldsymbol{\omega}_0 = \nabla\times\mathbf{u}_0.$$

Note $\nabla\mathbf{u}$ is formally of the same order as $\boldsymbol{\omega}$. Thus the vortex stretching term $\nabla\mathbf{u}\cdot\boldsymbol{\omega}\approx\boldsymbol{\omega}^2$.

History and review

- Classical existence theorems.
$\mathbf{u}_0 \in H^m(\mathbb{R}^3)$, $m > 5/2 \Rightarrow \mathbf{u} \in H^m$ up to $T_0 = T_0(\|u_0\|_{H^m})$. (Swann 1971, Kato 1972, see also Lichtenstein, Kato, Ebin-Marsden-Fischer, etc.)
- (Beale-Kato-Majda criterion, 1984)
u ceases to be classical at T^* if and only if

$$\int_0^{T^*} \|\boldsymbol{\omega}\|_\infty(t)\, dt = \infty.$$

Improvement of B-K-M criteria: BMO norm instead of L^∞ norm. Kozomo and Taniuchi, 2000.

Non-blowup conditions by Constantin-Fefferman-Majda

- Geometry of direction field of $\boldsymbol{\omega}$:
  Constantin, Fefferman and Majda. 1996.
  Let $\boldsymbol{\omega} = |\boldsymbol{\omega}|\boldsymbol{\xi}$, no blow-up if
  - (Bounded velocity) $\|\mathbf{u}\|_\infty$ is bounded in a $O(1)$ region of large vorticity;
  - (Regular orientedness) $\int_0^t \|\nabla\boldsymbol{\xi}\|_\infty^2 d\tau$ is uniformly bounded;

Local non-blowup conditions by Deng-Hou-Yu

Theorem 1 (Deng-Hou-Yu, 2005 and 2006, CPDE)

- Denote by $L(t)$ the arclength of a vortex line segment L_t around the maximum vorticity. If

  1. $\max_{L_t}(|\mathbf{u}\cdot\boldsymbol{\xi}| + |\mathbf{u}\cdot\mathbf{n}|) \le C_U(T-t)^{-A}$ with $A < 1$;
  2. $C_L(T-t)^B \le L(t) \le C_0/\max_{L_t}(|\kappa|,\ |\nabla\cdot\xi|)$ with $B < 1-A$;

  then the solution of the 3D Euler equations remains regular up to T.
- When $B = 1-A$, if in addition, the scaling constants C_U, C_0 and C_L satisfy an algebraic inequality, the solution will remain regular.
- The blowup scenario described by Kerr falls into the critical case.

Numerical evidence of Euler singularity

In 1993 (and 2005), R. Kerr [Phys. Fluids] presented numerical evidence of 3D Euler singularity for two anti-parallel vortex tubes:

- Pseudo-spectral in x and y, Chebyshev in z direction;
- Best resolution: $512 \times 256 \times 192$;
- $\|\boldsymbol{\omega}\|_{L^\infty} \approx (T-t)^{-1}$;
- $\|\mathbf{u}\|_{L^\infty} \approx (T-t)^{-1/2}$;
- Anisotropic scaling: $(T-t) \times \sqrt{T-t} \times \sqrt{T-t}$;
- Vortex lines: relatively straight, $|\nabla \boldsymbol{\xi}| \approx (T-t)^{-1/2}$;

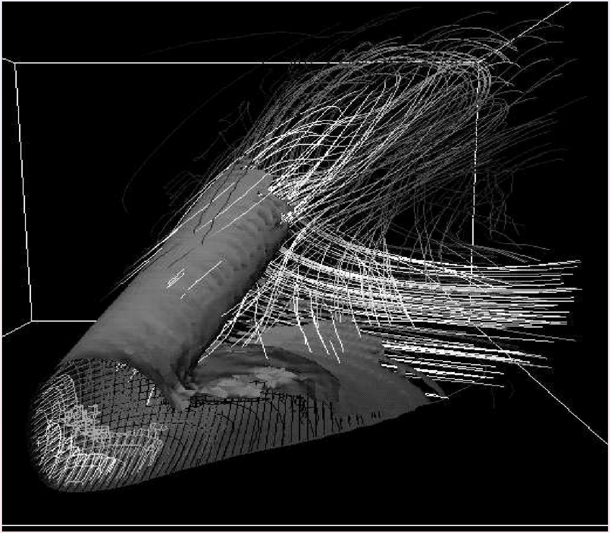
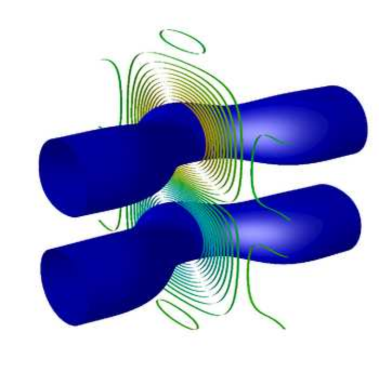


Figure: From: R.Kerr, Euler singularities and turbulence, 19th ICTAM Kyoto '96, 1997, pp57-70.



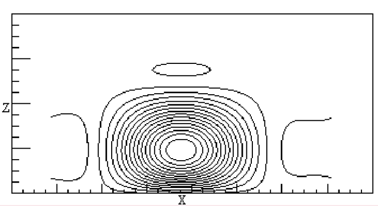


Figure: The 3D vortex tube and axial vorticity on the symmetry plane for initial value.

Numerical implementation

- A pseudo-spectral method is used in all three dimensions;
- Four step Runge-Kutta scheme for time integration with adaptive time stepping;
- A 36th order Fourier smoothing is used to remove aliasing error;
- Careful resolution study is performed: $768 \times 512 \times 1536$, $1024 \times 768 \times 2048$ and $1536 \times 1024 \times 3072$.
- 256 parallel processors with maximal memory comsumption 120Gb.

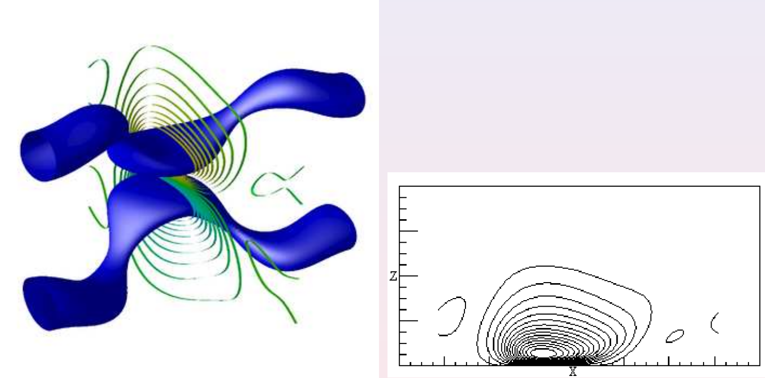


Figure: The 3D vortex tube and axial vorticity on the symmetry plane when $t = 6$.

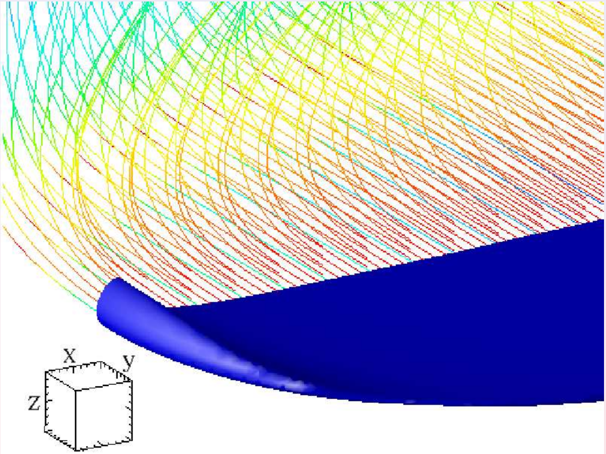


Figure: The local 3D vortex structures and vortex lines around the maximum vorticity at $t = 17$.

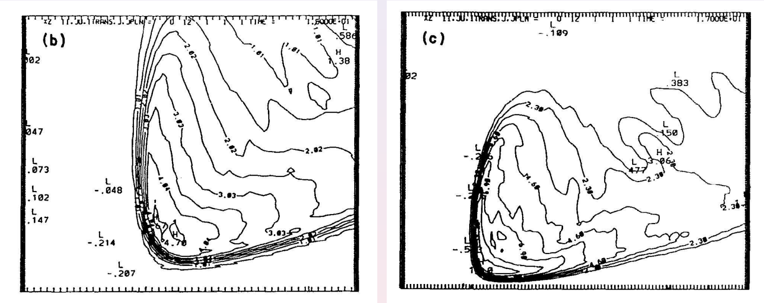


Figure: From: Kerr, Phys. Fluids A 5(7), 1993, pp1725-1746. $t = 15$(left) and $t = 17$(right).

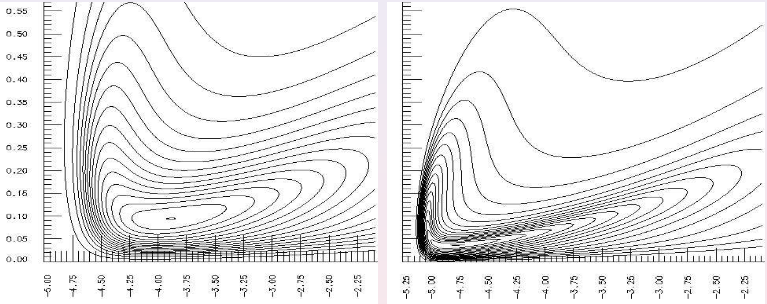


Figure: The contour of axial vorticity around the maximum vorticity on the symmetry plane at $t = 15, 17$.

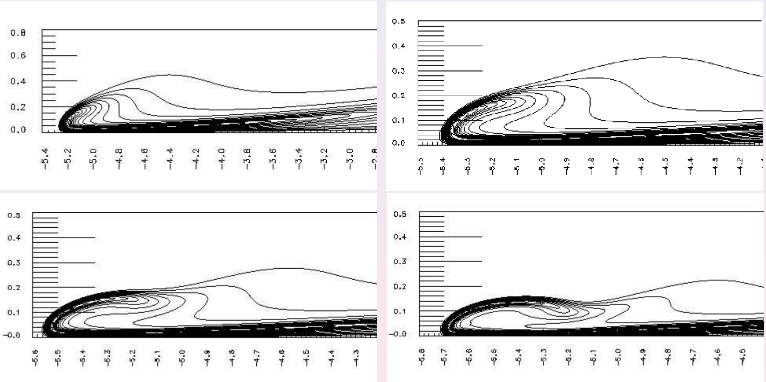


Figure: The contour of axial vorticity around the maximum vorticity on the symmetry plane (the *xz*-plane) at $t = 17.5, 18, 18.5, 19$.

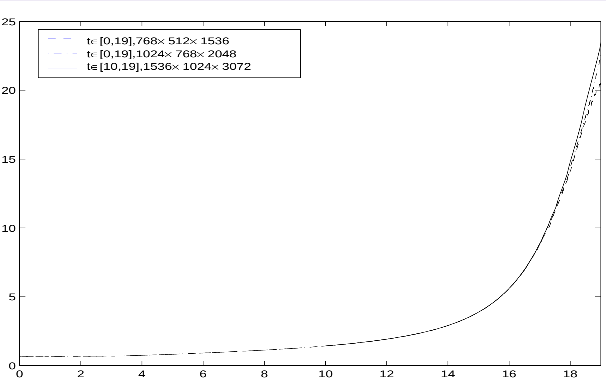


Figure: The maximum vorticity $\|\omega\|_\infty$ in time using different resolutions.

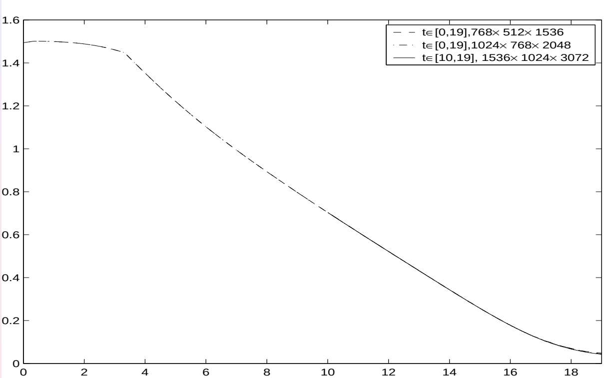


Figure: The inverse of maximum vorticity $\|\omega\|_\infty$ in time using different resolutions.

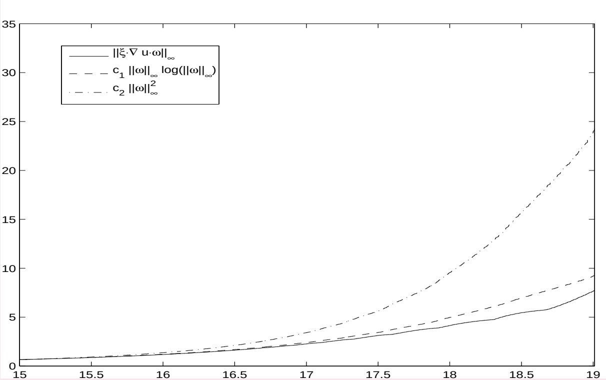


Figure: Study of the vortex stretching term in time, resolution $1536 \times 1024 \times 3072$. The fact $|\xi \cdot \nabla \mathbf{u} \cdot \omega| \leq c_1 |\omega| log |\omega|$ implies $|\omega|$ bounded by doubly exponential.

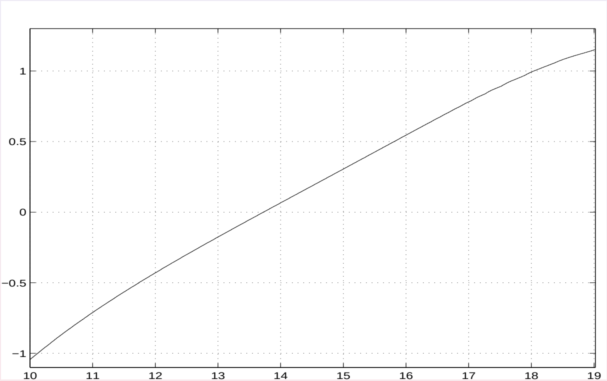


Figure: The plot of $\log\log\|\boldsymbol{\omega}\|_\infty$ vs time, resolution $1536\times1024\times3072$.

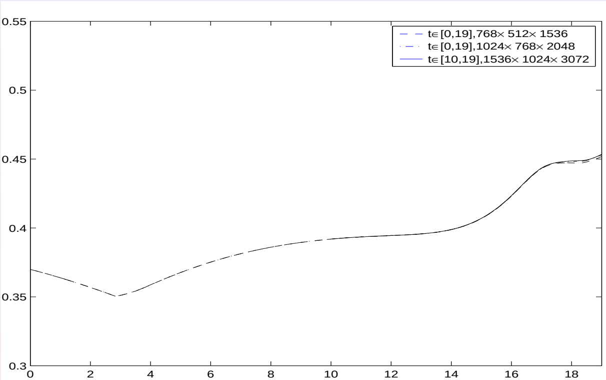


Figure: Maximum velocity $\|\mathbf{u}\|_\infty$ in time using different resolutions.

The local geometric criteria applies

Recall the local geometric criteria by Deng-Hou-Yu:

1. $\max_{L_t}(|\mathbf{u} \cdot \boldsymbol{\xi}| + |\mathbf{u} \cdot \mathbf{n}|) \leq C_U(T - t)^{-A}$ for some $A < 1$;
2. $C_L(T - t)^B \leq L(t) \leq C_0 / \max_{L_t}(|\kappa|,\ |\nabla \cdot \xi|)$ for some $B < 1 - A$,

then the solution of the 3D Euler equations remains regular up to T.

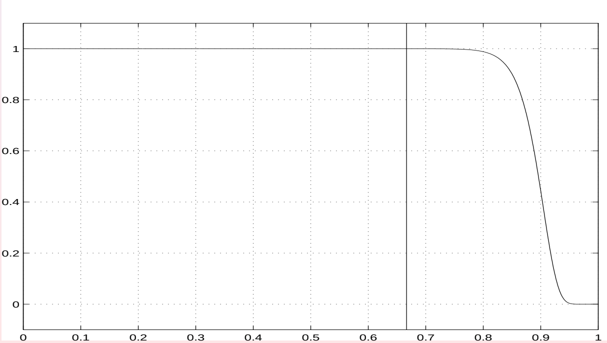
- Since u is bounded, we have $A = 0$. Therefore, we can take $B = 1/2 < 1 - A$, the theory applies.

2/3 Dealiasing vs high order Fourier smoothing

- A 36-order Fourier smoother is used to remove aliasing error;
- The Fourier smoother is shaped as along the x_j direction

$$\rho(2k_j/N_j) \equiv \exp(-36(2k_j/N_j)^{36})$$

where k_j is the wave number ($|k_j| \leqslant N_j/2$).



Comparison of spectra with resolution $768 \times 512 \times 1024$

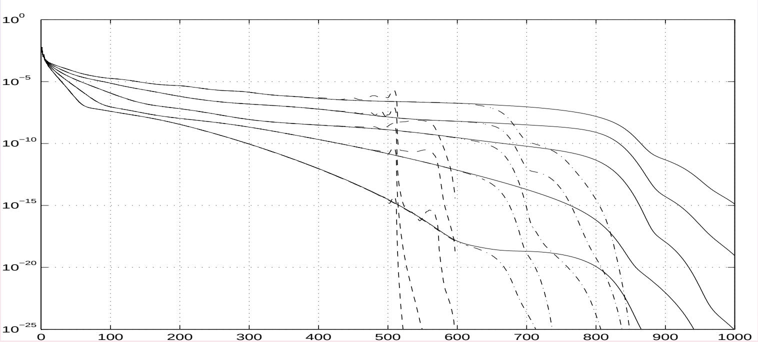
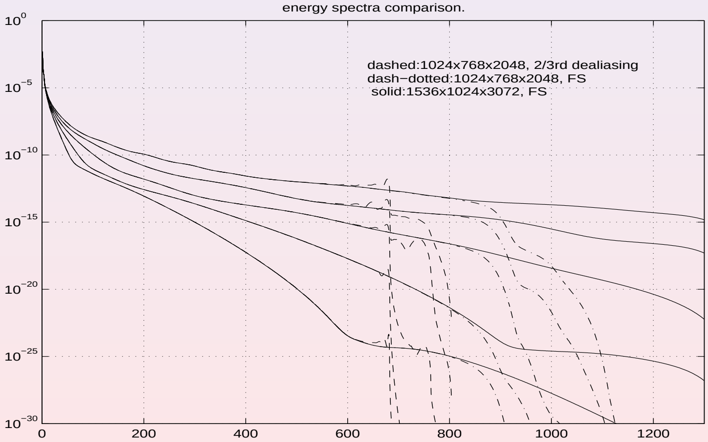
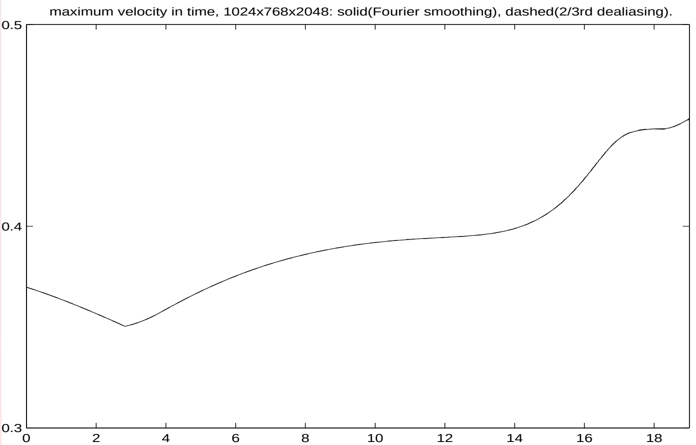


Figure: The enstrophy spectra versus wave numbers. The dashed lines and dashed-dotted lines are solutions with $768 \times 512 \times 1024$ using the 2/3 dealiasing rule and the Fourier smoothing, respectively. The times for the spectra lines are at $t = 15, 16, 17, 18, 19$ respectively.

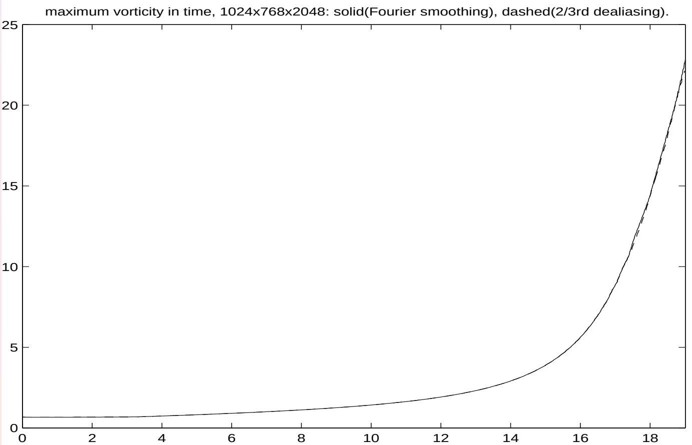
Comparison of spectra with resolution $1024 \times 768 \times 2048$



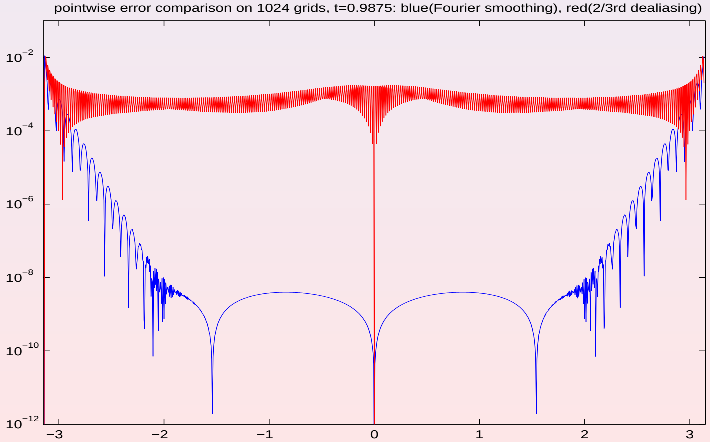
Comparison of maximum velocity with resolution $1024 \times 768 \times 2048$



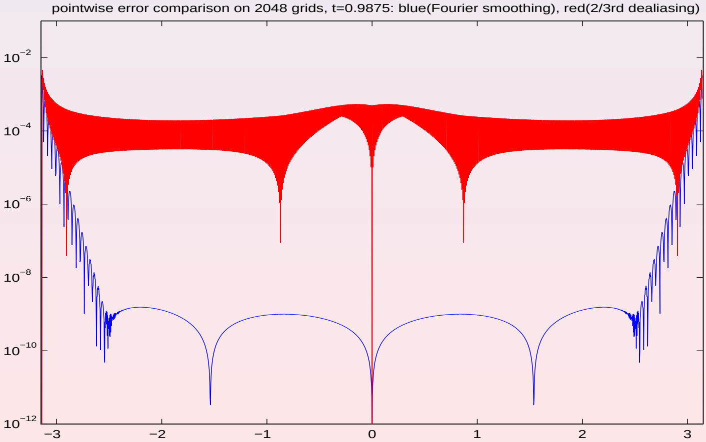
Comparison of maximum vorticity with resolution $1024 \times 768 \times 2048$



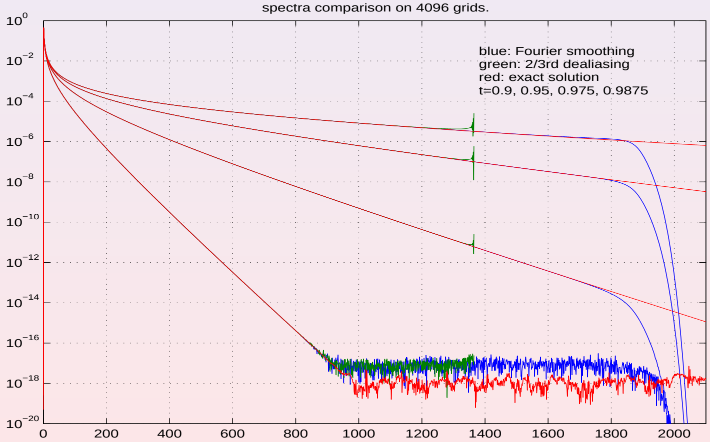
Burgers equation: maximum errors comparison with $N = 1024$, $u_0(x) = sin(x)$, $T_{\text{shock}} = 1$.



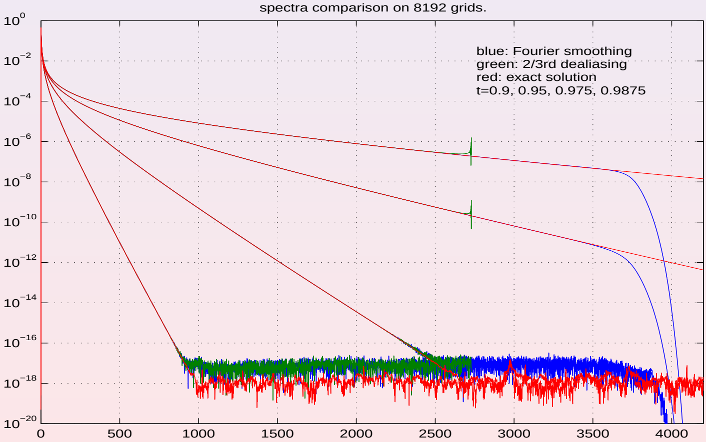
Burgers equation: maximum errors comparison with $N = 2048$, $u_0(x) = sin(x)$, $T_{shock} = 1$.



Burgers equation: spectra comparison with $N = 4096$



Burgers equation: spectra comparison with $N = 8192$



3D axisymmetric Navier-Stokes equations with swirl

Consider the 3D axi-symmetric incompressible Navier-Stokes equations

$$u_t^\theta + v^r u_r^\theta + v^z u_z^\theta = \nu \left(\nabla^2 - \frac{1}{r^2}\right) u^\theta - \frac{1}{r} v^r u^\theta, \quad (1)$$

$$\omega_t^\theta + v^r \omega_r^\theta + v^z \omega_z^\theta = \nu \left(\nabla^2 - \frac{1}{r^2}\right) \omega^\theta + \frac{1}{r}\left((u^\theta)^2\right)_z + \frac{1}{r} v^r \omega^\theta, \quad (2)$$

$$-\left(\nabla^2 - \frac{1}{r^2}\right)\psi^\theta = \omega^\theta, \quad (3)$$

where u^θ, ω^θ and ψ^θ are the angular components of the velocity, vorticity and stream function respectively, and

$$v^r = -\frac{\partial \psi^\theta}{\partial z}, \quad v^z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi^\theta).$$

Note that equations (1)-(3) completely determine the evolution of the 3D axisymmetric Navier-Stokes equations.

A 1D model for the 3D Navier-Stokes equations

Note that any singularity must occur along the symmetry axis [Caffarelli-Kohn-Nirenberg].
Expand the solution u^θ, ω^θ and ψ^θ around $r = 0$ as follows [Liu-Wang]:

$$u^\theta(r,z,t) = ru_1(z,t) + \frac{r^3}{3!}u_3(z,t) + \frac{r^5}{5!}u_5(z,t) + \cdots ,$$
$$\omega^\theta(r,z,t) = r\omega_1(z,t) + \frac{r^3}{3!}\omega_3(z,t) + \frac{r^5}{5!}\omega_5(z,t) + \cdots ,$$
$$\psi^\theta(r,z,t) = r\psi_1(z,t) + \frac{r^3}{3!}\psi_3(z,t) + \frac{r^5}{5!}\psi_5(z,t) + \cdots .$$

Substitute the above expansions into (1)-(3). After cancelling r from both sides and setting $r = 0$, we obtain

$$(u_1)_t + 2\psi_1 (u_1)_z = \nu \left(4/3u_3 + (u_1)_{zz}\right) + 2 (\psi_1)_z u_1,$$
$$(\omega_1)_t + 2\psi_1 (\omega_1)_z = \nu \left(4/3\omega_3 + (\omega_1)_{zz}\right) + \left(u_1^2\right)_z ,$$
$$-\left(4/3\psi_3 + (\psi_1)_{zz}\right) = \omega_1 .$$

Note that $u_3 = u^{\theta}_{rrr}(0,z,t)$, $(u_1)_{zz} = u^{\theta}_{rzz}(0,z,t)$. If we further assume

$$u^{\theta}_{rzz} \gg u^{\theta}_{rrr}, \quad \omega^{\theta}_{rzz} \gg \omega^{\theta}_{rrr}, \quad \psi^{\theta}_{rzz} \gg \psi^{\theta}_{rrr},$$

we can ignore the coupling to u_3, ω_3, ψ_3, and obtain our 1D model:

$$(u_1)_t + 2\psi_1 (u_1)_z = \nu(u_1)_{zz} + 2(\psi_1)_z u_1, \tag{4}$$

$$(\omega_1)_t + 2\psi_1 (\omega_1)_z = \nu(\omega_1)_{zz} + \left(u_1^2\right)_z, \tag{5}$$

$$-(\psi_1)_{zz} = \omega_1. \tag{6}$$

Let $\tilde{u} = u_1$, $\tilde{v} = -(\psi_1)_z$, and $\tilde{\psi} = \psi_1$. The above system becomes

$$(\tilde{u})_t + 2\tilde{\psi}(\tilde{u})_z = \nu(\tilde{u})_{zz} - 2\tilde{v}\tilde{u}, \tag{7}$$

$$(\tilde{v})_t + 2\tilde{\psi}(\tilde{v})_z = \nu(\tilde{v})_{zz} + (\tilde{u})^2 - (\tilde{v})^2 + c(t), \tag{8}$$

where $\tilde{v} = -(\tilde{\psi})_z$, $\tilde{v}_z = \tilde{\omega}$, and $c(t)$ is an integration constant to enforce the mean of $\tilde{v}$ equal to zero.

The 1D model is exact!

A surprising result is that the above 1D model is exact.

Theorem 2. Let u_1, ψ_1 and ω_1 be the solution of the 1D model (4)-(6) and define

$$u^\theta(r,z,t) = ru_1(z,t), \quad \omega^\theta(r,z,t) = r\omega_1(z,t), \quad \psi^\theta(r,z,t) = r\psi_1(z,t).$$

Then $(u^\theta(r,z,t),\ \omega^\theta(r,z,t),\ \psi^\theta(r,z,t))$ is an exact solution of the 3D Navier-Stokes equations.

Theorem 2 tells us that the 1D model (4)-(6) preserves some essential nonlinear structure of the 3D axisymmetric Navier-Stokes equations.

The ODE model

Consider an ODE model by ignoring the convection and diffusion terms.

$$(\tilde{u})_t = -2\tilde{v}\tilde{u}, \tag{9}$$
$$(\tilde{v})_t = (\tilde{u})^2 - (\tilde{v})^2. \tag{10}$$

Theorem 3. Assume that $\tilde{u}_0 \neq 0$. Then the solution $(\tilde{u}(t), \tilde{v}(t))$ of the ODE system (9)-(10) exists for all times. Moreover, we have

$$\lim_{t\to\infty} \tilde{u}(t) = 0, \quad \lim_{t\to\infty} \tilde{v}(t) = 0.$$

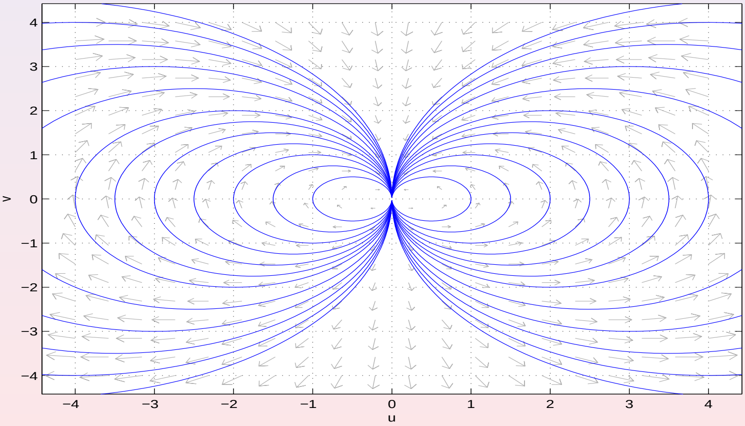
Proof. Let $w = \tilde{u} + i\tilde{v}$. Then the ODE system is reduced to a complex nonlinear ODE:

$$\frac{dw}{dt} = iw^2, \quad w(0) = w_0,$$

which can be solved analytically. The solution has the form

$$w(t) = \frac{w_0}{1 - iw_0 t}.$$

The phase diagram for the ODE system



The Reaction Diffusion Model

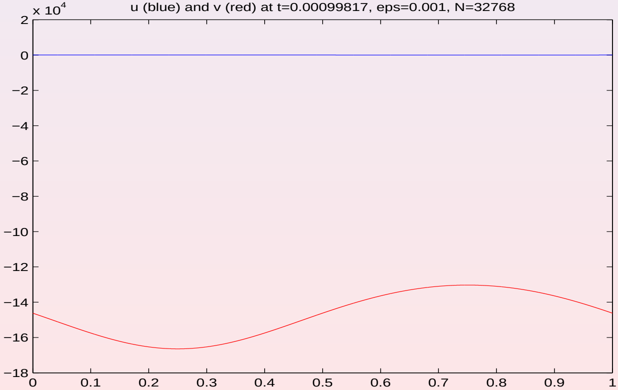
Consider the reaction-diffusion system:

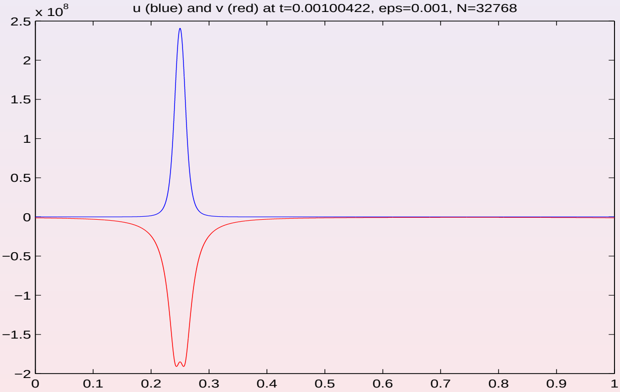
$$(\tilde{u})_t = \nu \tilde{u}_{zz} - 2\tilde{v}\tilde{u}, \tag{11}$$

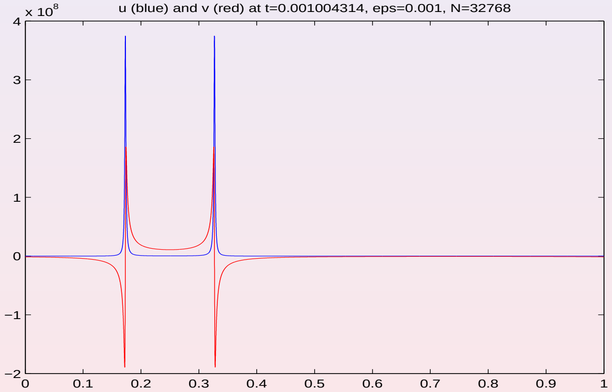
$$(\tilde{v})_t = \nu \tilde{v}_{zz} + (\tilde{u})^2 - (\tilde{v})^2. \tag{12}$$

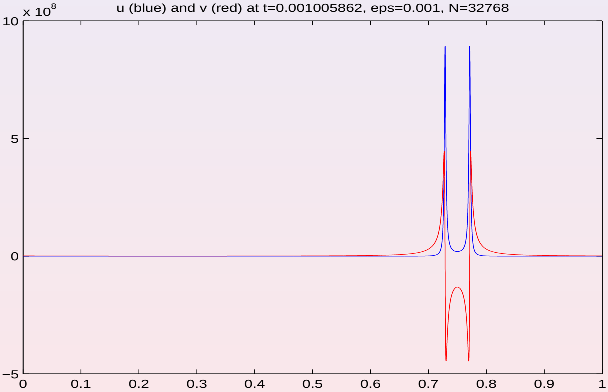
- Intuitively, one may think that the diffusion term would help to stabilize the dynamic growth induced by the nonlinear terms.
- However, because the nonlinear ODE system in the absence of viscosity is very unstable, the diffusion term can actually have a destabilizing effect.

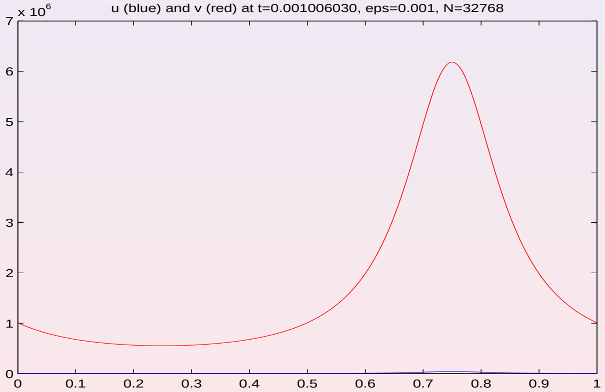
Growth at early times: $\tilde{u}_0(z) = (2 + sin(2\pi z))/1000$, $\tilde{v}_0(z) = -1000 - sin(2\pi z)$, $\nu = 1$.

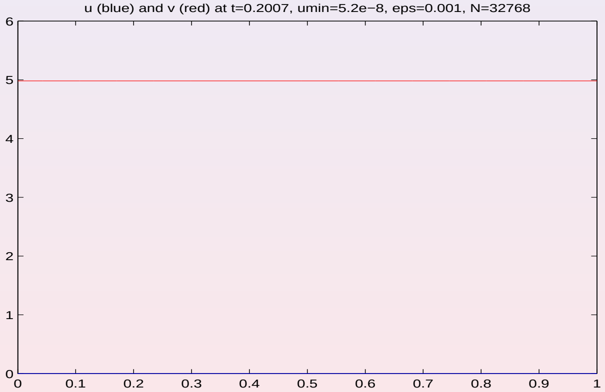



u (blue) and v (red) at t=0.00100422, eps=0.001, N=32768
x 10^8
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


u (blue) and v (red) at t=0.001004314, eps=0.001, N=32768
x 10^8
4
3
2
1
0
−1
−2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


u (blue) and v (red) at t=0.001005862, eps=0.001, N=32768
x 10^8
10
5
0
−5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


u (blue) and v (red) at t=0.001006030, eps=0.001, N=32768
x 10^6
7
6
5
4
3
2
1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


u (blue) and v (red) at t=0.2007, umin=5.2e−8, eps=0.001, N=32768
0
1
2
3
4
5
6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Energy method does not work for the 1D model!

- If we multiply the $\tilde{u}$-equation by $\tilde{u}$, and the $\tilde{v}$-equation by $\tilde{v}$, and integrate over z, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \tilde{u}^2 dz = -3\int_0^1 (\tilde{u})^2 \tilde{v}\, dz - \nu \int_0^1 \tilde{u}_z^2 dz,$$
$$\frac{1}{2}\frac{d}{dt}\int_0^1 \tilde{v}^2 dz = \int_0^1 \tilde{u}^2 \tilde{v}\, dz - 3\int_0^1 (\tilde{v})^3 dz - \nu \int_0^1 \tilde{v}_z^2 dz.$$

- Even for this 1D model, the energy estimate shares the some essential difficulty as the 3D Navier-Stokes equations.
- It is not clear how to control the nonlinear vortex stretching like terms by the diffusion terms, unless we assume

$$\int_0^T \|\tilde{v}\|_{L^\infty} dt < \infty, t \le T.$$

Global Well-Posedness of the full 1D Model

Theorem 4. *Assume that* $\tilde{u}(z,0)$ *and* $\tilde{v}(z,0)$ *are in* $C^m[0,1]$ *with* $m \geq 1$ *and periodic with period 1. Then the solution* $(\tilde{u}, \tilde{v})$ *of the 1D model will be in* $C^m[0,1]$ *for all times and for* $\nu \geq 0$*.*

Proof. The key is to obtain *a priori* **pointwise** estimate for the nonlinear term $\tilde{u}_z^2 + \tilde{v}_z^2$. Differentiating the $\tilde{u}$ and $\tilde{v}$-equations w.r.t z, we get

$$(\tilde{u}_z)_t + 2\tilde{\psi}(\tilde{u}_z)_z - 2\tilde{v}\tilde{u}_z = -2\tilde{v}\tilde{u}_z - 2\tilde{u}\tilde{v}_z + \nu(\tilde{u}_z)_{zz},$$
$$(\tilde{v}_z)_t + 2\tilde{\psi}(\tilde{v}_z)_z - 2\tilde{v}\tilde{v}_z = 2\tilde{u}\tilde{u}_z - 2\tilde{v}\tilde{v}_z + \nu(\tilde{v}_z)_{zz}.$$

Note that the **convection term contributes to stability** by cancelling one of the nonlinear terms on the right hand side. This gives

$$(\tilde{u}_z)_t + 2\tilde{\psi}(\tilde{u}_z)_z = -2\tilde{u}\tilde{v}_z + \nu(\tilde{u}_z)_{zz}, \quad (13)$$
$$(\tilde{v}_z)_t + 2\tilde{\psi}(\tilde{v}_z)_z = 2\tilde{u}\tilde{u}_z + \nu(\tilde{v}_z)_{zz}. \quad (14)$$

Multiplying (13) by $2\tilde{u}_z$ and (14) by $2\tilde{v}_z$, we have

$$(\tilde{u}_z^2)_t + 2\tilde{\psi}(\tilde{u}_z^2)_z = -4\tilde{u}\tilde{u}_z\tilde{v}_z + 2\nu\tilde{u}_z(\tilde{u}_z)_{zz}, \qquad (15)$$

$$(\tilde{v}_z^2)_t + 2\tilde{\psi}(\tilde{v}_z^2)_z = 4\tilde{u}\tilde{u}_z\tilde{v}_z + 2\nu\tilde{v}_z(\tilde{v}_z)_{zz}. \qquad (16)$$

Now, we add (15) to (16). **Surprisingly, the nonlinear vortex stretching-like terms cancel each other.** We get

$$\left(\tilde{u}_z^2 + \tilde{v}_z^2\right)_t + 2\tilde{\psi}\left(\tilde{u}_z^2 + \tilde{v}_z^2\right)_z = 2\nu\left(\tilde{u}_z(\tilde{u}_z)_{zz} + \tilde{v}_z(\tilde{v}_z)_{zz}\right).$$

Moreover we can rewrite the diffusion term in the following form:

$$\left(\tilde{u}_z^2 + \tilde{v}_z^2\right)_t + 2\tilde{\psi}\left(\tilde{u}_z^2 + \tilde{v}_z^2\right)_z = \nu\left(\tilde{u}_z^2 + \tilde{v}_z^2\right)_{zz} - 2\nu\left[(\tilde{u}_{zz})^2 + (\tilde{v}_{zz})^2\right].$$

Thus, $(\tilde{u}_z^2 + \tilde{v}_z^2)$ satisfies a **maximum principle** for all $\nu \geq 0$:

$$\|\tilde{u}_z^2 + \tilde{v}_z^2\|_{L^\infty} \leq \|(\tilde{u}_0)_z^2 + (\tilde{v}_0)_z^2\|_{L^\infty}.$$

Construction of a family of globally smooth solutions

Theorem 5. *Let $\phi(r)$ be a smooth cut-off function and u_1, ω_1 and ψ_1 be the solution of the 1D model. Define*

$$
\begin{aligned}
u^\theta(r,z,t) &= r u_1(z,t)\phi(r) + \tilde{u}(r,z,t), \\
\omega^\theta(r,z,t) &= r\omega_1(z,t)\phi(r) + \tilde{\omega}_1(r,z,t), \\
\psi^\theta(r,z,t) &= r\psi_1(z,t)\phi(r) + \tilde{\psi}(r,z,t).
\end{aligned}
$$

Then there exists a family of globally smooth functioons $\tilde{u}$, $\tilde{\omega}$ and $\tilde{\psi}$ such that u^θ, ω^θ and ψ^θ are globally smooth solutions of the 3D Navier-Stokes equations with finite energy.

Concluding Remarks

- Our analysis and computation demonstrate a subtle dynamic depletion of vortex stretching due to local geometric regularity of vortex lines.
- Our analysis also reveals a subtle dynamic stability property due to the special structure of nonlinearity.
- Nonlinear vortex stretching on one hand can lead to large dynamic growth, but on the other hand has a surprising stabilizing effect.
- Convection term also plays an essential role in stabilizing the nonlinear growth due to vortex stretching.
- New analytic tools that exploit the local structure of the singularity and nonlinearity are needed.

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