Good reasons to study Euler equation.

Applications often correspond to very large Reynolds number =ratio between *the strenght of the non linear effects* and *the strenght of the linear viscous effects.*

$$\Re = \frac{UL}{\nu} \Rightarrow \Re \sim 2 \times 10^7 \text{airplanes} \tag{1}$$

A theorem valid for any finite Reynolds number should be compatible with results concerning infinite Reynolds number. In fact it is the case Reynolds$= \infty$ which drive other results.

The parabolic structure and the scalings does not carry enough information to deal with the $3d$ Navier-Stokes equations. *Simple examples with the same scalings but with no conservation of energy may (concerning regularity) exhibit very different behaviour.*

1. Hamilton Jacobi type equation

$$\partial_t \phi - \nu \Delta \phi + \frac{1}{2}|\nabla \phi|^2 = 0 \ \text{ in}\, \Omega \times \mathbb{R}_t^+ \,, \tag{2}$$

$$\partial_t \nabla \phi - \nu \Delta \nabla \phi + \nabla \phi \cdot \nabla(\nabla \phi) = 0 \ \text{ in } \Omega \times \mathbb{R}_t^+ \tag{3}$$

$$\phi(x,t) = 0 \text{ for } x \in \partial\Omega\,, \phi(.,0) = \phi_0(.) \in L^\infty(\Omega)\,, \tag{4}$$

For $\nu > 0$ global smooth solution. May become singular (with shocks) for $\nu = 0$. .

2. Kuramoto Sivashinsky equation :

$$\partial_t u + \Delta^2 u + \Delta u + u \cdot \nabla u = 0 \tag{5}$$

Open problem in $2d$

2. Montgomery-Smith example :

$$\partial_t u - \nu\Delta_D u + \frac{1}{2}(-\Delta)^{\frac{1}{2}}u^2 = 0 \text{ in } \Omega \times \mathbb{R}_t^+ , \tag{6}$$

$$u(x,t) = 0 \text{ for } x \in \partial\Omega , u(.,0) = u_0(.) \in L^\infty(\Omega) , \tag{7}$$

$$m(t) = -\int_\Omega u(x,t)\phi_1(x)dx - \Delta\phi_1 = \lambda_1\phi_1 , \phi_1(x) \geq 0 \tag{8}$$

$$\frac{d}{dt}m(t) + \lambda_1 m(t) = \sqrt{\lambda_1}\int_\Omega u(x,t)^2\phi_1(x)dx . \tag{9}$$

With Cauchy Schwartz relation :

$$m(t)^2 = (\int u(x,t)\phi_1(x)dx)^2 \leq \int_\Omega u(x,t)^2\phi_1(x)dx \int_\Omega \phi_1(x)dx$$

Blows up for $m(0) > 0$ large enough. Introduced with $\Omega = \mathbb{R}^3$ by Montgomery-Smith. Proof shows that the same blow up property may appear in any space dimension for the solution of the "cheap hyper viscosity equations"

$$\partial_t u + \nu(-\Delta)^m u + \frac{1}{2}|\nabla|u^2 = 0 \tag{10}$$

Pathologies for Euler equations.

Blow up withInfinite Energy

P. Constantin, Ohkitani J. Gibbon and C. Oseen (1927)

$$u = (u_1(x_1, x_2, t), u_2(x_1, x_2, t), x_3\gamma(x_1, x_2, t)) = (\tilde{u}, x_3\gamma)$$
$$\partial_t(\nabla \wedge \tilde{u}) + \tilde{u}\nabla(\nabla \wedge \tilde{u}) = \gamma\nabla \wedge \tilde{u}$$
$$\nabla \cdot \tilde{u} = -\gamma \Rightarrow \nabla \cdot u = 0$$
$$\partial_t\gamma + \tilde{u} \cdot \nabla\gamma = -\gamma^2 + I(t)$$
$$I(t) = -\frac{2}{L^2}\int_{(\mathbb{R}^2/L)^2}(\gamma(x_1, x_2, t))^2 dx_1 dx_2 \text{ to enforce periodicity}$$
$$\partial_t\gamma + \tilde{u}\nabla\gamma = -\gamma^2 - \frac{2}{L^2}\int_{(\mathbb{R}^2/L)^2}(\gamma(x_1, x_2, t))^2 dx_1 dx_2 \quad (11)$$

Proof of the blow up including explicit nature of this blow up follows. Non physical because the initial energy :

$$\int_{(\mathbb{R}^2/L)^2\times\mathbb{R}} |u(x_1, x_2, x_3, 0)|^2 dx_1 dx_2 dx_3 = \infty$$

is infinite.

The Di Perna Lions pressureless example.

Proposition *For* $1 < p < \infty$ *there is no continuous function* $\tau \mapsto \phi(\tau)$ *such that :*

$$\|u(.,t)\|_{W^{1,p}(\Omega)} \le \phi(\|u(.,0)\|_{W^{1,p}(\Omega)})\,. \tag{12}$$

No contradiction with $s > \frac{5}{2}$ $\|u(t)\|_{H^s(\Omega)} \le \|u(0)\|_{H^s(\Omega)}/1 - Ct\|u(0)\|_{H^s(\Omega)}$

$$u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2), x_2))\,, \text{in}\,, (\mathbb{R}/Z^3), \tag{13}$$

$$\nabla \cdot u = 0\,, \quad \partial_t u + u \cdot \nabla u = 0 \tag{14}$$

$$\|u(.,t)\|^p_{W^{1,p}(\Omega)} \simeq \int |\partial_{x_2} u_1(x_2)|^p dx_1 dx_2 dx_3 + \int |\partial_{x_1} u_3(x_2)|^p dx_1 dx_2 dx_3 + t^p \int |\partial_{x_2} u_1(x_2)|^p |\partial_{x_1} u_3(x_2)|^p dx_1 dx_2 dx_3\,. \tag{15}$$

Proof completed by regularisation. Shows also that the vorticity may grow at least linearly.

Weak limits. Reynolds stress tensor.

Weak limits of solutions of Navier-Stokes Dynamics

$$\partial_t u_\nu + \nabla \cdot (u_\nu \otimes u_\nu) - \nu \Delta u_\nu + \nabla p_\nu = 0\,, \tag{16}$$

$$u_\nu(x,0) = u_0(x) \in L^2(\Omega)\,,\ \ \nabla \cdot u_\nu = 0\,,\ u_\nu = 0 \text{ on } \partial\Omega \tag{17}$$

$$\int_\Omega |\overline{u}(x,t)|^2 dx + 2\nu \int_0^t\int_\Omega |\nabla u|^2 dx dt \le \int_\Omega |\overline{u}_0(x)|^2 dx \tag{18}$$

$$\lim_{\nu\to 0} (u_\nu \otimes u_\nu) = \overline{u} \otimes \overline{u} + \lim_{\nu \to 0} (u_\nu - \overline{u}) \otimes (u_\nu - \overline{u})\,, \tag{19}$$

$$\partial_t \overline{u} + \nabla \cdot (\overline{u} \otimes \overline{u}) + \lim_{\nu\to 0} \nabla \cdot \Big((\overline{u} - u_\nu) \otimes (\overline{u} - u_\nu) \Big) + \nabla p = 0 \tag{20}$$

$$RT(x,t) = \lim_{\nu \to 0} \Big(\nabla \cdot (\overline{u} - u_\nu) \otimes (\overline{u} - u_\nu) \Big) \tag{21}$$

Basic properties (if any) of RT Reynolds tensor ? $RT(x,t) \equiv 0$? $RT(x,t)$ Generated by high frequency oscillations should be intrinsic and in *particular independent of orthogonal change of coordinates. In $2d \Rightarrow$*

$$RT(x,t) = \alpha(x,t)Id + \frac{1}{2}\beta(x,t)(\nabla\overline{u} + (\nabla\overline{u})^T) \quad (22)$$

$$\partial_t\overline{u} + \nabla\cdot(\overline{u}\otimes\overline{u}) + \lim_{\nu\to 0}\nabla\cdot(\beta(x,t)\frac{1}{2}(\nabla\overline{u} + (\nabla\overline{u})^T)) + \nabla p = 0 \quad (23)$$

"Soft information" does not indicate where $\beta(x,t)$ is not zero. It does not indicate if this coefficient is positive and it does not say how to compute it. But this turns out to be the turbulent diffusion coefficient which is present in classical engineering models like Smagorinskii or $k\epsilon$.

A Counter example of Cheverry : a sequence of oscillating solutions :

$$u_\epsilon(x,t) = U(x,t,\frac{\phi(x,t)}{\epsilon}) + O(\epsilon)\,, \overline{u} = w - \lim_{\epsilon\to 0} u_\epsilon = \int_0^1 U(x,t,\theta)d\theta \quad (24)$$

indicates that the isotropy property may not be always true.

Dissipative solutions

Let $w(x,t)$ be divergence free smooth test functions, which satisfies $w\cdot\vec{n}=0$ on the boundary $\partial\Omega$, test functions not assumed to be solution of the Euler equations and therefore generating the (may be not zero) tensor :

$$E(w)=\partial_t w+P(w\cdot\nabla w)\neq 0?\,,\ P \text{ Leray projector} \tag{25}$$

With $u(x,t)$ smooth divergence free, tangent to the boundary solution

$$\partial_t u+\nabla\cdot(u\otimes u)+\nabla p=0$$
$$\partial_t w+\nabla\cdot(w\otimes w)+\nabla q=E(w)$$
$$\frac{d|u-w|^2}{dt}+2(S(w)(u-w),(u-w))=2(E(w),u-w) \tag{26}$$
$$S(w)=\frac{1}{2}(\nabla w+(\nabla w)^T)\,. \tag{27}$$
$$|u(t)-w(t)|^2\le e^{\int_0^t 2\|S(w)\|_\infty(s)ds}|u(0)-w(0)|^2$$
$$+2\int_0^t e^{\int_s^t 2\|S(w)\|_\infty(\tau)d\tau}(E(w),u-w)(s)ds \tag{28}$$

• *Any classical solution is a dissipative solution. Every dissipative solution satisfies the relation*

$$|u(t)|^2 \leq |u(0)|^2 \,. \tag{29}$$

• *The dissipative solutions are "stable with respect to classical solutions."*

$$|u(t) - w(t)|^2 \leq e^{\int_0^t 2\|S(w)\|_\infty(s)ds} |u(0) - w(0)|^2$$

• *In the absence of physical boundaries any weak limit of suitable Leray solutions of Navier-Stokes dynamics is a dissipative solution.*

$$\partial_t u_\nu + \nabla_x \cdot (u_\nu \otimes u_\nu) - \nu\Delta u_\nu + \nabla p_\nu = 0 \tag{30}$$

$$\partial_t w + \nabla_x \cdot (w \otimes w) - \nu\Delta w + \nabla p = -\nu\Delta w \tag{31}$$

$$\frac{d|u_\nu - w|^2}{dt} + 2(S(w)(u_\nu - w),(u_\nu - w)) - 2\nu \int \Delta(w - u_\nu),(w - u_\nu)dx$$
$$\leq 2(E(w), u_\nu - w) - (\nu\Delta w(u_\nu - w)) \tag{32}$$

No boundary & existence of a smooth solution of Euler⇒strong convergence and 0 *limit for the energy dissipation.*

• *Play a similar role for the Boltzmann Limit, L. Saint Raymond.*

Two cases where dissipative solution bring no information even in $2d$.

• $u_\epsilon(x,0)$ converging weakly but not strongly in $L^2(\Omega)$ to an initial data $\overline{u}(x,0)$

$$|u_\epsilon(t)-w(t)|^2 \le e^{\int_0^t 2\|S(w)\|_\infty(s)ds}|u_\epsilon(0)-w(0)|^2$$
$$+2\int_0^t e^{\int_s^t 2\|S(w)\|_\infty(\tau)d\tau}(E(w),u_\epsilon - w)(s)ds \quad (33)$$
$$\text{However } |\overline{u}(0)-w(0)|^2 \le \liminf_{\epsilon\to 0}|u_\epsilon(0)-w(0)|^2 \quad (34)$$

Counter example of Cheverry : a sequence of oscillating solutions :

$$u_\epsilon(x,t) = U(x,t,\frac{\phi(x,t)}{\epsilon}) + O(\epsilon)\,, \overline{u} = w - \lim_{\epsilon\to 0} u_\epsilon = \int_0^1 U(x,t,\theta)d\theta \quad (35)$$

Reynolds stress tensor is not invariant under rotation.

• Physical boundary and no slip boundary condition⇒ vorticity generation.

$$\frac{1}{2}\frac{d|u_\nu - w|^2}{dt} + (S(w)(u_\nu - w),(u_\nu - w)) + \nu\int|\nabla(w-u_\nu)|^2dx$$
$$\le (E(w),u_\nu - w) - (\nu\Delta w(u_\nu - w)) + \nu\int_{\partial\Omega}\partial_n u_\nu w d\sigma \quad (36)$$

Theorem, Kato *Let* $u(x,t) \in W^{1,\infty}((0,T)\times\Omega)$ *be a solution of the Euler dynamics and introduce a sequence of Leray solutions* u_ν *of the Navier-Stokes dynamics with no slip boundary condition :*

$$\partial_t u_\nu - \nu\Delta u_\nu + \nabla\cdot(u_\nu \otimes u_\nu) + \nabla p_\nu = 0\,,\ u_\nu(x,t) = 0 \text{ on } \partial\Omega \tag{37}$$

with initial data $u_\nu(x,0) = u(x,0)$ *then the following facts are equivalent :*

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\partial\Omega} (\nabla \wedge u_\nu)\cdot(\vec{n}\wedge u) d\sigma dt = 0 \tag{38}$$

$$u_\nu(t) \to u(t) \text{ in } L^2(\Omega) \text{ uniformly in} t \in [0,T] \tag{39}$$

$$u_\nu(t) \to u(t) \text{ weakly in } L^2(\Omega) \text{ for each } t \in [0,T] \tag{40}$$

$$\lim_{\nu\to 0} \nu \int_0^T \int_\Omega |\nabla u_\nu(x,t)|^2 dx dt = 0 \tag{41}$$

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\Omega\cap\{d(x,\partial\Omega)<\nu\}} |\nabla u_\nu(x,t)|^2 dx dt = 0\,. \tag{42}$$

Proof Main point construction of a divergence free solution $v_\nu(x,t)$ with support in the region $\{x \in \Omega / dist(x, \partial\Omega) \leq \nu\} \times [0, T[$ which coincides with u on $\partial\Omega \times [0, T]$

$$\|v_\nu\|_{L^\infty(\Omega)\times]0,T]} + \|dist(x,\partial\Omega)\nabla v_\nu\|_{L^\infty(0,T\times\Omega} \leq K\,, \tag{43}$$

$$\|(dist(x,\partial\Omega))^2\nabla v_\nu\|_{L^\infty(0,T\times\Omega)} \leq K\nu \tag{44}$$

$$\|v_\nu\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t v_\nu\|_{L^\infty(0,T;L^2(\Omega))} \leq K\nu^{\frac{1}{2}} \tag{45}$$

$$\|\nabla v_\nu\|_{L^\infty(0,T;L^2(\Omega))} \leq K\nu^{-\frac{1}{2}} \tag{46}$$

$$\|\nabla v_\nu\|_{L^\infty(\Omega)\times]0,T]} \leq K\nu^{-1} \tag{47}$$

Clear cut difference between two situations in the presence u a smooth solution of Euler equations

- *The dissipation of energy*

$$\lim_{\nu\to 0}\epsilon(\nu)=\lim_{\nu\to 0}\nu\int_0^T\|\nabla u_\nu(x,t)\|^2 dxdt=0\,, u_\nu\to u \text{ strong} \tag{48}$$

- *Otherwise* $\overline{u}=w\lim u_\nu$ *does not conserve energy. One of the two situations .*

a) $\overline{u}$ *a weak solution (not strong) solution energy decay. Compatible with uniform estimate for the Fourier spectra*

$$E_\nu(k,t)=|u_\nu(\widehat{k},t)|^2|k|^{d-1}\leq C|k|^{-\beta}\,,0<\beta<5/3 \tag{49}$$

Otherwise contradiction with the Onsager and Constantin Eyink Titi.

b) No estimate of the type (49) $\overline{u}$ *solution of an equation with non trivial Reynolds tensor.*

Spectra and Wigner transform

Start the energy estimate

$$\frac{1}{2}|u_\nu(.,t)|^2 + \nu\int_0^t\int_\Omega |\nabla u_\nu(x,t)|^2 dx \le \frac{1}{2}|u(.,0)|^2 \tag{50}$$

Localise in $\Omega' \subset\subset \Omega$ $v_\nu = a(x)(u_\nu - \overline{u})$.

$$R\widehat{T(}v_\nu)(x,t,k) = \frac{1}{2\pi^d}\int_{\mathbb{R}^d_y} e^{iky} v_\nu(x - \frac{\sqrt{\nu}}{2}y) \otimes v_\nu(x + \frac{\sqrt{\nu}}{2}y)dy \tag{51}$$

With the inverse Fourier transform one has

$$v_\nu(x,t)\otimes v_\nu(x,t) = \int_{\mathbb{R}^d_k} R\widehat{T(}v_\nu)(x,t,k)dk \tag{52}$$

introduced by Gerard Markowich Mauser and Poupaud and converges weakly to a non negative symmetric matrix-valued measure $\hat{RT}(x,t,dk)$ *Wigner measure or Wigner spectra* and inside

$$\partial_t\overline{u} + \nabla\cdot(\overline{u}\otimes\overline{u}) + \nabla\cdot\int_{\mathbb{R}^d_k}\hat{RT}(x,t,k)dk + \nabla p = 0\,. \tag{53}$$

The Wigner spectra has the following properties :

- *Defined by a two point correlation formula and local :*

$$\lim_{\nu\to 0}\left(\frac{1}{2\pi^d}\int_{\mathbb{R}^d_y} e^{iky}(\phi v_\nu)(x-\frac{\sqrt{\nu}}{2}y)\otimes(\phi v_\nu)(x+\frac{\sqrt{\nu}}{2}y)dy\right)$$
$$= |\phi|^2 RT(\widehat{x,} t, dk)\,\forall\phi\in\mathcal{D} \quad (54)$$

- *Turbulence criteria :* u_ν *smooth near* $(x,t)\Rightarrow RT(\widehat{x,}t,dk)=0$.

- *A microlocal object depends Fourier spectra in the range* $A\le|k|\le\frac{B}{\sqrt{\nu}}$

$$\int_0^\infty\int\psi(k)|\phi(x)|^2\theta(t)RT\widehat{(}u_\nu)(x,t,dk)dxdt$$
$$=\lim_{\nu\to 0}\int_0^\infty\theta(t)\int_{A\le|k|\le\frac{B}{\sqrt{\nu}}}(\psi(\sqrt{\nu}k)(\widehat{\phi v_\nu})\otimes(\widehat{\phi v_\nu}))dt \quad (55)$$

Statistic theory of turbulence : Start with hypothesis : Isotropy, homogeneity, power law scaling and ergodicity then prove properties.

Wigner measures should be compared :

- *Local object determination of its support should be very hard and problem dependent.*

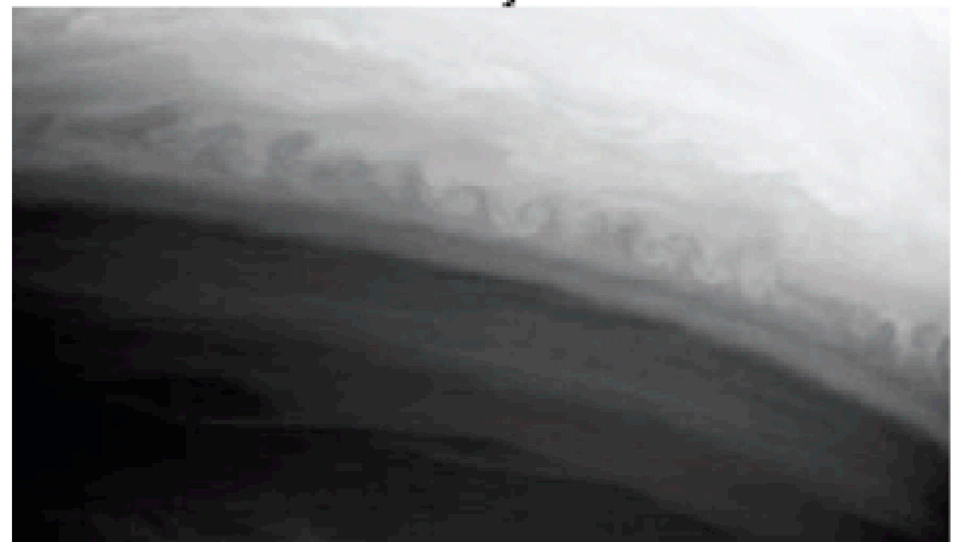
- *Sufficients conditions to make it isotropic ? Not always true Cheverry counter example.*

- *Like in statistical theory there is a range. Assuming isotropy and a power law is there any chance for a formula of the type :*

$$\text{for } A \le |k| \le \frac{B}{\sqrt{\nu}}\,, \frac{\sum_i R\widehat{T}_{ii}(u_\nu)}{|k|^2} \simeq C\epsilon(\nu)^{\frac{2}{3}}|k|^{-\frac{5}{3}}\,? \qquad (56)$$

Prandlt= Kelvin Helmohltz??

Kelvin-Helmholtz instability



A KHI on the planet Saturn, formed at the interaction of two bands of the planet's atmosphere

Kelvin Helmholtz type interfaces.

Wavecloudsduval.jpg (62KB, MIME type: image/jpeg)

Prandlt equations

$$\partial_t u_1^\nu - \nu\Delta u_1^\nu + u_1^\nu \partial_{x_1} u_1^\nu + u_2^\nu \partial_{x_2} u_1^\nu + \partial_{x_1} p^\nu = 0\,, \tag{57}$$

$$\partial_t u_2^\nu - \nu\Delta u_2^\nu + u_1^\nu \partial_{x_1} u_2^\nu + u_2^\nu \partial_{x_2} u_2^\nu + \partial_{x_2} p^\epsilon \nu = 0\,, \tag{58}$$

$$\partial_{x_1} u_1^\nu + \partial_{x_2} u_2^\nu = 0 \quad u_1^\nu(x_1,0) = u_2^\nu(x_1,0) = 0 \text{ on } x_1 \in \mathbb{R}\,. \tag{59}$$

$$\epsilon = \sqrt{\nu}\,, \;\; X_1 = x_1, X_2 = \frac{x_2}{\epsilon}, \tag{60}$$

$$\tilde{u}_1(x_1, X_2) = u_1(x_1, X_2), \tilde{u}_2(x_1, X_2) = \epsilon u_2(x_1, X_2)\,. \tag{61}$$

$$\tilde{u}_1(x_1, 0, t) + U_1(x_1, 0, t) = 0 \tag{62}$$

$$\partial_{x_2} p(x_1, x_2) = 0 \Rightarrow \tilde{p}(x_1, x_2, t) = \tilde{P}(x_1, t) \tag{63}$$

$$\partial_t \tilde{u}_1 - \partial_{x_2}^2 \tilde{u}_1 + \tilde{u}_1 \partial_{x_1} \tilde{u}_1 + \tilde{u}_2 \partial_{x_2} \tilde{u}_1 = \partial_{x_1} \tilde{P}(x_1, t) \tag{64}$$

$$\partial_{x_1} \tilde{u}_1 + \partial_{x_2} \tilde{u}_2 = 0\,, \; \tilde{u}_1(x_1, 0) = \tilde{u}_2(x_1, 0) = 0 \text{ for } x_1 \in \mathbb{R} \tag{65}$$

$$\lim_{x_2\to\infty} \tilde{u}_1(x_1, x_2) = \lim_{x_2\to\infty} \tilde{u}_2(x_1, x_2) = 0, \tag{66}$$

$$\begin{pmatrix} \mathrm{u}_1^\nu(x_1, x_2) \\ \mathrm{u}_2^\nu(x_1, x_2) \end{pmatrix} = \begin{pmatrix} \mathrm{u}_1^\nu(x_1, \frac{x_2}{\sqrt{\nu}}) \\ \sqrt{\nu}\, u_2^\nu(x_1, \frac{x_2}{\sqrt{\nu}}) \end{pmatrix} + u_{\mathrm{int}}(x_1, x_2) \tag{67}$$

• *Prandlt approximation is consistant with the Kato theorem*

$$\nu \int_0^T \int_{\Omega\cap d(x,\partial\Omega)\leq c\nu} |\nabla \wedge u_\nu(x,s)|^2 dxds \leq C\sqrt{\nu}\,. \tag{68}$$

• *Prandlt expansion cannot be always valid Grenier solution not of finite energy. Would be interesting to compare with Kato criteria*

• *Smooth solution for Prandlt* $\Leftrightarrow$ *Convergence of solutions of Navier Stokes (with boundary layer) to Euler*

• *Initial conditions with "recirculation properties" may lead to a finite time blow up. PE highly unstable.*

$$\partial_{x_1}\tilde{u}_1 + \partial_{x_2}\tilde{u}_2 = 0\,. \tag{69}$$

• *Only with analytic initial data (analytic respect to the tangential variable) existence of a smooth solution of the Prandlt equation for a finite time and the convergence to the solution of the Euler equation during this time*

Kelvin Helmholtz equations

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0\,, \quad \nabla \cdot u = 0 \tag{70}$$

$$\partial_t (\nabla \wedge u) + u \cdot \nabla (\nabla \wedge u) = 0\,, \tag{71}$$

$$u(x,t) = \frac{1}{2\pi} R_{\frac{\pi}{2}} \int \frac{x - r'}{|x - r'|^2} \omega(t, r(t,\lambda'))ds'$$

$$= \frac{1}{2\pi} R_{\frac{\pi}{2}} \int \frac{x - r(t,\lambda')}{|x - r(t,\lambda')|^2} \omega(t, r(t,\lambda')) \frac{\partial s(\lambda',t)}{\partial \lambda'} d\lambda' \tag{72}$$

$$\omega_t - \partial_s \left(\omega (\partial_t r - v) \cdot \tau \right) = 0\,, \tag{73}$$

$$(r_t - v) \cdot \nu = 0\,, \tag{74}$$

$$v(t,r) = R_{\pi/2} \frac{1}{2\pi} p.v. \int \frac{r - r(t,s')}{\|r - r(t,s')\|^2} \omega(t,s')ds'\,. \tag{75}$$

$$\partial_t \overline{z} = \frac{1}{2\pi i} p.v. \int \frac{d\lambda'}{z(t,\lambda) - z(t,\lambda')}\,. \tag{76}$$

- *No possible to use Delort construction. Lack of uniqueness.*

- *For the interface problem well posed for small analytic perturbation of rest and small time.*

- *Appearance of singularities Using reversibility.*

- *Reason is that above some regularity threshold* **T** *the problem becomes locally elliptic*

$$z(\lambda,t)=(\alpha t+\beta(\lambda+\epsilon f(t,\lambda))\,,\; f(0,0)=\nabla f(0,0)=0\,. \tag{77}$$

$$\epsilon|\beta|^2\partial_t\overline{f}=\frac{1}{2\pi i}p.v.\int_{z(t,\lambda')\in}\frac{d\lambda'}{(\lambda-\lambda')(1-\epsilon\frac{f(t,\lambda)-f(t,\lambda')}{\lambda-\lambda'})}+E(r(t,\lambda)) \tag{78}$$

$$\frac{1}{2\pi}pv\int\frac{d\lambda'}{(\lambda-\lambda')(1+\epsilon\frac{f(\lambda,t)-f(\lambda',t)}{\lambda-\lambda'})}d\lambda'= \tag{79}$$

$$\frac{\epsilon}{2\pi}\int\frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}d\lambda'+\sum_{n\geq 2}\frac{\epsilon^n}{2\pi}\int\frac{(f(\lambda,t)-f(\lambda',t))^n}{(\lambda-\lambda')^{(n+1)}}d\lambda'\,. \tag{80}$$

$$\frac{1}{2\pi}\int\frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}d\lambda'=-\frac{i}{2}sign(D)f\,, \tag{81}$$

$$\frac{1}{2\pi}vp\int\frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}d\lambda'=|D|f\,. \tag{82}$$

$$\partial_t X=\frac{1}{2|\beta|^2}|D_\lambda|Y+\epsilon R_1(X,Y)+E_1(X,Y)\,, \tag{83}$$

$$\partial_t Y=\frac{1}{2|\beta|^2}|D_\lambda|X+\epsilon R_2(X,Y)+E_2(X,Y)\,, \tag{84}$$

• *The above computation need some regularity assumption. Bigger assumption on regularity $\Rightarrow$proof easier. This is standard in Free boundary problems*

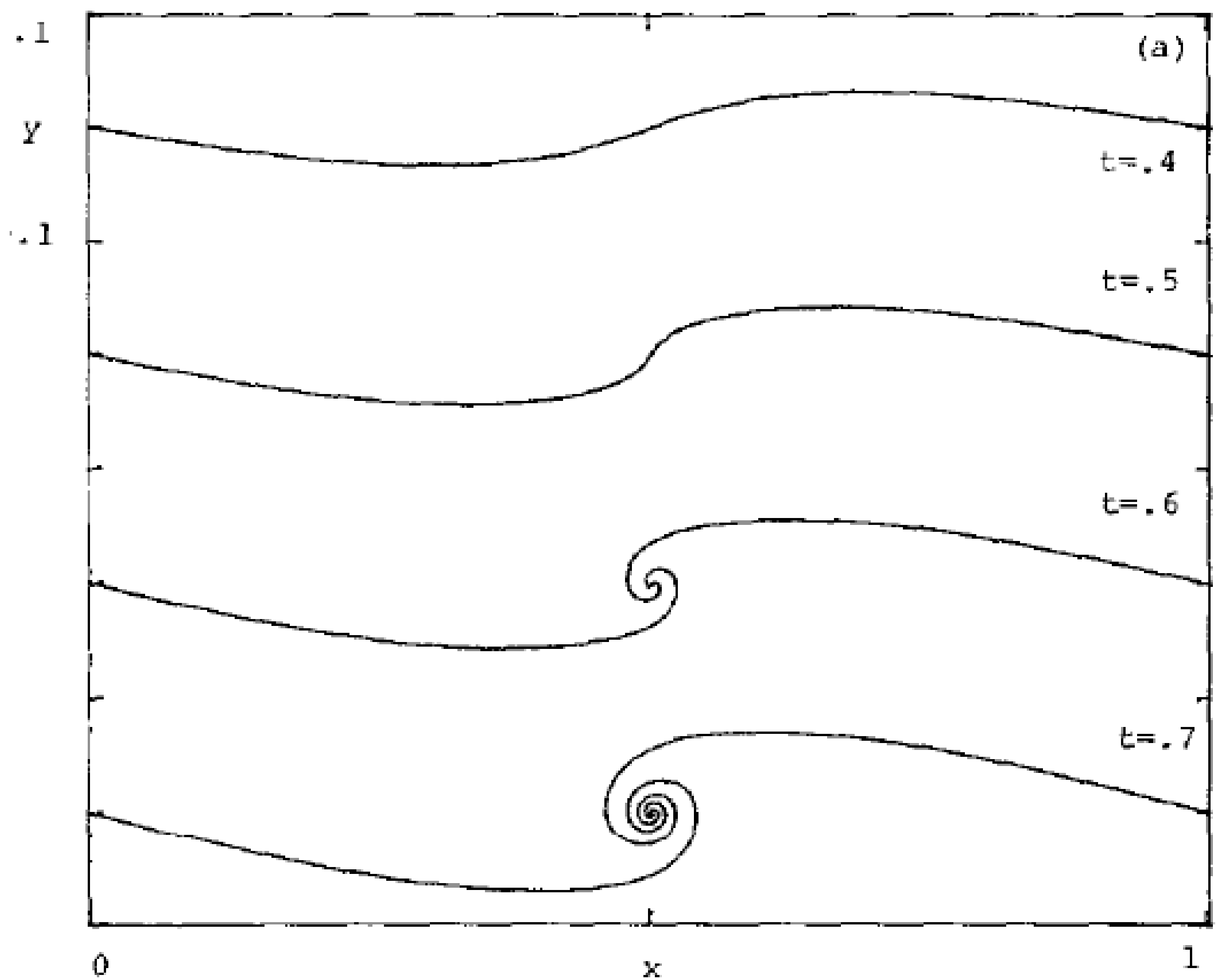
• *Locally in $C^\alpha_t(C^{1+\alpha}_\lambda)$ Lebeau compatible with Caflisch Orellana examples*

• *What after the singularity. It has to be less regular than the threshold* **T** *. People observe spirals with infinite lenght (shape of such spiral). Result of S. Wu with chord arc hypothesis and solution in $H^1_{loc}(\mathbb{R}_t \times \mathbb{R}_\lambda)$ which allows finite length spirals. Therefore no finite lenght spirals may exist after the singularity*

• *Problems with weak solution of Birkhoff Rott. No proof of existence of such solution.*

• *There exist solutions of Birkoff Rott which are not weak solution of Euler*

Kransy simulations in agreement with S. Wu



Prandlt-Munk example : Start from the vortex sheet

$$\omega_0(x_1,x_2) = \frac{x_1}{\sqrt{1-x_1^2}}\Xi(-1,1)(x_1)\otimes\delta(x_2) \tag{85}$$

with $\Xi(-1,1)$ the characteristic function of the interval $(-1,1)$. With the Biot Savard law the velocity is constant

$$\langle v\rangle = (0,-\frac{1}{2})\,. \tag{86}$$

The solution of the Birkhoff-Rott equation is given by the formula

$$x_1(t) = x_1(0)\,, x_2(t) = \frac{t}{2}\omega(x_1,x_2) = \omega_0(x_1,x_2+\frac{t}{2})\,. \tag{87}$$

u associated to this vorticity is *not* even a weak solution of the Euler equation. In fact one has

$$\nabla\cdot u = 0 \qquad \text{and } \partial_t u + \nabla_x\cdot(u\otimes u) + \nabla p = F \tag{88}$$

With F given by

$$F = \frac{\pi}{8}[(\delta(x_1+1,x_2+\frac{t}{2}) - \delta(x_1-1,x_2+\frac{t}{2})),0]\,. \tag{89}$$