# Interaction Dynamics of Singular Wave Fronts Computed by Particle Methods

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*joint work with Philip Du Toit and Jerrold Marsden*

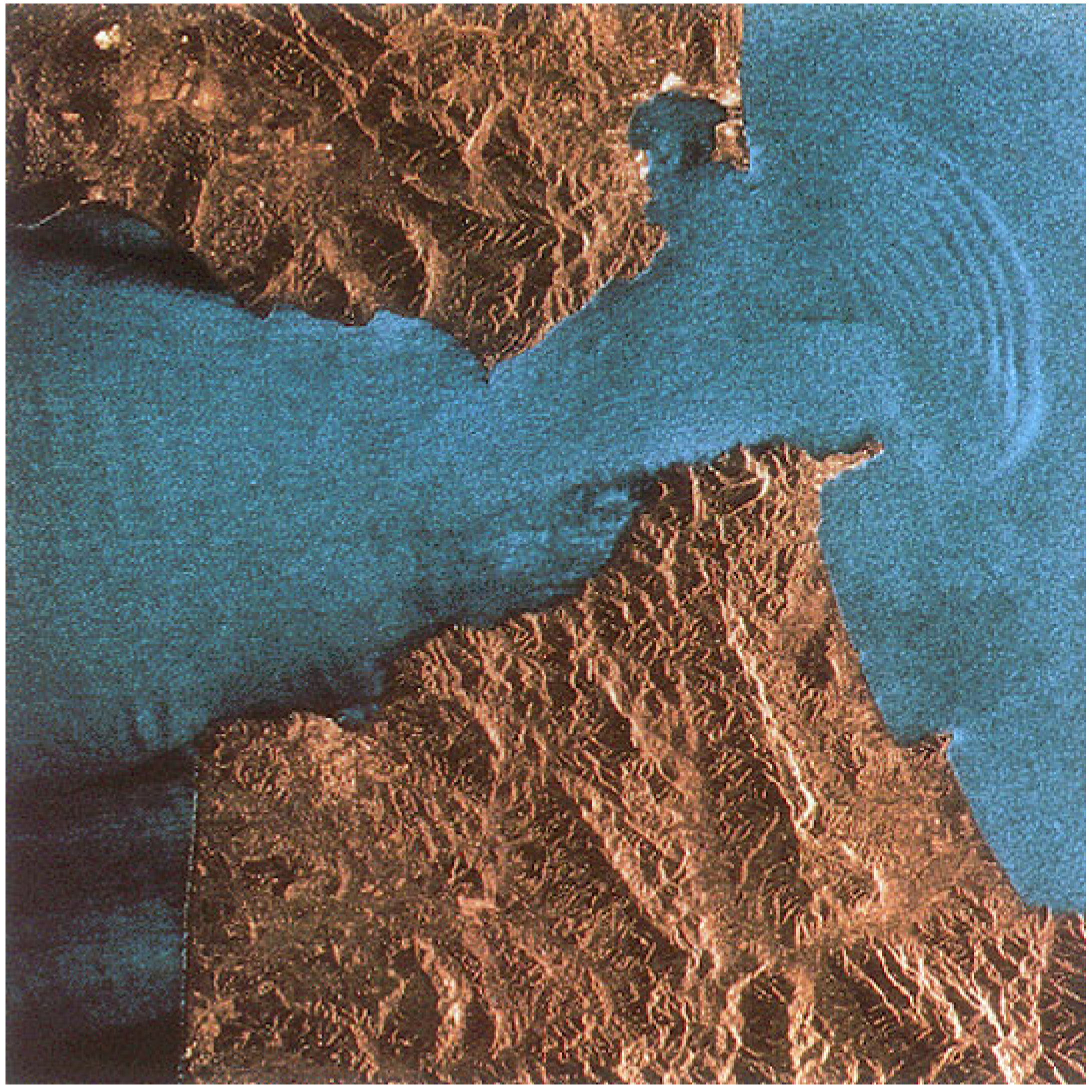
# Internal Waves

- Gravity waves that oscillate within, rather than on the surface, of a fluid medium.
- Arise from perturbations to hydrostatic equilibrium:

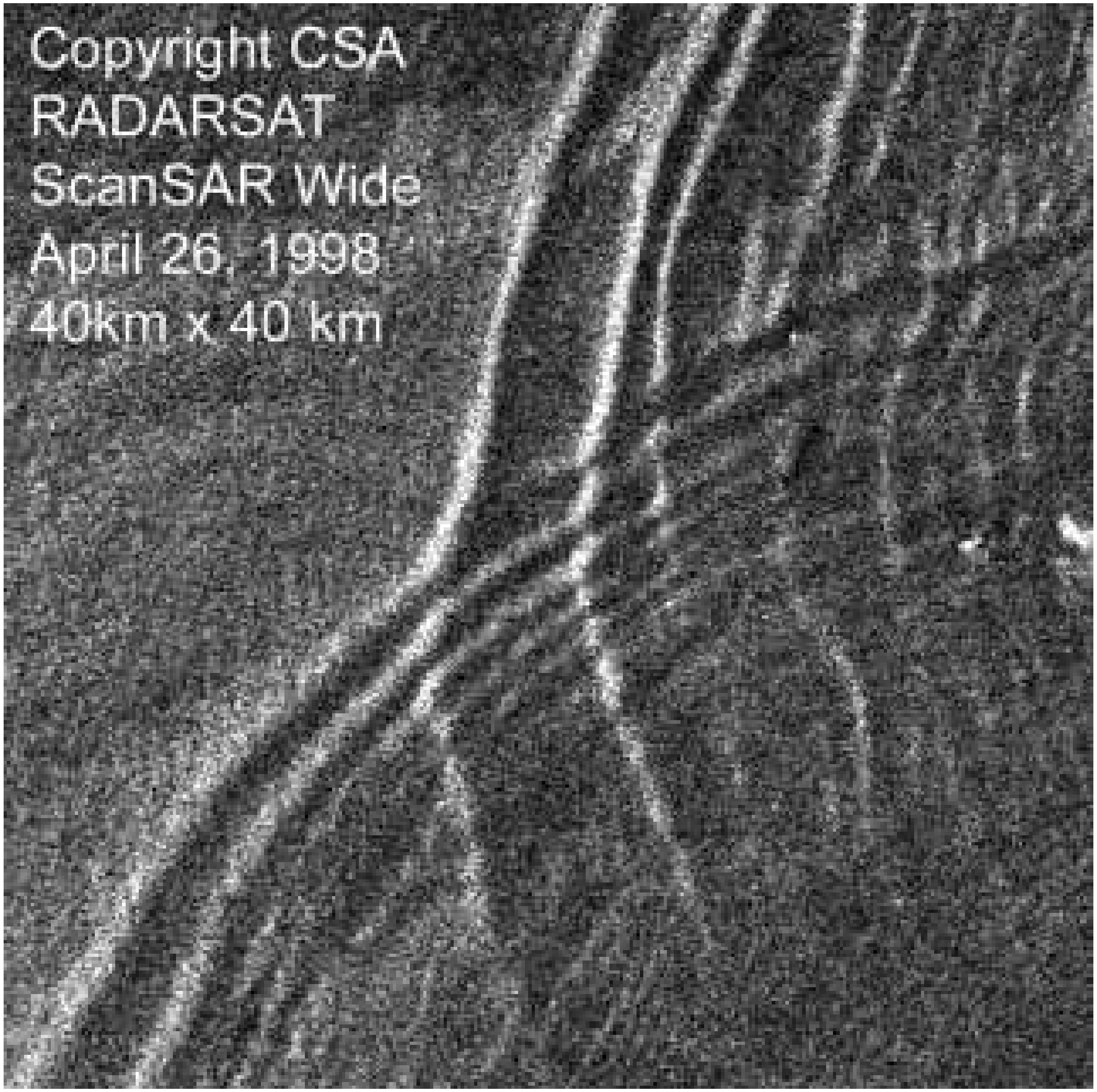
  a simple example – wave propagation on the interface between two different fluids of different densities.
- Typically have much lower frequencies and higher amplitudes than surface waves.
- Appear in both the ocean and atmosphere (100-200km long, 100m tall):

  Propagate deep down in the oceans where denser, colder and saltier deep waters meet warmer, fresher and less dense upper waters.

  Play an important role in maintaining the large-scale, deep circulation, by providing downward mixing of heat.



This artificially colored image from space of the Strait of Gibraltar shows internal waves (wavelength about 2 km) which seem to move from the Atlantic ocean to the Mediterranean Sea, at the east of Gibraltar and Ceuta.




Synthetic Aperture Radar (SAR) images from Space Shuttle observations of the South China Sea surface. [*Holm & Staley (2005)*]

# EPDiff Equation – Model of Active Fluid Transport

*[Holm & Marsden (2005), Holm & Staley (2003,05)]*

$$\frac{\partial \mathbf{m}}{\partial t} + \underbrace{\mathbf{u} \cdot \nabla \mathbf{m}}_{\text{convection}} + \underbrace{\nabla \mathbf{u}^T \cdot \mathbf{m}}_{\text{streching}} + \underbrace{\mathbf{m}(\operatorname{div} \mathbf{u})}_{\text{expansion}} = 0, \quad \mathbf{m} = \mathbf{u} - \alpha^2 \Delta \mathbf{u}.$$

$\mathbf{u}$ – fluid velocity; $\mathbf{m}$ – wave momentum; $\alpha$ – a constant parameter.

- models internal wave fronts as delta functions of momentum distributed on moving curves in the plane $\Longrightarrow$ this corresponds to modeling internal waves as contact discontinuities in the velocity.
- it has a characteristic velocity, but the relation between fluid's velocity and momentum is nonlocal.
- the velocity profile is obtained via the given relation between wave momentum and fluid velocity (the elliptic equation arises from nonhydrostatic processes).
- weak solutions – contact discontinuities that carry momentum $\rightarrow$ the front interactions – are *collisions*, in which momentum is exchanged.

# EPDiff Equation

$$\frac{\partial \mathbf{m}}{\partial t} + \underbrace{\mathbf{u}\cdot\nabla\mathbf{m}}_{\text{convection}} + \underbrace{\nabla\mathbf{u}^T\cdot\mathbf{m}}_{\text{streching}} + \underbrace{\mathbf{m}(\operatorname{div}\mathbf{u})}_{\text{expansion}} = 0, \quad \mathbf{m} = \mathbf{u} - \alpha^2\Delta\mathbf{u}.$$

Some other applications:

- Camassa-Holm (CH) equation of shallow water in 1-D and 2-D [*Camassa & Holm (1993), Kruse, Schuerle & Du (2001)*];
- Averaged template matching (ATM) equation for computer vision [*Hirani, Marsden & Arvo (2001), Holm & Marsden (2005)*];
- Geometrical structures in computational anatomy, such as landmarks and image outlines, can also be described by singular solutions of the EPDiff equation [*Holm, Ratnanather, Trouvé, & Younes (2004)*];
- Applying the proper viscosity and enforcing incompressibility produces the Navier-Stokes-alpha model of turbulence [*Chen, Foias, Holm, Olson, Titi & Wynne (1998)*].

# 1-D EPDiff Equation

$$m_t + (um)_x + u_x m = 0, \quad m = u - \alpha^2 u_{xx}, \quad \lim_{|x|\to\infty} u = 0.$$

- Fokas & Fuchssteiner (1981) – a formally bi-Hamiltonian nonlinear PDE.

- **EPDiff equation** is the dispersionless limit of the CH equation [Camassa & Holm (1993)] – a model for shallow water waves:

$$m_t + (um)_x + u_x m = \underbrace{-c_0 u_x - \gamma u_{xxx}}_{\text{dispersion}}, \qquad m = u - \alpha^2 u_{xx},$$

  If $\alpha \to 0$, then CH equation $\to$ KdV equation.

- **EPDiff equation** $+$ viscosity:

$$m_t + (um)_x + u_x m = \nu m_{xx}, \qquad m = u - \alpha^2 u_{xx}.$$

  When $\alpha \to 0$, then EPDiff with viscosity $\to$ Burgers' equation.

# 1-D EPDiff Equation – Properties [CH, 1993]

$$m_t + (um)_x + u_x m = 0, \qquad m = u - \alpha^2 u_{xx}$$

- it is conservative
  - it is bi-Hamiltonian

$$H_1 = \frac{1}{2\alpha}\int_{\mathbb{R}} \left(u^2 + u_x^2\right)\, dx \quad \text{and} \quad H_2 = \frac{1}{2\alpha}\int_{\mathbb{R}} \left(u^3 + u u_x^2\right)\, dx$$

  - it possesses an infinite number of conservation laws
  - it is completely integrable
- it is nonlocal
- has interesting and unusual solution properties:
  - admits peaked solitary waves (peakons)
  - it features breaking phenomena

# Traveling Wave Solutions of the 1-D EPDiff equation

CH equation: $m_t + (um)_x + u_x m = 0, \qquad m = u - \alpha^2 u_{xx}$

or

$$u_t - \alpha^2 u_{xxt} + 3uu_x = \alpha^2 \left(2u_x u_{xx} + uu_{xxx}\right)$$

We seek a solution of the form

$$u(x,t) = U(x-ct), \quad U(\pm\infty) = U'(\pm\infty) = U''(\pm\infty) = 0$$

and obtain the following ODE for $U$:

$$-cU' + \alpha^2 cU''' + 3UU' = \alpha^2 \left(U'U'' + UU'''\right).$$

After solving the above ODE, we obtain:

$$u(x,t) = ce^{-|x-ct|/\alpha}, \qquad c = U_{\max}.$$

# $N$-Peakon Solutions, [CH, 1993]

$$m_t + (um)_x + u_x m = 0, \qquad m = u - \alpha^2 u_{xx}$$

Solution ansatz for $N$ interacting peakons:

$$u(x,t) = \sum_{i=1}^{N} p_i(t) e^{-|x-q_i(t)|/\alpha}$$

Hamilton's canonical equations for $q_i$ and $p_i$:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, N,$$

where

$$H = \frac{1}{2\alpha} \sum_{i=1}^{N} \sum_{j=1}^{N} p_i(t) p_j(t) e^{-|q_i(t)-q_j(t)|/\alpha}.$$

# Two-Peakon Dynamics

Consider

- $2$ peakons, $\alpha = 1$: $u(x,t) = p_1 e^{-|x-q_1(t)|} + p_2 e^{-|x-q_2(t)|}$.
- $\gamma$ is the initial separation of peaks.
- initial speeds $c_1 > 0$ and $c_1 > c_2$, so the peakons collide.

Denote

$$P = p_1 + p_2, \qquad p = p_1 - p_2,$$
$$Q = q_1 + q_2, \qquad q = q_1 - q_2,$$

Then

$$\dot{P} = 0, \qquad \dot{p} = \frac{1}{2}\left[P^2 - p^2\right] \operatorname{sign}(q) e^{-|q|},$$
$$\dot{Q} = P\left(1 + e^{-|q|}\right), \qquad \dot{q} = p\left(1 - e^{-|q|}\right).$$

Solving the system of ODEs, we obtain ...

# Two-Peakon Dynamics – Continued

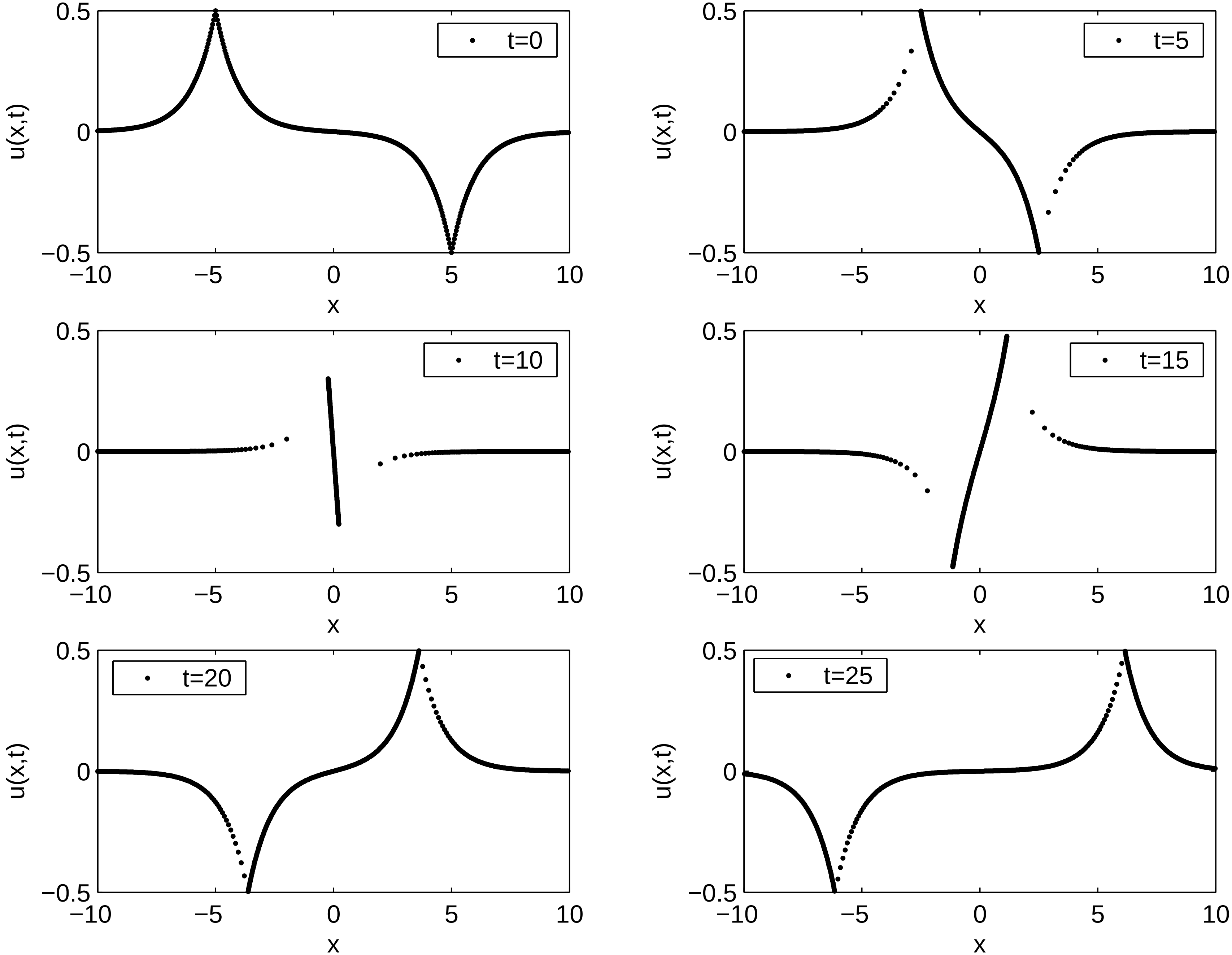
$$q_1 - q_2 = -\ln\left[\frac{4\gamma(c_1-c_2)^2 e^{(c_1-c_2)t}}{\left(\gamma e^{(c_1-c_2)t}+4c_1^2\right)\left(\gamma e^{(c_1-c_2)t}+4c_1^2\right)}\right],$$

$$p_1 - p_2 = \pm(c_1-c_2)\frac{\gamma e^{-(c_1-c_2)t}-4c_1c_2}{\gamma e^{-(c_1-c_2)t}+4c_1c_2}\ .$$

Also,

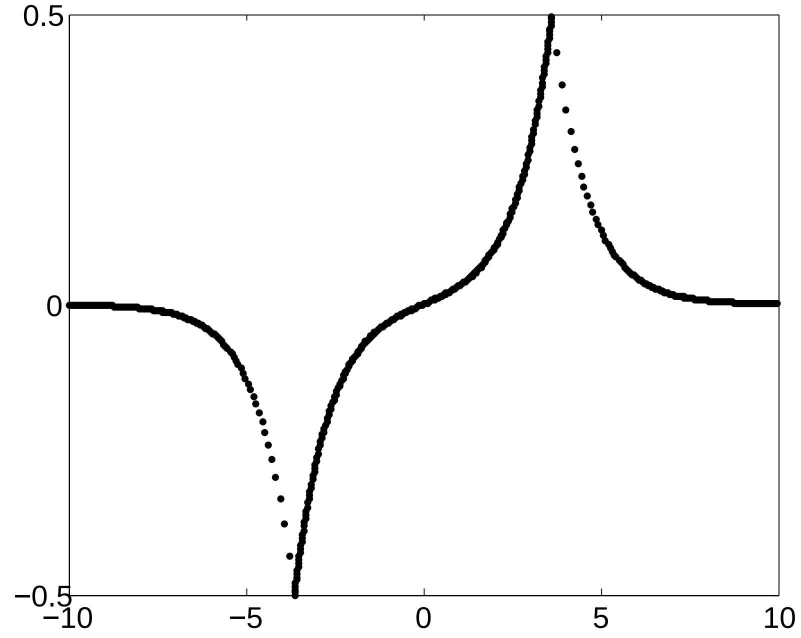
$$p_1 + p_2 = c_1 + c_2.$$

- **overlapping peaks**: may occur only $c_1$ and $c_2$ have opposite signs.


t=0
t=5
t=10
t=15
t=20
t=25
u(x,t)
x
0.5
0
−0.5
−10
−5
0
5
10

# Numerical Challenges

- Solutions are contact discontinuities in the fluid (are discontinuous in the gradient of the velocity that move along with the flow).
- Various methods exist that accurately capture shocks and vortices.
- Considerably less is known about designing numerical methods for
  - capturing contacts
  - characterizing their nonlinear interactions, especially in higher dimensions.



# Numerical Methods

There are only a few numerical works to solve the EPDiff equation.

- 1-D:
  - FD method [*Coclite & Karlsen & Riseboro (2008)*];
  - FD method [*Holden & Raynaud (2006)*];
  - Adaptive upwinding [*Artebrant & Schroll (2006)*];
  - DG method [*Shu & Xu, 2008*];

  Computationally demanding; require a large number of grid points along with adaptivity techniques; unable to capture peakon-antipekon interaction.

- 2-D and 3-D: the level of numerical complexity increases even further, since the nonlinear interaction between the waves may lead to an extremely complicated structure of the solutions.
  - Compatible differencing algorithm (CDA) [*Holm & Staley*, unpublished];

# Particle Method

$$\frac{\partial \mathbf{m}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{m} + \nabla \mathbf{u}^T \cdot \mathbf{m} + \mathbf{m}(\mathrm{div}\, \mathbf{u}) = 0, \quad \mathbf{m} = \mathbf{u} - \alpha^2 \Delta \mathbf{u},$$
$$\mathbf{m}_0(\mathbf{x}) = \mathbf{m}(\mathbf{x}, 0).$$

1. approximate the initial data

$$\mathbf{m}^N(\mathbf{x}, 0) = \sum_{i=1}^{N} \mathbf{p}_i(0)\delta(\mathbf{x} - \mathbf{x}_i(0))$$

2. follow the time evolution of particles

$$\frac{d\mathbf{x}_i}{dt} = \cdots, \qquad \frac{d\mathbf{p}_i}{dt} = \cdots$$

3. a particle solution is given by

$$\mathbf{m}^N(\mathbf{x}, t) = \sum_{i=1}^{N} \mathbf{p}_i(t)\delta(\mathbf{x} - \mathbf{x}_i(t))$$

# Particle Methods for the EPDiff Equation

- Methods derived using a discretization of a variational principle:
  - Integral and integrable algorithms in 1-D [*Camassa & Huang & Li (2005,2006)*];
  - Study of the dynamics of $N$ point particles ("blobs") [*McLachlan & Marsland (2007)*];

- An equivalent representation of the particle system, which is obtained by considering a weak formulation of the problem.

[*A.C. & Philip Du Toit & Jerrold Marsden*]

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# Particle Method for EPDiff – 1-D

Consider

$$m_t + (um)_x + u_x m = 0, \qquad m(x,0) = m_0(x)$$

Substitute

$$m^N(x,t) = \sum_{i=1}^{N} p_i(t)\delta(x - x_i(t))$$

into the weak formulation ($\phi \in C_0^1(\mathbb{R} \times [0,T))$)

$$\int_0^T\int_{\mathbb{R}} m_t\phi\,dxdt + \int_0^T\int_{\mathbb{R}} (um)_x\phi\,dxdt + \int_0^T\int_{\mathbb{R}} u_x m\phi\,dxdt = 0$$

or

$$-\int_{\mathbb{R}} m(x,0)\phi(x,0)\,dx - \int_0^T\int_{\mathbb{R}} m\left[\phi_t + u\phi_x\right]\,dx\,dt + \int_0^T\int_{\mathbb{R}} u_x m\phi\,dx\,dt = 0$$

# Particle Method for EPDiff – 1-D

$$
\begin{aligned}
&-\sum_{i=1}^{N} p_i(0)\phi(x_i(0),0) - \sum_{i=1}^{N}\int_0^T p_i(t)\ \left[\phi_t(x_i(t),t) + u(x_i(t),t)\phi_x(x_i(t),t)\right]\ dt \\
&\qquad + \sum_{i=1}^{N}\int_0^T p_i(t)u_x(x_i(t),t)\phi(x_i(t),t)\,dt = 0
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=1}^{N} p_i(0)\phi(x_i(0),0) - \sum_{i=1}^{N}\int_0^T p_i(t)\ \left[\phi_t(x_i(t),t) + \frac{dx_i(t)}{dt}\phi_x(x_i(t),t)\right]\ dt \\
&\qquad + \sum_{i=1}^{N}\int_0^T p_i(t)\ \left[\frac{dx_i(t)}{dt} - u(x_i(t),t)\right]\ \phi_x(x_i(t),t)\,dt \\
&\qquad + \sum_{i=1}^{N}\int_0^T p_i(t)u_x(x_i(t),t)\phi(x_i(t),t)\,dt = 0
\end{aligned}
$$

# Particle Method for EPDiff – 1-D

$$
\begin{aligned}
-\sum_{i=1}^{N} p_i(0)\phi(x_i(0),0) &- \sum_{i=1}^{N}\int_0^T p_i(t)\frac{d\phi_i(x_i(t),t)}{dt}\,dt \\
&+ \sum_{i=1}^{N}\int_0^T p_i(t)\left[\frac{dx_i(t)}{dt} - u(x_i(t),t)\right]\phi_x(x_i(t),t)\,dt \\
&+ \sum_{i=1}^{N}\int_0^T p_i(t)u_x(x_i(t),t)\phi(x_i(t),t)\,dt = 0
\end{aligned}
$$

Integrating by parts ...

$$
\begin{aligned}
\int_0^T \sum_{i=1}^{N} \frac{dp_i(t)}{dt}\phi(x_i(t),t)\,dt + \int_0^T \sum_{i=1}^{N} p_i(t)\left[\frac{dx_i(t)}{dt} - u(x_i(t),t)\right]\phi_x(x_i(t),t)\,dt \\
+ \sum_{i=1}^{N}\int_0^T p_i(t)u_x(x_i(t),t)\phi(x_i(t),t)\,dt = 0
\end{aligned}
$$

# Particle Method for EPDiff – 1-D

Consider

$$m_t + (um)_x + u_x m = 0, \qquad m(x,0) = m_0(x)$$

Looking for a solution in the form:

$$m^N(x,t) = \sum_{i=1}^{N} p_i(t)\delta(x - x_i(t))$$

yields

$$\begin{cases} \dfrac{dx_i}{dt} = u(x_i(t),t), & x_i(0) = x_i^0, \\[2ex] \dfrac{dp_i}{dt} + u_x(x_i(t),t)p_i = 0, & p_i(0) = p_i^0. \end{cases}$$

$$m_t + (um)_x + u_x m = 0, \qquad m = u - \alpha^2 u_{xx}$$

Solution of the form:

$$m^N(x,t) = \sum_{i=1}^{N} p_i(t)\delta(x - x_i(t)),$$

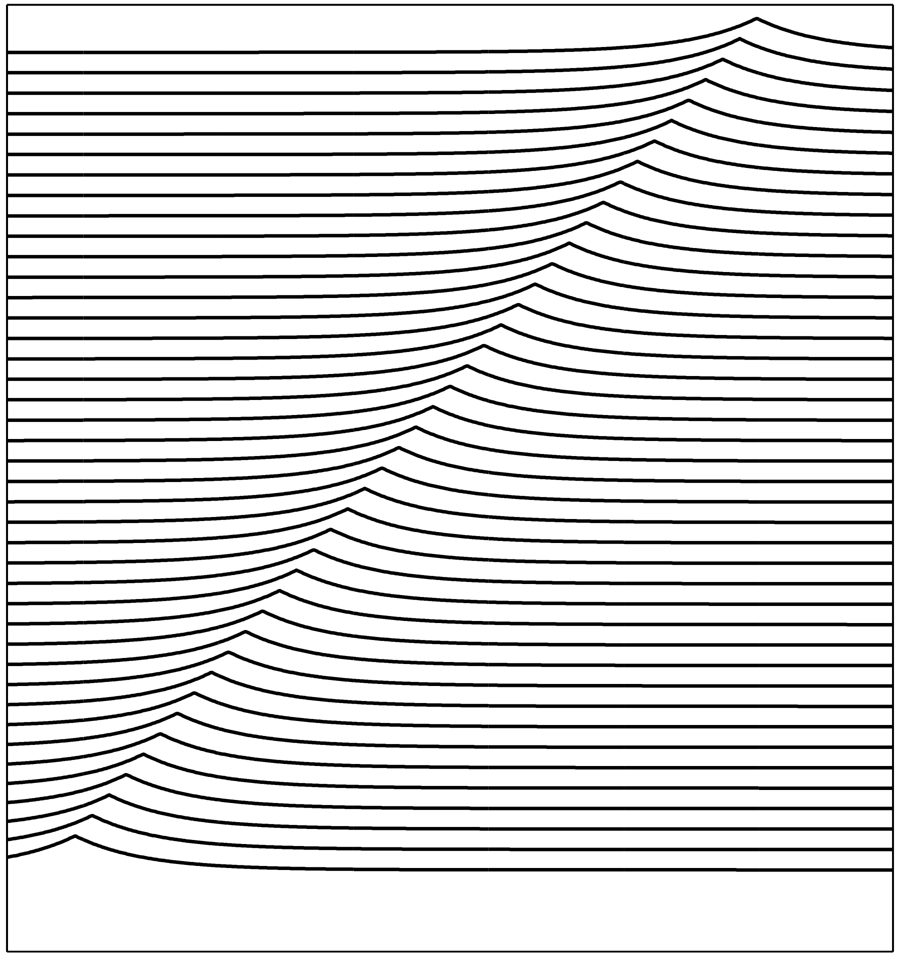
$$\frac{dx_i}{dt} = u^N(x_i(t),t), \qquad \frac{dp_i}{dt} + u_x^N(x_i(t),t)p_i = 0.$$

The velocities:

$$m^N = \sum_{i=1}^{N} p_i(t)\delta(x - x_i(t)) = u^N - \alpha^2 u_{xx}^N$$

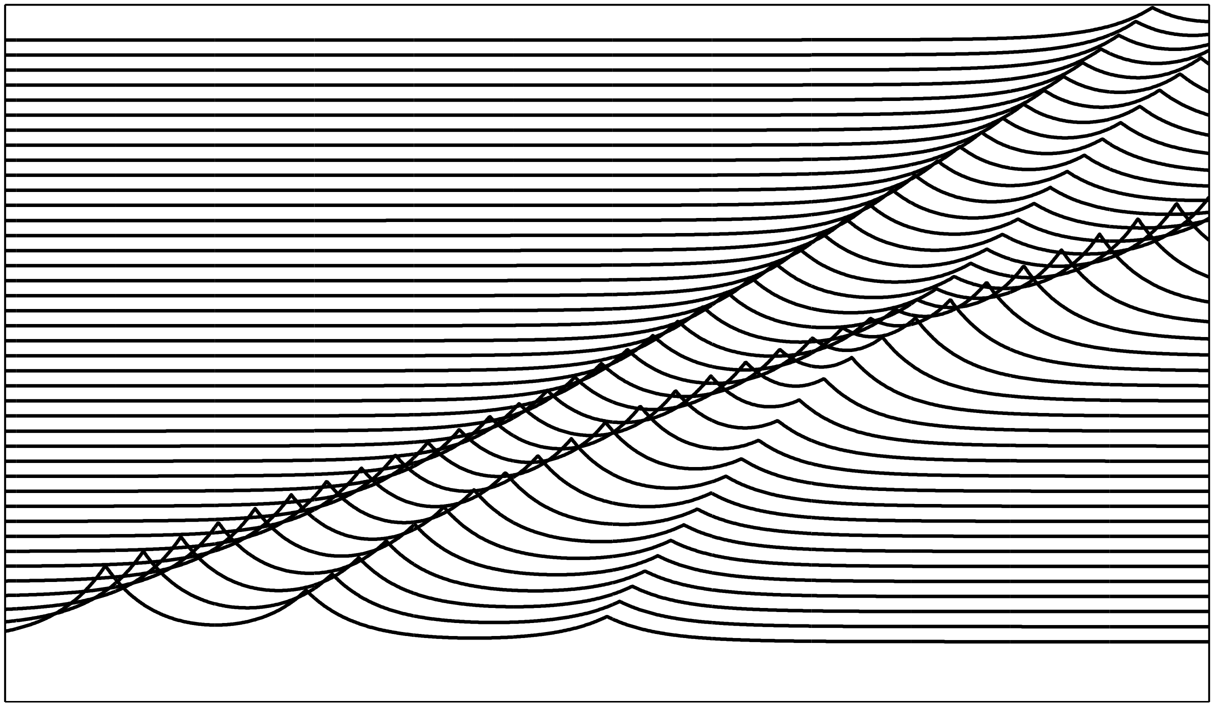
$$u^N(x,t) = \frac{1}{2}\sum_{i=1}^{N} p_i(t) e^{-|x - x_i(t)|/\alpha}$$

$$u_x^N(x,t) = \frac{1}{2}\sum_{i=1}^{N} p_i(t)\,\mathsf{sgn}(x - x_i(t)) e^{-|x - x_i(t)|/\alpha}.$$



$$x_i(0) \quad \Longrightarrow \quad x_i(t)$$

$$p_i(0) = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases} \qquad \Longrightarrow \qquad p_i(t) = p_i(0).$$

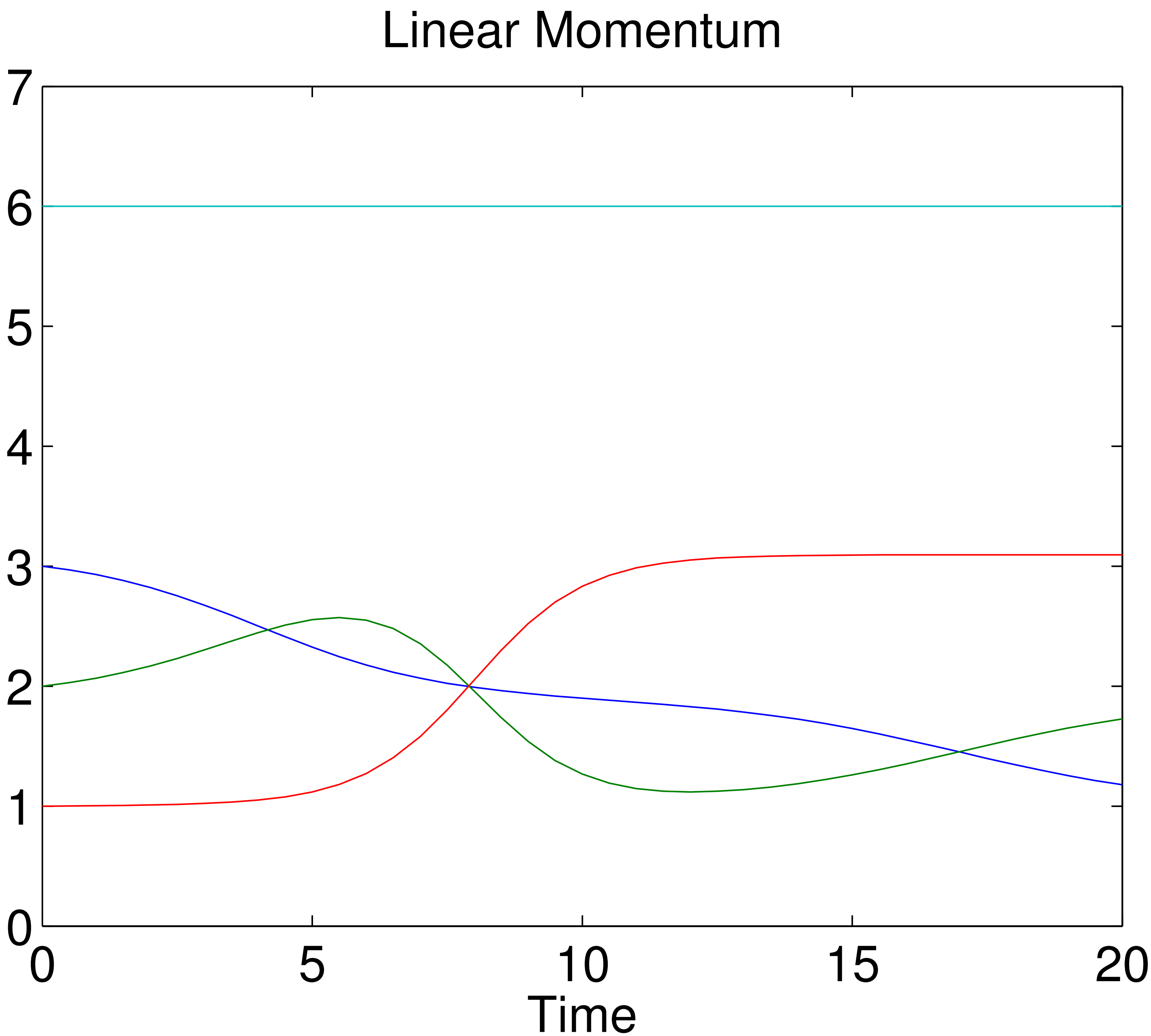


The three "nonzero" peakons:

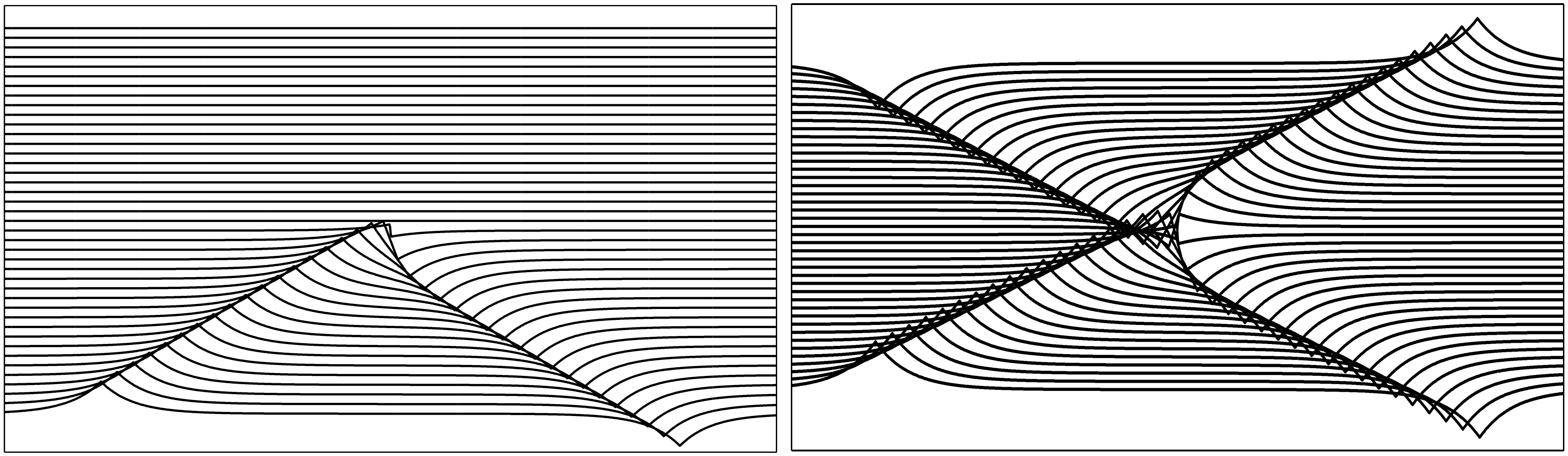
$$x_{n_1}(0) = 0,\ x_{n_2}(0) = 4 \text{ and } x_{n_3}(0) = 10$$

with the weights

$$p_{n_1}(0) = 3,\ p_{n_2}(0) = 2 \text{ and } p_{n_3}(0) = 1.$$


Linear Momentum
0
1
2
3
4
5
6
7
0
5
10
15
20
Time

movie



Peakon-antipeakon interaction:

$$x_{n_1}(0) = 0 \text{ and } x_{n_2}(0) = 18$$

have momenta of equal magnitude but opposite sign

$$p_{n_1}(0) = 2, \; p_{n_2}(0) = -2$$

so that the total momentum is zero.

# Properties of the Particle System

$$m_t + (um)_x + u_x m = 0, \qquad m = u - \alpha^2 u_{xx}$$

Particle solution: $m^N(x,t) = \sum_{i=1}^{N} p_i(t)\delta(x - x_i(t))$

$$\frac{dx_i}{dt} = u^N(x_i(t),t), \qquad \frac{dp_i}{dt} + u_x^N(x_i(t),t)p_i = 0.$$

Hamiltonian function:

$$H^N(t;x,p) = \frac{1}{2\alpha}\sum_{i=1}^{N}\sum_{j=1}^{N} p_i(t)p_j(t)e^{-|x_i(t)-x_j(t)|/\alpha}$$

Canonical Hamiltonian equations:

$$\frac{dx_i}{dt} = \frac{\partial H^N}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H^N}{\partial x_i}, \qquad j = 1, \ldots, N.$$

# Same Equations via Variational Method Approach

$$m_t + (um)_x + u_x m = 0, \qquad m = u - \alpha^2 u_{xx}$$

Solution: $\quad m^N(x,t) = \sum_{i=1}^{N} p_i(t)\delta(x - x_i(t))$

Hamiltonian function: $H^N(t;x,p) = \frac{1}{2\alpha}\sum_{i=1}^{N}\sum_{j=1}^{N} p_i(t)p_j(t)e^{-|x_i(t)-x_i(t)|/\alpha}$

*The function $m^N$ is a weak solution of EPDiff equation if $x_i(t)$ and $p_i(t)$ satisfy the Hamilton's variational principle in the phase space, that is,*

$$\delta\int_0^T \sum_{i=1}^{N}\left[p_i\dot{x}_i - H^N(t;x_i,p_i)\right]dt =$$

$$\int_0^T \sum_{i=1}^{N}\left[\left(\dot{x}_i - \frac{\partial H^N}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\partial H^N}{\partial x_i}\right)\delta x_i\right]dt = 0, \quad \forall\ \delta x_i, \delta p_i$$

# Two-Dimensional Case

$$\mathbf{m}_t + \mathbf{u}\cdot\nabla\mathbf{m} + \nabla\mathbf{u}^T\cdot\mathbf{m} + \mathbf{m}(\operatorname{div}\mathbf{u}), \quad \mathbf{m} = \mathbf{u} - \alpha^2\Delta\mathbf{u}$$

A singular momentum solution is

$$\mathbf{m}(\mathbf{x},t) = \sum_{i=1}^{N} \mathbf{p}_j(t)\delta(\mathbf{x}-\mathbf{x}_i(t))$$

The Green's function for the Helmholtz operator:

$$\mathbf{u} = G * \mathbf{m}, \qquad G(|\mathbf{x}|) = \frac{1}{2\pi}K_0\left(\frac{|\mathbf{x}|}{\alpha}\right),$$

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{2\pi\alpha^2}\sum_{i=1}^{N}\mathbf{p}_i(t)K_0\left(\frac{|\mathbf{x}-\mathbf{x}_i|}{\alpha}\right),$$

where $K_0$ is the modified Bessel function.

# Particle Method

$$\mathbf{m}_t + \mathbf{u}\cdot\nabla\mathbf{m} + \nabla\mathbf{u}^T\cdot\mathbf{m} + \mathbf{m}(\operatorname{div}\mathbf{u}), \quad \mathbf{m} = \mathbf{u} - \alpha^2\Delta\mathbf{u}$$

Here $\mathbf{u} = (u, v)^T$, $\mathbf{m} = (m_1, m_2)^T$, and $\mathbf{x} = (x, y)^T$.

We rewrite the equation in the coordinate form,

$$\frac{\partial m_1}{\partial t} + (um_1)_x + (vm_1)_y + m_1u_x + m_2v_x = 0,$$
$$\frac{\partial m_2}{\partial t} + (um_2)_x + (vm_2)_y + m_1u_y + m_2v_y = 0,$$

and seek a solution $\mathbf{m} = (m_1, m_2)^T$ of the form

$$m_k^N(x, y, t) = \sum_{i=1}^{N} p_{k,i}(t)\delta(x - x_i(t))\delta(y - y_i(t)), \quad k = 1, 2.$$

# Particle Method – Continued

$$
\begin{cases}
\dfrac{dx_i(t)}{dt} = u^N(\mathbf{x}_i(t), t), \\[2ex]
\dfrac{dy_i(t)}{dt} = v^N(\mathbf{x}_i(t), t), \\[2ex]
\dfrac{dp_{1,i}(t)}{dt} = -u_x^N(\mathbf{x}_i(t), t)\, p_{1,i}(t) - v_x^N(\mathbf{x}(t), t)\, p_{2,i}(t), \\[2ex]
\dfrac{dp_{2,i}(t)}{dt} = -u_y^N(\mathbf{x}_i(t), t)\, p_{1,i}(t) - v_y^N(\mathbf{x}_i(t), t)\, p_{2,i}(t),
\end{cases}
$$

$$
u^N(\mathbf{x}_i, t) = \frac{1}{2\pi\alpha^2} \sum_{j=1}^{N} p_{1,j}(t) K_{0,\varepsilon}\left(\frac{|\mathbf{x}_i - \mathbf{x}_j|}{\alpha}\right),
$$

$$
v^N(\mathbf{x}_i, t) = \frac{1}{2\pi\alpha^2} \sum_{j=1}^{N} p_{2,j}(t) K_{0,\varepsilon}\left(\frac{|\mathbf{x}_i - \mathbf{x}_j|}{\alpha}\right).
$$

# Cutoff Example and Construction of Mollified Kernel

Simple example:

$$\zeta_\varepsilon(\mathbf{x}) = \frac{1}{2\pi\varepsilon^2} e^{-\frac{|\mathbf{x}|^2}{2\varepsilon^2}},$$

Smoothed kernel $K_{0,\varepsilon}$:

$$K_{0,\varepsilon}\left(\frac{|\mathbf{x}|}{\alpha}\right) = K_0 * \zeta_\varepsilon(\mathbf{x}) = \int\limits_{\mathbb{R}^2} K_0\left(\frac{|\mathbf{x}-\mathbf{y}|}{\alpha}\right)\zeta_\varepsilon(\mathbf{y})\, d\mathbf{y}.$$

Using Fourier transform, one can obtain

$$K_{0,\varepsilon}(|\mathbf{x}|) = \int_0^\infty r J_0(r|\mathbf{x}|)\, \frac{\alpha^2 e^{-\frac{r^2\varepsilon^2}{2}}}{1+\alpha^2 r^2}\, dr,$$

where $J_0(x)$ is the Bessel function of the first kind given by

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2}\, x^{2k}.$$

# Properties of the Particle System

The Hamiltonian function:

$$H^N(t;\mathbf{x},\mathbf{p}) = \frac{1}{4\pi\alpha^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[p_{1,i}p_{1,j} + p_{2,i}p_{2,j}\right] K_{0,\varepsilon}\left(\frac{|\mathbf{x}_i - \mathbf{x}_j|}{\alpha}\right).$$

- $\mathbf{x}_j(t)$ and $\mathbf{p}_j(t)$ satisfy the canonical Hamiltonian equations:

$$\frac{d\mathbf{x}_j}{dt} = \frac{\partial H^N}{\partial \mathbf{p}_j}, \qquad \frac{d\mathbf{p}_j}{dt} = -\frac{\partial H^N}{\partial \mathbf{x}_j}, \qquad j = 1,\ldots,N.$$

The same Hamiltonian equations can be obtained using the variational method approach!

# Properties of the Particle System

- **Lemma** [Conservation of Linear Momentum]:

$$\frac{d}{dt}\left[\sum_{i=1}^{N} p_{1,i}(t) + \sum_{i=1}^{N} p_{2,i}(t)\right] = 0, \qquad \forall t \geq 0.$$

- **Lemma** [Conservation of Angular Momentum]:

$$\sum_{i=1}^{N}\left[\frac{dx_i}{dt} p_{2,i} - \frac{dy_i}{dt} p_{1,i} + x_i \frac{dp_{2,i}}{dt} - y_i \frac{dp_{1,i}}{dt}\right] = 0.$$

- **Theorem** : The Hamiltonian

$$H^N(t; \mathbf{x}, \mathbf{p}) = \frac{1}{4\pi\alpha^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[p_{1,i} p_{1,j} + p_{2,i} p_{2,j}\right] K_{0,\varepsilon}\left(\frac{|\mathbf{x}_i - \mathbf{x}_j|}{\alpha}\right)$$

is invariant under translation and rotation.

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- Convergence of the particle method in 1-D (with Jian-Guo Liu and Terrance Pendleton):
  - Use the fact that the particle ODE system has a unique global solution [*Camassa & Huang & Li (2005)*];
  - Define a weak solution of the EPDiff equation and show that the particle solution $(u^N, m^N)$ is a weak solution of the equation;
  - Show that there exist a limit $(u^N, m^N) \to (u, m)$ as $N \to \infty$:
    * show that $u^N$ and $u_x^N$ are BV functions in space and time.
    * use a compactness result, associated with $BV$ functions, to show that the limit exists .
  - Show that the limit $(u, m)$ is also a weak solution to the EPDiff equation.
- Numerical experiments for general initial data and convergence of the particle method in 2-D.
- Integrability of the 2-D equation.