Fast Reconstruction from Interleaved Uniform Samples

Jared Tanner

Stanford University
Department of Statistics

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∗ in collaboration with Thomas Strohmer

Bandlimited signals and Shannon's sampling theorem

- Bandlimited signals, function space for signal transmission

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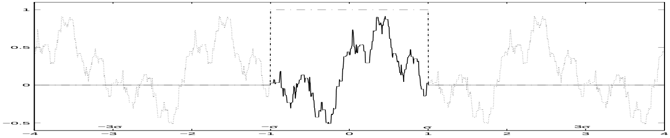

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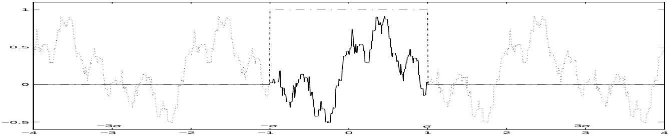

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- Real world applications have only a finite number of samples

Truncation error and oversampling

- Characteristic filter, $\chi_{[-\sigma,\sigma]}$, gives slow approx. first order error

$$\left| f(t) - \sqrt{2\pi}\, T \sum_{|k|\leq L} f(kT)\, \mathrm{sinc}\,(t - kT) \right| \lesssim (LT - |t|)^{-1}$$

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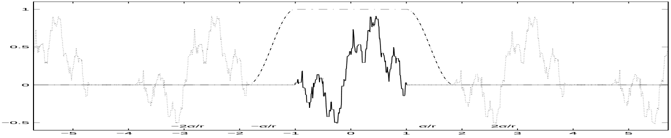
- Sampling faster, $T < 1/2\sigma$, allows smooth filter, $\Psi(w)$



$$\left| f(t) - \sqrt{2\pi}\, T \sum_{|k| \le L} f(kT)\, \psi(t - kT) \right| \lesssim \sum_{|k| > L} |\psi(t - kT)|$$

- Fourier domain smoothness yields localization in time domain

Smoothness and localization

- Localization from smooth filter, $|\psi(t)| \leq (2\pi t)^{-s} \|\Psi^{(s)}\|_{L^\infty}$
- Filter reconstruction condition

$$\Psi(w) = \begin{cases} 1 & |w| \leq \sigma \\ 0 & |w| > T^{-1} - \sigma \\ \text{smooth connection} & \text{else,} \end{cases}$$

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- Industry standard, raised cosine, $\frac{1}{2}(1 + cos(w))$
- Compact support and infinitely smooth, Gevrey regularity

$$|\Psi^{(s)}| \leq Const \frac{(s!)^\alpha}{\eta^s} \quad \Rightarrow \quad |\psi(t)| \leq \exp(-\alpha(2\pi\eta|t|)^{1/\alpha})$$

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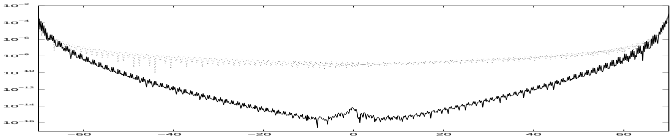

- Fast, $L \log L$ algorithm for L samples, exponential accuracy

Non-uniform sampling structures

- Non-uniform sampling, low order reconstructions

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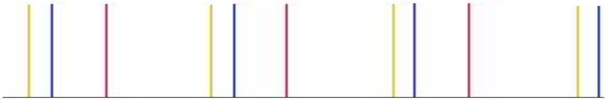
- Uniform interleaved sampling, union of N undersampled sets



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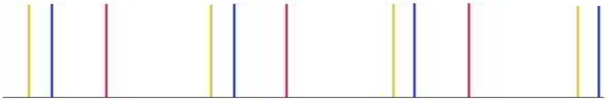

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- Can sampling structure be exploited for a fast algorithm?
- Can smooth filters be used for exponential accuracy?

Removing the periodization, an example

$$S_{T_n}(w) := e^{2\pi i T_n w} \sum_{l=-\infty}^{\infty} e^{-2\pi i l T_n T^{-1}} F(w - lT^{-1})$$

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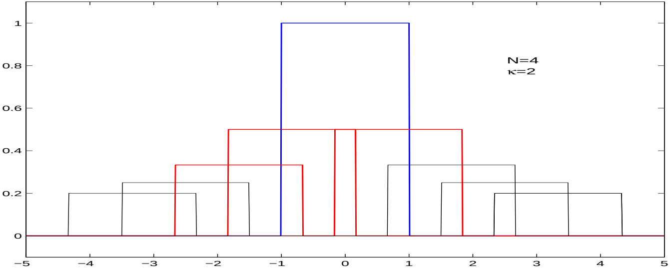

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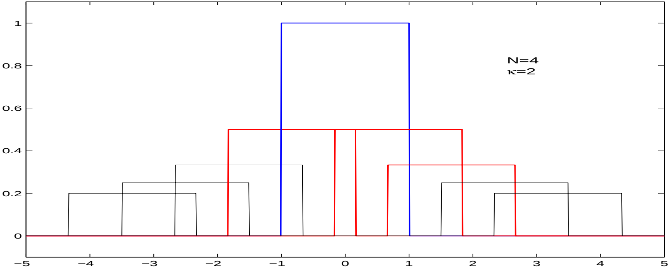

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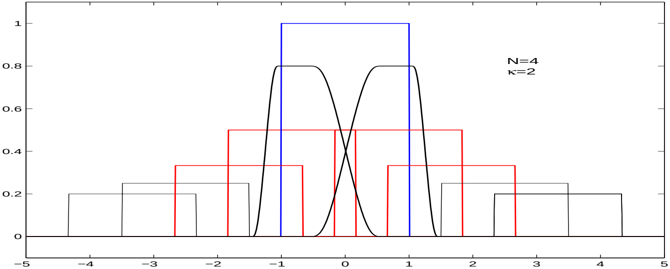
Partitions, an example

- Extract $F_k(w)$ on I_k and splice together to recover $F(w)$

$$F(w) = \sum_{j=1}^{\kappa} F_{k_j}(w)\Phi_j(w), \qquad |w| \leq \sigma$$



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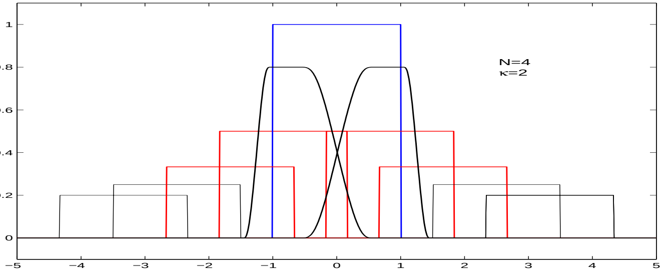

- Smoothness for localization, high order accuracy
- How to determine filter coefficients, $c_{k,n}$?
- How to select partitioning of unity, $\{\Phi(w)\}_{k=1}^{\kappa}$ for $N \gg 1$?

Filter coefficients

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- Resulting system of equations for $c(k) = (c_{k,1}\ c_{k,2}\ \cdots\ c_{k,N})^T$

$$AR(k)c(k) = e_{N-k+1}; \quad A_{m,n} := e^{2\pi i T_n T^{-1} m}, \quad m, n = 1, \ldots, N,$$

$$R(k) := \mathrm{diag}(\gamma_1(k)\ \cdots\ \gamma_N(k)), \quad \gamma_n(k) := e^{2\pi i T_n T^{-1}(k-N-1)}.$$

Partitions

- Largest overlap with $I_{k_j} \cap I_{k_{j+2}} = \emptyset \quad \Rightarrow \quad \kappa, k_j$
  With effective oversampling rate $r := T/N$
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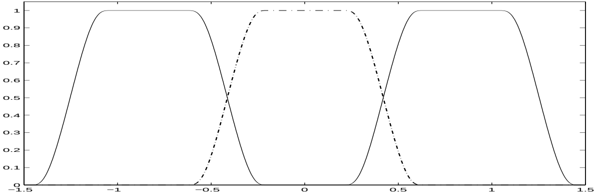

$N = 7$
$\kappa = 3$
$k_j = 1, 4, 7$

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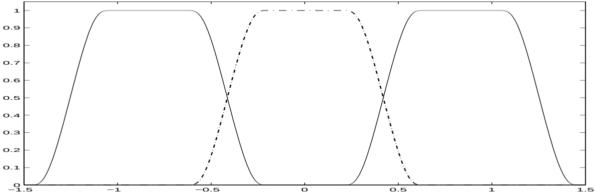

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- Action on a single uniform undersampled set?
- Partitions and coefficients, $\{c_{k,n}\}_{k=1}^{\kappa}$ determine filter

$$\Psi_n(w) := \sum_{j=1}^{\kappa} c_{k_j,n} \Phi_{k_j}(w)$$

Atomic decomposition

- Extension of Shannon's sampling theorem to bunched samples
- Exact atomic decomposition from bunched samples

$$f(t) = T \sum_{n=1}^{N} \sum_{l=-\infty}^{\infty} f(lT - T_n)\psi_n(t - (lT + T_n))$$

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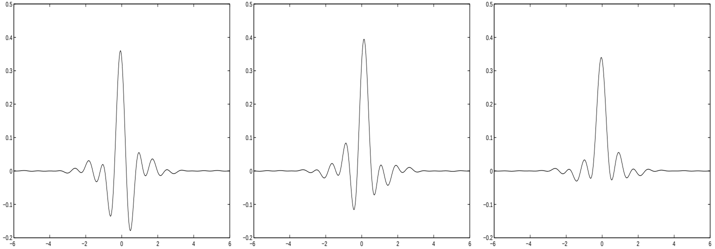
- Characteristic atoms for well separated translates, T_n



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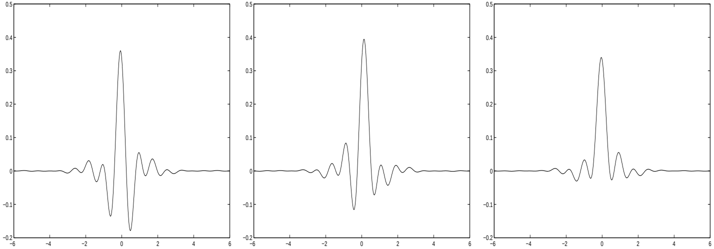
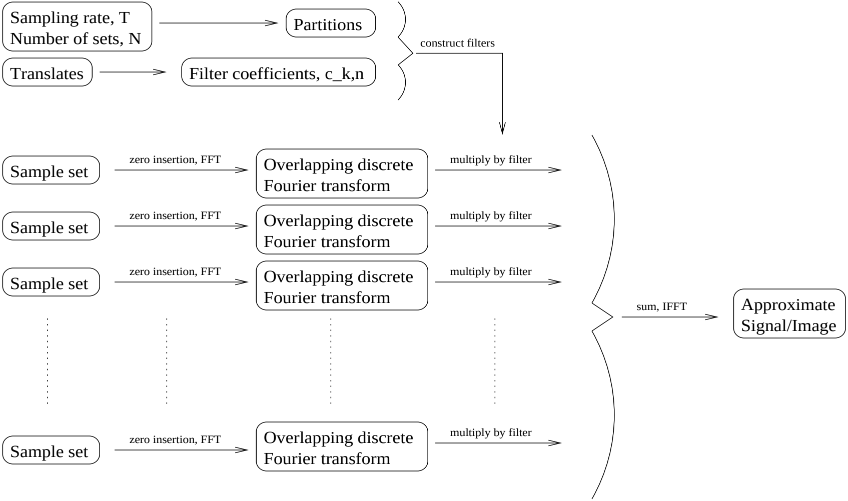


- Exponentially localized atoms, same as uniform oversampling
- Discrete sample algorithm for real world implementation

Algorithm via Filterbank, summary



$$|f(t) - \text{Approx } f(t)| \leq Const \cdot \|A^{-1}\| \exp(-\eta(LT - |t|)^{1/2})$$

A note on robustness

- Filter coefficient system yields sum of filters fixed

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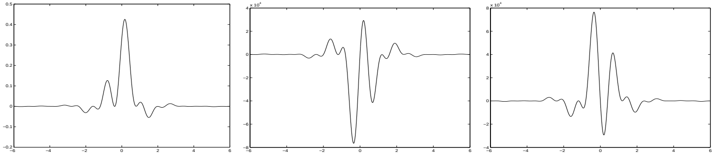

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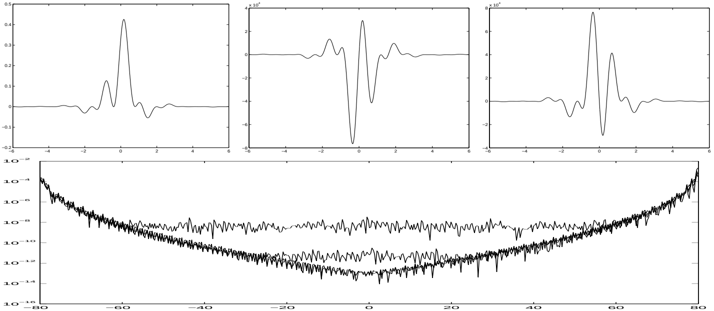
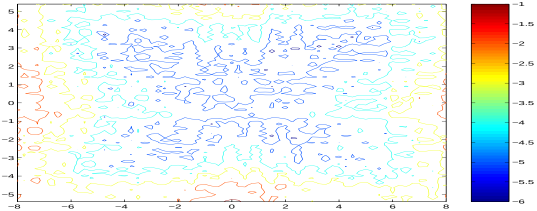


$T_n/T = \{-3^{-1}, 0, 10^{-3j}\}$, $cond(A) = 5 \times 10^{3j-1}$, $j = 1, 2, 3$.

Summary and extensions

- Fast computation, $NL\log(L)$ for N sets, L total samples
- Stable under quantization and/or jitter errors
- Extension to images, super-resolution
  - Direct extension via Cartesian products



12 images of size 13×17, using 6 filters

  - Rotations, how to exploit polar structure, fast algorithm?

Thank you for your time