# On the Global Regularity of the 3D Navier-Stokes Equations and Relevant Geophysical Models

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# Rayleigh Be'nard Convection / Boussinesq Approximation

- Conservation of Momentum

$$\frac{\partial}{\partial t}\vec{u} - \upsilon \Delta \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \frac{1}{\rho_0}\nabla p + f\vec{k} \times \vec{u} = gT\vec{k}$$

- Incompressibility

$$\nabla \cdot \vec{u} = 0$$

- Heat Transport and Diffusion

$$\frac{\partial}{\partial t}T - \kappa \Delta T + (\vec{u} \cdot \nabla)T = 0$$

# Temperature Estimates

- Maximum Principle

$$\|T\|_{L^\infty} \le C_0 + C_1 \|T_0\|_{L^\infty}$$

- Gradient Estimates

$$\frac{1}{2}\frac{d}{dt}\|\nabla T\|_{L^2}^2 + \kappa\|\Delta T\|_{L^2}^2 = \int (\vec{u}\cdot\nabla)T\cdot\Delta T dx$$

- Estimate of the Nonlinear Term

$$\left|\int (\vec{u}\cdot\nabla)T\cdot\Delta T dx\right| \le c\|\vec{u}\|_{L^6}\|\nabla T\|_{L^3}\|\Delta T\|_{L^2}$$

- Interpolation/Calculus Inequality

$$\|\varphi\|_{L^3} \le c\|\varphi\|_{L^2}^{1/2}\|\nabla\varphi\|_{L^2}^{1/2} \qquad \forall\, \varphi\in \dot{H}^1$$

$$\Rightarrow$$

$$\left|\int (\vec{u}\cdot\nabla)T\cdot\Delta T dx\right| \le c\|\vec{u}\|_{L^6}\|\nabla T\|_{L^2}^{1/2}\|\Delta T\|_{L^2}^{3/2}$$

- Young's Inequality

$$|a\cdot b| \le \frac{1}{p}|a|^p + \frac{1}{q}|b|^q, \qquad \frac{1}{p}+\frac{1}{q}=1$$

$$\left|\int (\vec{u}\cdot\nabla)T\cdot\Delta T\,dx\right| \le \frac{c}{\kappa^3}\|\vec{u}\|_{L^6}^4\,\|\nabla T\|_{L^2}^2 + \frac{\kappa}{2}\|\Delta T\|_{L^2}^2$$

$$\Rightarrow$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla T\|_{L^2}^2 + \frac{\kappa}{2}\|\Delta T\|_{L^2}^2 \le \frac{c}{\kappa^3}\|u\|_{L^6}^4\,\|\nabla T\|_{L^2}^2$$

By Gronwall's inequality

$$\|\nabla T(t)\|_{L^2}^2 \le \|\nabla T(0)\|_{L^2}^2\, e^{\frac{c}{\kappa^3}\int_0^t \|u(\tau)\|_{L^6}^4\,d\tau}$$

# Question:

Is

$$\int_0^t \|u(\tau)\|_{L^6}^4 \, d\tau \le K \,?$$

To answer this question we have to deal with the Navier-Stokes equations.

# The Navier-Stokes Equations

$$\frac{\partial}{\partial t}\vec{u} - \upsilon \Delta \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \frac{1}{\rho_0}\nabla p = \vec{f}$$

$$\nabla \cdot \vec{u} = 0$$

Plus Boundary conditions, say periodic in the box

$$\Omega = [0, L]^3$$

- We will assume that $\rho_0 = 1$
- Denote by $\langle \varphi \rangle = \int_\Omega \varphi(x)\,dx$
- Observe that if $\langle \vec{u}_0 \rangle = \langle \vec{f} \rangle = 0$ then $\langle \vec{u} \rangle = 0$.
- Poncare' Inequality

For every $\varphi \in H^1$ with $\langle \varphi \rangle = 0$ we have

$$\|\varphi\|_{L^2} \leq cL \|\nabla \varphi\|_{L^2}$$

# Sobolev Spaces

$$H^{s}(\Omega) = \{\varphi = \sum_{\vec{k} \in \mathrm{Z}^{d}} \hat{\varphi}_{\vec{k}} e^{i\vec{k} \cdot \vec{x} \frac{2\pi}{L}}$$

such that

$$\sum_{\vec{k} \in \mathrm{Z}^{\mathrm{d}}} \left|\hat{\varphi}_{\vec{k}}\right|^{2} (1 + \left|\vec{k}\right|^{2})^{s} < \infty\}$$

# Navier-Stokes Equations Estimates

- Formal Energy estimate

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}\|_{L^2}^2 + \upsilon\|\nabla\vec{u}\|_{L^2}^2 + \int(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{u} + \int\nabla p\cdot\vec{u} = \left(\vec{f},\vec{u}\right)$$

- Observe that since $\nabla\cdot\vec{u} = 0$ we have

$$\int(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{u}\,dx = \int\nabla p\cdot\vec{u}\,dx = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|\vec{u}\|_{L^2}^2 + \upsilon\|\nabla\vec{u}\|_{L^2}^2 = \left(\vec{f},\vec{u}\right)$$

By the Cauchy-Schwarz and Poincare' inequalities

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}\|_{L^2}^2+\upsilon\|\nabla\vec{u}\|_{L^2}^2\leq\left\|\vec{f}\right\|_{L^2}^2\|\vec{u}\|_{L^2}^2\leq cL\left\|\vec{f}\right\|_{L^2}\|\nabla\vec{u}\|_{L^2}$$

By the Young's inequality

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}\|_{L^2}^2+\upsilon\|\nabla\vec{u}\|_{L^2}^2\leq\frac{cL^2}{\upsilon}\left\|\vec{f}\right\|_{L^2}^2+\frac{\upsilon}{2}\|\nabla\vec{u}\|_{L^2}^2$$

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}\|_{L^2}^2+\frac{\upsilon}{2}\|\nabla\vec{u}\|_{L^2}^2\leq\frac{cL^2}{\upsilon}\left\|\vec{f}\right\|_{L^2}^2$$

## By Poincare' inequality

$$\frac{d}{dt}\|\vec{u}\|_{L^2}^2 + c\,\frac{\upsilon}{L^2}\|\vec{u}\|_{L^2}^2 \le \frac{cL^2}{\upsilon}\|\vec{f}\|_{L^2}^2$$

## By Gronwall's inequality

$$\|\vec{u}(t)\|_{L^2}^2 \le e^{-c\upsilon L^{-2}t}\|\vec{u}(0)\|_{L^2}^2 + \frac{cL^4}{\upsilon^2}\left(1-e^{-c\upsilon L^{-2}t}\right)\|\vec{f}\|_{L^2}^2 \qquad \forall t \in [0,\mathrm{T}]$$

and

$$\upsilon\int_0^{\mathrm{T}} \|\nabla\vec{u}(\tau)\|_{L^2}^2\, d\tau \le K\left(L, \|\vec{u}_0\|_{L^2}, \|\vec{f}\|_{L^2}, \upsilon, \mathrm{T}\right)$$

# Theorem (Leray 1932-34)

For every $\mathrm{T} > 0$ there exists a weak solution (in the sense of distribution) of the Navier-stokes equations, which also satisfies

$$\vec{u} \in C_w([0,\mathrm{T}], L^2(\Omega)) \bigcap L^2([0,\mathrm{T}], H^1(\Omega))$$

The uniqueness of weak solutions in the three dimensional Navier-Stokes equations case is still an open question.

# Strong Solutions of Navier-Stokes

$$\vec{u} \in C([0,\mathrm{T}], H^1(\Omega)) \bigcap L^2([0,\mathrm{T}], H^2(\Omega))$$

## Enstrophy

$$\left\| \nabla \times \vec{u} \right\|_{L^2}^2 = \left\| \vec{\omega} \right\|_{L^2}^2 = \left\| \nabla \vec{u} \right\|_{L^2}^2$$

# Formal Enstrophy Estimates

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla\vec{u}\right\|_{L^2}^2 + \upsilon\left\|\Delta\vec{u}\right\|_{L^2}^2 + \int(\vec{u}\cdot\nabla)\vec{u}\cdot(-\Delta\vec{u}) + \int\nabla p(-\Delta\vec{u}) = \int\vec{f}\cdot(-\Delta\vec{u})$$

Observe that $\int\nabla p\cdot(-\Delta\vec{u})dx = 0$

By Cauchy-Schwarz $\left|\int\vec{f}\cdot(-\Delta\vec{u})\right| \le \frac{\left\|\vec{f}\right\|_{L^2}^2}{\upsilon} + \frac{\upsilon}{4}\left\|\Delta\vec{u}\right\|_{L^2}^2$

By Hőlder inequality

$$\left|\int(\vec{u}\cdot\nabla)\vec{u}\cdot(-\Delta\vec{u})\right| \le \left\|\vec{u}\right\|_{L^4}\left\|\nabla\vec{u}\right\|_{L^4}\left\|\Delta\vec{u}\right\|_{L^2}$$

# Calculus/Interpolation (Ladyzhenskaya) Inequatlities

$$\|\varphi\|_{L^4} \le \begin{cases} c\,\|\varphi\|_{L^2}^{1/2}\ \|\nabla\varphi\|_{L^2}^{1/2} & 2-D \\ c\,\|\varphi\|_{L^2}^{1/4}\ \|\nabla\varphi\|_{L^2}^{3/4} & 3-D \end{cases}$$

Denote by $\quad y = e_0 + \|\nabla\vec{u}\|_{L^2}^2$

# The Two-dimensional Case

$$\dot{y} \le c\, y^2 \qquad \& \qquad \int_0^{\mathrm{T}} y(\tau)\,d\tau \le K(\mathrm{T})$$

$$\Rightarrow y(t) \le \tilde{K}(\mathrm{T})$$

Global regularity of strong solutions to the two-dimensional Navier-Stokes equations.

# Navier-Stokes Equations

- Two-dimensional Case

  * Global Existence and Uniqueness of weak and strong solutions

  * Finite dimension global attractor

# The Three-dimensional Case

Recall that $y = e_0 + \|\nabla \vec{u}\|_{L^2}^2$

One can show that

$$\dot{y} \le c(\|u\|_{L^6}^4 + e_0^2)\, y$$

Which implies that

$$y(t) \le y(0)\, e^{c\int_0^t (\|u(\tau)\|_{L^6}^4 + e_0^2)\, d\tau}$$

# The Question Is Again Whether:

$$\int_0^T \|u(\tau)\|_{L^6}^4 \, d\tau \le K\,?$$

One can instead use the following Sobolev inequality

$$\left\|\vec{u}\right\|_{L^6} \leq c\left\|\nabla \vec{u}\right\|_{L^2}$$

Which leads to $\dot{y} \leq c y^3 \quad \& \quad \int_0^{\mathrm{T}} y(\tau) d\tau \leq K$

**Theorem (Leray 1932-1934)**

There exists $\mathrm{T}_*(\left\|\vec{u}_0\right\|_{L^2}, \left\|\vec{f}\right\|_{L^2}, \upsilon, L)$ such that

$y(t) < \infty$ for every $t \in [0, \mathrm{T}_*)$.

# Navier-Stokes Equations

- The Three-dimensional Case
  * Global existence of the weak solutions
  * Short time existence of the strong solutions
  * Uniqueness of the strong solutions

- Open Problems:
  * Uniqueness of the weak solution
  * Global existence of the strong solution.

# Vorticity Formulation

$$\frac{\partial \vec{\omega}}{\partial t} - \nu \Delta \vec{\omega} + (\vec{u} \cdot \nabla) \vec{\omega} - \underline{\underline{(\vec{\omega} \cdot \nabla) \vec{u}}} = \nabla \times \vec{f}$$

**Vorticity Stretching Term** $(\vec{\omega} \cdot \nabla) \vec{u}$

**Two dimensional case** $(\vec{\omega} \cdot \nabla) \vec{u} \equiv \vec{0}$

$$\frac{\partial \vec{\omega}}{\partial t} - \nu \Delta \vec{\omega} + (\vec{u} \cdot \nabla) \vec{\omega} = \nabla \times \vec{f}$$

$\left| \vec{\omega}(x,t) \right|^2$ Satisfies a maximum principle.

## The Three-dimensional Case

$$(\vec{\omega} \cdot \nabla)\vec{u} \not\equiv 0$$

$$\vec{\omega} \sim \mathrm{z}$$

$$(\vec{\omega} \cdot \nabla)\vec{u} \sim \mathrm{z}^2$$

For large initial data $\vec{\omega}_0$ the vorticity balance takes the form

$$\dot{\mathrm{z}} \sim \mathrm{z}^2 \Rightarrow$$ **Potential "Blow Up"!!**

# Euler Equations $\upsilon = 0$

## • Three-Dimensional case

$\exists \mathrm{T}_*(\vec{u}_0)$ such that we have existence and uniqueness on $[0, \mathrm{T}_*)$.

**• Beale-Kato-Majda**

If $\int_0^{\mathrm{T}} \|\vec{\omega}(t)\|_{L^\infty} \, dt < \infty$ then we have existence and uniqueness on the interval $[0, \mathrm{T}]$

- That is, one has to "control" the $\|\vec{\omega}(t)\|_{L^\infty}$ in some way!!

- **Constantin and Fefferman**:

Provided sufficient condition involving the Lipschitz regularity of the direction of the vorticity:

$$\vec{\xi} = \frac{\vec{\omega}}{|\vec{\omega}|}$$

## Two-Dimensions Euler

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla)\vec{\omega} = 0$$

$$\vec{u} = \nabla \times (\psi \vec{k})$$

$$\Delta \psi \vec{k} = \vec{\omega}$$

- Yudovich proved a weak version of the maximum principle, that is $\|\omega(t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty}$.

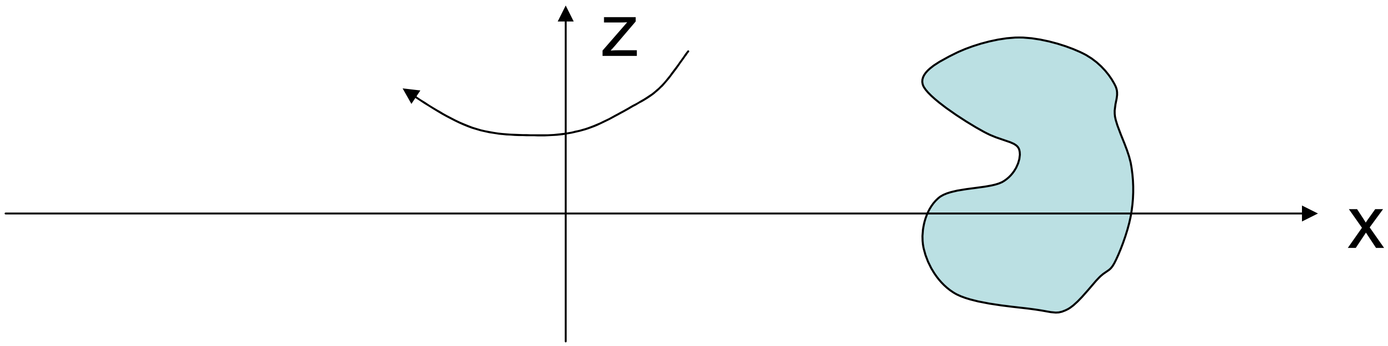
IDEA

$$\|\psi\|_{W^{2,p}} = \sum_{|\alpha| \le 2} \|D^\alpha \psi\|_{L^p} \le c \cdot p \|\Delta \psi\|_{L^p}$$

# Special Results of Global Existence for the three-dimensional Navier-Stokes

Theorem (Kato)

Let $\|u_0\|_{H^{1/2}}$ be small enough. Then the 3D Navier-Stokes equations are globally well-posed for all time with such initial data. The same result holds if the initial data is small in $L^3(\Omega)$ (Kato, Giga & Miyakawa)




• $\Omega$ – Revolution Domain around the $z$ - axis [away from $z$ - axis]

• Let us move to Cylindrical coordinates

**Theorem (Ladyzhenskaya)** Let

$$\vec{u}_0(x,y,z) = (\varphi_r^0(r,z), \varphi_\theta^0(r,z), \varphi_z^0(r,z))$$

be axi-symmetric initial data. Then the three-dimensional Navier-Stokes equations have globally (in time) strong solution corresponding to such initial data. Moreover, such strong solution remains axi-symmetric.

## Theorem (Leiboviz, Mahalov and E.S.T.)

Consider the three-dimensional Navier-Stokes equations in an infinite Pipe. Let

$$\vec{u}_0 = (\varphi_r^0(r, n\theta + \alpha z), \varphi_\theta^0(r, n\theta + \alpha z), \varphi_z^0(r, n\theta + \alpha z))$$

(Helical symmetry). For such initial data we have global existence and uniqueness. Moreover, such a solution remains helically symmetric.

# Remarks

- For axi-symmetric and helical flows the vorticity stretching term is nontrivial, and the velocity field is three-dimensional.
- In the inviscid case, i.e. $\upsilon = 0$, the question of global regularity of the three-dimensional helical or axi-symmetrical Euler equations is still open. Except the invariant sub-spaces where the vorticity stretching term is trivial.

- **Theorem [Cannone, Meyer & Planchon] [Bondarevsky] 1996**

*Let M be given, as large as we want. Then there exists K(M) such that for every initial data of the form*

$$\vec{u}_0 = \sum_{|\vec{k}| \geq K(M)} \hat{\vec{u}}^0_{\vec{k}} e^{i\vec{k}\cdot\vec{x}\frac{2\pi}{L}}$$

[VERY OSCILLATORY]

*the three-dimensional Navier-Stokes equations have global existence of strong solutions.*

**Remark** Such initial data satisfies $\|u_0\|_{H^{1/2}} << 1.$

So, this is a particular case of Kato's Theorem.

# The Effect of Rotation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + \nabla p + \vec{\Omega} \times \vec{u} = 0$$

$$\nabla \cdot \vec{u} = 0$$

- There is $\Omega_0(\mathrm{T}, \vec{u}_0)$ such that if $|\Omega| > \Omega_0$ the solution exists on $[0, \mathrm{T})$.
- That is there exists $\mathrm{T}_0\ (\vec{u}_{0,} |\vec{\Omega}|)$ such that the solution

exists on $[0, \mathrm{T}_0)$. Observe that

$$\mathrm{T}_0 \to \infty \text{ as } |\vec{\Omega}| \to \infty$$

- Babin - Mahalov - Nicolaenko.
- Embid - Majda.
- Chemin, Ghalagher, Granier, Masmoudi,...
- Liu and Tadmor.

## An Illustrative Example

Inviscid Burgers Equation

$$u_t + uu_x = 0 \quad \text{in } \mathbb{R}$$

$$u(x,0) = u_0(\mathrm{x})$$

- If $u_0(x)$ is decreasing function on some subinterval of R then the solution of the above equation develops a singularity (Shock) in finite time.

The solution is given implicitly by the relation:

$$u(x,t) = u_0(x - tu(x,t))$$

## The Effect of the Rotation

$$u \in \mathrm{C} \quad z \in \mathrm{C}$$

$$u_t + uu_z + i\Omega u = 0$$

$$u_0(z) = u(z,0)$$

$$v(z,t) = e^{i\Omega t} u(z,t)$$

$$v_t + e^{-i\Omega t} v v_z = 0$$

$$v(z,t) = v_0\left(z - \frac{e^{-i\Omega t} - 1}{-i\Omega} v(z,t)\right)$$

$$\frac{\partial}{\partial z} v = \frac{v_0'\left(z - \frac{e^{-i\Omega t} - 1}{-i\Omega} v(z,t)\right)}{1 + \frac{e^{-i\Omega t} - 1}{-i\Omega} v_0'\left(z - \frac{e^{-i\Omega t} - 1}{-i\Omega} v(z,t)\right)}$$

If $\Omega >> 1$, (i.e. $\Omega > \Omega_0(u_0)$)

$\frac{\partial}{\partial z} v$ remains finite and the

solution remains regular for all $t \geq 0$.

## The above complex system is equivalent to 2D Rotating Burgers:

$$u = u_1 + iu_2, \qquad z = x + iy$$

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \Omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{u} = 0$$

More generally

- $u_t + div\ F(u) = 0$ (Short time existence)
- $v_t + \cos(\Omega t)\ div\ F(v) = 0$

For $\Omega > \Omega_0(v_0)$ we have global existence.

Let $\tau = \dfrac{\sin \Omega t}{\Omega}$ and denote by $w(\tau, x) = v(t, x)$

Then

$$\begin{cases} w_\tau + div\ F(w) = 0 \\ w(x,0) = v_0(x) = u_0(x) \end{cases}$$

For $\tau$ in the interval $-T_*(v_0) \le \tau \le T_*(v_0)$ the solution $w$ exists. That is whenever $t$ satisfies $-T_*(v_0) \le \dfrac{\sin(\Omega t)}{\Omega} \le T_*(v_0)$

# Bénard Convection Porous Medium

$$
\begin{cases}
\gamma \dfrac{\partial}{\partial t}\vec{u} + \vec{u} + \nabla p - RT\vec{k} = 0 \\
\nabla \cdot \vec{u} = 0 \\
\dfrac{\partial}{\partial t}T - \kappa \Delta T + (\vec{u} \cdot \vec{\nabla})T = 0 \\
\text{Subject to certain physical boundary conditions.}
\end{cases}
$$

- P. Fabrie [1986] Global Existence & Uniqueness
- H.V. Ly E.S.T. [1999] $(\gamma = 0)$

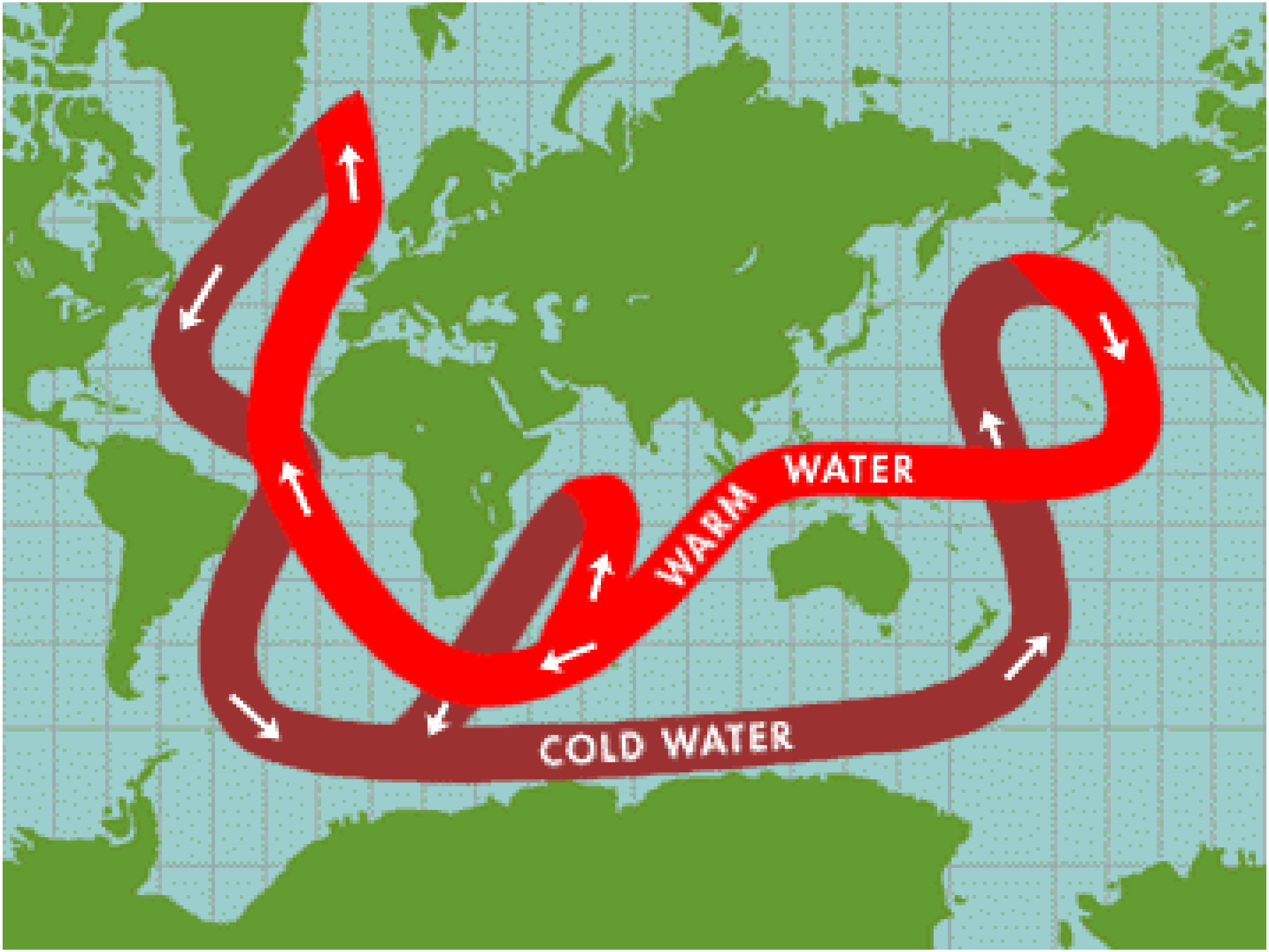
  Same result based on Galerkin numerical procedure.

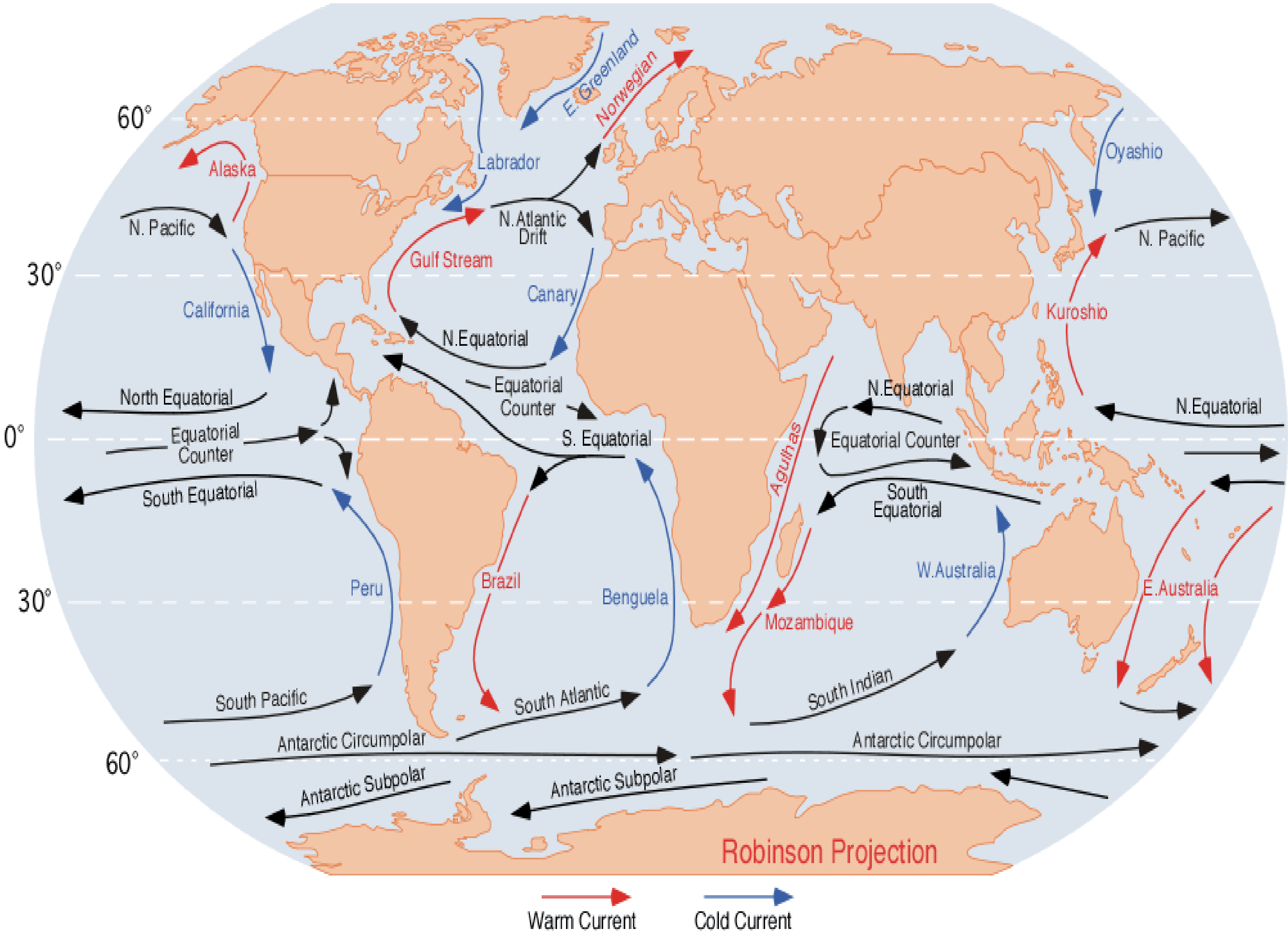
  This gived leads to Spatial Analyticity, and exponential rate of convergence of the Galerkin procedure.

- M. Oliver and E.S.T. $(\gamma > 0)$

  Spatial analyticity of the attractor.

# Large Scale Oceanic Circulations


WARM WATER
COLD WATER


60°
30°
0°
30°
60°
Alaska
N. Pacific
California
Labrador
E. Greenland
Norwegian
N. Atlantic Drift
Gulf Stream
Canary
N. Equatorial
Oyashio
N. Pacific
Kuroshio
North Equatorial
Equatorial Counter
South Equatorial
Equatorial Counter
S. Equatorial
N. Equatorial
Equatorial Counter
N. Equatorial
South Equatorial
Agulhas
Peru
Brazil
Benguela
W. Australia
E. Australia
Mozambique
South Pacific
South Atlantic
South Indian
Antarctic Circumpolar
Antarctic Circumpolar
Antarctic Subpolar
Antarctic Subpolar
Robinson Projection
Warm Current
Cold Current

# Be'nard Convection/Boussinesq Approximation

$$\frac{\partial}{\partial t} v_H - \upsilon\left(\Delta_H + \frac{\partial^2}{\partial z^2}\right) v_H + (v_H \cdot \nabla_H) v_H + w\frac{\partial}{\partial z} v_H + \frac{1}{\rho_0}\nabla_H p + f\,\vec{k} \times v_H = 0$$

$$\frac{\partial}{\partial t} w - \upsilon\left(\Delta_H + \frac{\partial^2}{\partial z^2}\right) w + (v_H \cdot \nabla_H) w + w\frac{\partial}{\partial z} w + \frac{1}{\rho_0}\frac{\partial}{\partial z} p + Tg = 0$$

$$\nabla_H \cdot v_H + \frac{\partial}{\partial z} w = 0$$

$$\frac{\partial}{\partial t} T - \kappa \Delta T + (v_H \cdot \nabla_H) T + w\frac{\partial}{\partial z} T = \rho_0 Q$$

Here $(v_H, w) = \vec{u}.$

# Typical Scales in the Ocean

- horizontal distance $L \sim 10^6$ m
- horizontal velocity $U \sim 10^{-1}$ m/s
- depth $H \sim 10^3$ m
- Coriolis parameter $f \sim 10^{-4}$ 1/s
- gravity $g \sim 10$ m/s$^2$
- density $\rho_0 \sim 10^3$ kg/m$^3$

# Calculating the typical values

- Typical vertical velocity $W = UH/L \sim 10^{-4}$ m/s
- Typical pressure $P = \rho_0 g H \sim 10^{7}$ Pa
- Typical time scale $T = L/U \sim 10^{7}$ s

# Scale Analysis of Vertical Motion – The Ideal Case

$$\frac{\partial}{\partial t}w+(v_H\cdot\nabla_H)w+w\frac{\partial}{\partial z}w+\frac{1}{\rho_0}\frac{\partial}{\partial z}p+Tg=0$$

$$\frac{W}{\mathrm{T}}+\frac{UW}{L}+\frac{W^2}{H}+\frac{P}{H\rho_0}+Tg=0$$

$$10^{-11}+10^{-11}+10^{-11}+10+10=0$$

# Hydrostatic Balance

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}p + Tg = 0$$

# Scale Analysis – The Ideal Case

$$\frac{\partial}{\partial t}v_H + (v_H \cdot \nabla_H)v_H + w\frac{\partial}{\partial z}v_H + \frac{1}{\rho_0}\nabla_H p + f\,\vec{k} \times v_H = 0$$

$$\frac{U}{\mathrm{T}} + \frac{U^2}{L} + \frac{UW}{H} + \frac{P}{L\rho_0} + UF = 0$$

$$10^{-8} + 10^{-8} + 10^{-8} + 10^{-2} + 10^{-5} = 0$$

# Rossby Number

$$R = \frac{U}{F \quad L}$$

# Geostrophic Balance

- When $R << 1$

$$\frac{1}{\rho_0}\nabla_H p + f\vec{k}\times v_H = 0$$

# The Ideal Planetary Geostrophic equations

$$\frac{1}{\rho_0}\nabla_H p + f\,\vec{k}\times v_H = 0$$

$$\frac{1}{\rho_0}\partial_z p + Tg = 0$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H)T + wT_z = \rho_0 Q + \kappa \partial_{zz} T$$

# Rayleigh Friction and Horizontal-Diffusion

$$\frac{1}{\rho_0}\nabla_H p + f\,\vec{k}\times v_H = F(v_H)$$

$$\frac{1}{\rho_0}\partial_z p + Tg = 0$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H)T + w\partial_z T = \rho_0 Q + \kappa_v \partial_{zz} T + D(T)$$

# Friction, Viscosity and Diffusion Schemes

- Conventional eddy viscosity

$$F(v_H) = A_v \Delta_H v_H + A_h \partial_{zz} v_H \quad \text{and} \quad D(T) = \kappa_H \Delta_H T$$

- Linear drag

$$F(v_H) = -\varepsilon v_H$$

What should be the diffusion operator D?

# The Viscous PG Equations

$$\frac{1}{\rho_0}\nabla_H p + f\,\vec{k}\times v_H = K_v \Delta_H v_H + K_h \partial_{zz} v_H$$

$$\frac{1}{\rho_0}\partial_z p + Tg = 0$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H)T + wT_z = \rho_0 Q + \kappa_h \Delta_H T + \kappa_v \partial_{zz} T$$

# The Viscous PG Equations

## Weak Solutions

$$T \in C_w([0,\mathrm{T}], L^2) \cap L^2([0,\mathrm{T}], H^1)$$

## Strong Solutions

$$T \in C([0,\mathrm{T}], H^1) \cap L^2([0,\mathrm{T}], H^2)$$

# Results

- Samelson, Temam and Wang (1998)

  * the existence of the weak solutions,

  but no uniqueness,

  * the short time existence of the strong solutions.

- Samelson, Temam and Wang (2000)

  * global existence of the strong solution if

  initial data is bounded, i.e. in $L^\infty$.

# Results

- Cao and E.S.T. (2003)

  * the uniqueness of weak solutions
  * the global existence of the strong solutions for any initial data in $H^1$
  * existence of the global attractor.
  * upper bounds for the dimension of the global attractor.

# Existence of Global Attractor

- Absorbing Ball **B** in $L^2$ (energy estimate)
- Absorbing Ball B in $H^1$ (energy estimate and the uniform Gronwall inequality)

$$A = \bigcap_{s>0} \bigcup_{t>s} S(t)B \subset H^1.$$

# The Rayleigh Friction Case

$$\frac{1}{\rho_0}\nabla_H p + f\,\vec{k}\times v_H = -\varepsilon v_H$$

$$\frac{1}{\rho_0}\partial_z p + Tg = 0$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H)T + wT_z = \rho_0 Q + \kappa_h \Delta_H T + \kappa_v \partial_{zz} T$$

# Natural Boundary Conditions

- no normal flow

$\vec{v}_H \cdot \vec{n} = 0$ on side and $w = 0 \quad \text{when } z = -h, 0$

- no heat-flux

$\frac{\partial}{\partial \vec{n}} T = 0$ on the side and

$\partial_z T = 0 \;\text{ when } z = -h, 0$

The no-flow boundary condition

$\vec{v}_H \cdot \vec{n} \,|_{\Gamma_s} = 0$ implies that

$$\frac{\partial T}{\partial \vec{e}}\,|_{\Gamma_s} = 0 \;\; \text{where} \;\; \vec{e} = \frac{1}{\sqrt{\varepsilon^2 + f^2}}(\varepsilon n_1 - f n_{2,}\, f n_1 - \varepsilon f_2)$$

this is in addition to the no-heat flux boundary condition

$$\frac{\partial}{\partial \vec{n}} T \,|_{\Gamma_s} = 0$$

Therefore, there are two boundary conditions for the temperature which is governed by a second order parabolic PDE. So it is over-determined, and the problem is ill-posed. This is consistent with the numerical instability observed using this system.

# Rayleigh Friction and Temperature Horizontal Hyper-Diffusion Model

We therefore propose the following artificial Horizontal Hyper-diffusion model

$$\frac{1}{\rho_0}\nabla_H p + f\,\vec{k}\times v_H = -\varepsilon v_H$$

$$\frac{1}{\rho_0}\partial_z p + Tg = 0$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H)T + wT_z = \rho_0 Q + \nabla_H \cdot q(T) + \kappa\partial_{zz} T$$

# With the Boundary Conditions

- no normal flow

$\vec{v}_H \cdot \vec{n} = 0$ on side $\Gamma_s$ , & $w = 0$ when $z = -h,\ 0$

- no heat-flux

$q(T) \cdot \vec{n} = 0$ on the side and

$\partial_z T = 0$ when $z = -h, 0$

# Proposed Artificial Hyper-Diffusion

$$H = \begin{pmatrix} 1 & -f/\varepsilon \\ f/\varepsilon & 1 \end{pmatrix}$$

$$q(T) = \lambda\, H \nabla_H \left( \nabla_H \cdot (H^T \nabla_H T)\right) + \mu \nabla_H T_{zz} - K_h \nabla_H T$$

Which is positive definite (dissipative/stabilizing) with the associate boundary conditions.

# Hyper Horizontal Diffusion Model

## Weak Solutions

$$\vec{u} \in C_w([0,\mathrm{T}],L^2), \quad \Delta\vec{u} \in L^2([0,\mathrm{T}],L^2)$$

## Strong Solutions

$$\nabla\vec{u} \in L^\infty([0,\mathrm{T}],H^1), \quad \Delta\vec{u} \in L^2([0,\mathrm{T}],H^2)$$

# Results

- Cao, E.S.T., Ziane (2004)

  * The global existence and uniqueness of the weak solutions.

  * The global existence of the strong solutions.

  * Existence of the global attractor.

  * Provide upper bounds for the dimension of the global attractor.

# Recall Scale Analysis of Vertical Motion –The Ideal Case

$$\frac{\partial}{\partial t}w+(v_H\cdot\nabla_H)w+w\frac{\partial}{\partial z}w+\frac{1}{\rho_0}\frac{\partial}{\partial z}p+Tg=0$$

$$\frac{W}{\mathrm{T}}+\frac{UW}{L}+\frac{W^2}{H}+\frac{P}{H\rho_0}+Tg=0$$

$$10^{-11}+10^{-11}+10^{-11}+10+10=0$$

# Recall Scale Analysis for Horizontal Motion – The Ideal Case

$$\frac{\partial}{\partial t}v_H + (v_H \cdot \nabla_H)v_H + w\frac{\partial}{\partial z}v_H + \frac{1}{\rho_0}\nabla_H p + f\,\vec{k} \times v_H = 0$$

$$\frac{U}{\mathrm{T}} + \frac{U^2}{L} + \frac{UW}{H} + \frac{P}{L\rho_0} + UF = 0$$

$$10^{-8} + 10^{-8} + 10^{-8} + 10^{-2} + 10^{-5} = 0$$

# The Primitive Equations of Large Scale Oceanic and Atmospheric Dynamics

$$\partial_t v_H + (v_H \cdot \nabla_H) v_H + w \partial_z v_H + \nabla_H p + f \vec{k} \times v_H$$
$$= A_h \Delta_H v_H + A_v \partial_{zz} v_H$$

$$\partial_z p + gT = 0$$

$$\nabla_H \cdot v_H + \partial_z w = 0$$

$$T_t + (v_H \cdot \nabla_H) T + w T_z = Q + K_h \Delta_H T + K_v T_{zz}$$

- Introduced by Richardson (1922)

  For Weather Prediction

- J.L. Lions, R. Temam, S. Wang (1992)

Gave Some Asymptotic Derivation of the Model.

# Primitive Equations

## Weak Solutions

$$\vec{u} \in C_w([0,T], L^2) \bigcap L^2([0,T], H^1)$$

## Strong Solutions

$$\vec{u} \in L^\infty([0,T], H^1) \bigcap L^2([0,T], H^2)$$

# Previous Results

- J.L. Lions, Temam, S. Wang (1992), and Temam, Ziane (2003)
The global existence of the weak solutions (No Uniqueness).

- Guillen-Gonzalez, Masmoudi, Rodriquez-Bellido (2001), and Temam, Ziane (2003)
The short time existence of the strong solution

- Temam, Ziane (2003)
Global Existence of Strong Solution for the 2-D case.

- C. Hu, Temam, Ziane (2003)
Global Regularity for Restricted (Large) Initial Data in Thin Domains.

# Results

- Cao and E.S.T. *Annals of Mathematics* (2007) (to appear)

  * the global existence of the weak solutions (Galerkin method)
  * the global existence and uniqueness of the strong solutions.
  * existence of the global attractor.
  * upper bound for the dimension of the global attractor.

# A different formulation of the PE

$$w(x,y,z) = -\int_{-h}^{z} \nabla_H \cdot v_H(x,y,\xi)\, d\xi$$

$$p(x,y,z) = p_s(x,y) - g\int_{-h}^{z} T(x,y,\xi)\, d\xi$$

$$\bar{v}_H(x,y) = \frac{1}{h}\int_{-h}^{0} v_H(x,y,\xi)\, d\xi\,, \quad \nabla_H \cdot \bar{v}_H = 0$$

$$\tilde{v}_H(x,y,z) = v_H(x,y,z) - \bar{v}_H(x,y)$$

# The Barotropic Mode – The Averaged Part of the Horizontal Velocity

$$\partial_t \bar{v}_H + \overline{(v_H \cdot \nabla_H) v_H + w \partial_z v_H} + f \vec{k} \times \bar{v}_H + \nabla_H p_s$$

$$= A_h \Delta_H \bar{v}_H + \nabla_H \overline{\int_{-h}^{z} T \, dz}$$

# The Baroclinic Mode –The Fluctuation Part of the Horizontal Velocity

$$\partial_t \tilde{v}_H + (\tilde{v}_H \cdot \nabla_H)\tilde{v}_H + (\tilde{v}_H \cdot \nabla_H)\bar{v}_H + (\bar{v}_H \cdot \nabla_H)\tilde{v}_H +$$

$$\left(-\int_{-h}^{z} \nabla_H \cdot v_H \, dz\right) \partial_z \tilde{v}_H + f\,\vec{k} \times \tilde{v}_H -$$

$$\overline{(\tilde{v}_H \cdot \nabla_H)\tilde{v}_H + (\nabla_H \cdot \tilde{v}_H)\tilde{v}_H} =$$

$$A_h \Delta_H \tilde{v}_H + A_v \partial_{zz} \tilde{v}_H + \nabla_H \int_{-h}^{z} gT\, d\xi - \nabla_H \overline{\int_{-h}^{z} gT\, d\xi}$$

# The IDEA – Focus on Burgers Equation

$$u_t - \nu \Delta u + (u \cdot \nabla) u = 0$$

We have

$$\frac{1}{2}\partial_t |u(x,t)|^2 - \frac{1}{2}\Delta |u(x,t)|^2 + \sum_{i,j} \left( \frac{\partial u_i}{\partial x_j} \right)^2 + \frac{1}{2} u \cdot \nabla |u(x,t)|^2 = 0$$

A maximum principle for $|u(x,t)|^2$ and $L^\infty$ bound.

Global Regularity for 1D, 2D and 3D Burgers Equation.

# The Pressure Term!!

- Is the major difference between Burgers and the Navier-Stokes equations.

- What about in our system?

# The Averaged Equation is "like" the 2D Navier-Stokes.

$$\partial_t \bar{v}_H + \overline{(v_H \cdot \nabla_H) v_H + w \partial_z v_H} + f \vec{k} \times \bar{v}_H + \nabla_H p_s$$

$$= A_h \Delta_H \bar{v}_H + \nabla_H \overline{\int_{-h}^{z} T \, dz}$$

Where $p_s(x, y)!!$

# The Fluctuation Equation is "like" 3D Burgers Equations – Has No Pressure Term!!

$$\partial_t \tilde{v}_H + (\tilde{v}_H \cdot \nabla_H)\tilde{v}_H + (\tilde{v}_H \cdot \nabla_H)\bar{v}_H + (\bar{v}_H \cdot \nabla_H)\tilde{v}_H +$$

$$\left( -\int_{-h}^{z} \nabla_H \cdot v_H \, dz \right) \partial_z \tilde{v}_H + f\,\vec{k} \times \tilde{v}_H -$$

$$\overline{(\tilde{v}_H \cdot \nabla_H)\tilde{v}_H + (\nabla_H \cdot \tilde{v}_H)\tilde{v}}_H$$

$$= A_h \Delta_H \tilde{v}_H + A_v \partial_{zz} \tilde{v}_H + \nabla_H \int_{-h}^{z} gT \, d\xi - \nabla_H \overline{\int_{-h}^{z} gT \, d\xi}$$

# A-priori Estimates

- $\left\| \tilde{v}_H \right\|_{L^6} \le K$
- $\left\| \nabla_H \bar{v}_H \right\|_{L^2} \le K$

$\Rightarrow \left\| \bar{v}_H \right\|_{L^6} \le K$

$\Rightarrow \left\| v_H \right\|_{L^6} = \left\| \bar{v}_H + \tilde{v}_H \right\|_{L^6} \le K$

Q.E.D.

# One of the Main Estimates Used

$$\left| \int_\Omega \left[ \left( \int_{-h}^{0} u(x,y,z)\, dz \right) f(x,y,z)\; g(x,y,z) \right] dxdydz \right|$$

$$\leq C \| u \|_{L^2(\Omega)}^{1/2} \| u \|_{H^1(\Omega)}^{1/2} \| f \|_{L^2(\Omega)}^{1/2} \| f \|_{H^1(\Omega)}^{1/2} \| g \|_{L^2(\Omega)}$$

# Back to The 3D Navier-Stokes Equations

$$\frac{\partial}{\partial t}\vec{u} - \upsilon \Delta \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \frac{1}{\rho_0}\nabla p = \vec{f}$$

$$\nabla \cdot \vec{u} = 0$$

# New Criterion for Global Regularity of the 3D Navier-Stokes Equations

Theorem (C. Cao and E.S.T. 2005) :

The strong solution of the 3D Navier - Stokes equations exists on the the interval [0, T] for as long as $\partial_z p \in L^r((0,\mathrm{T}), L^s(\Omega))$, where $r > 3$ and $s > 2$.

This is different that the result of Y. Zhou (2005) where the assumption is on $\nabla p$.

# Inviscid Regularazation of the 3D Euler Equations

$$-\alpha^2 \Delta \frac{\partial}{\partial t}\vec{u} + \frac{\partial}{\partial t}\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho_0}\nabla p = 0$$

$$\nabla \cdot \vec{u} = 0$$

# Modified Energy

$$\int (|u(x,t)|^2 + \alpha^2 |\nabla u(x,t)|^2) dx = \text{const.}$$

# Inviscid Regularization of the Surface Quasi-Geostrophic

$$-\alpha^2 \Delta \theta_t + \theta_t + u \cdot \nabla \theta = 0$$

$$u = \nabla^{\perp} (-\Delta)^{-1/2} \theta$$