

# Quantum Chaos? Genericity and Nongenericity in the MHD Spectrum of Nonaxisymmetric Toroidal Plasmas

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The eigenmode spectrum is a fundamental starting point for the analysis of plasma stability and the onset of turbulence, but the characterization of the spectrum even for the simplest plasma model, ideal magnetohydrodynamics (MHD), is not fully understood. This is especially true in configurations with no continuous geometric symmetry, such as in a real tokamak when the discrete nature of the external magnetic field coils is taken into account, or the alternative fusion concept, the stellarator, where axisymmetry is deliberately broken to provide a nonzero winding number (rotational transform) on each invariant torus of the magnetic field line dynamics (assumed for present purposes to be an integrable Hamiltonian system). Quantum (wave) chaos theory provides tools for characterizing the spectrum statistically, from the regular spectrum of the separable case (integrable semiclassical dynamics) to that where the semiclassical ray dynamics is so chaotic that no simple classification of the individual eigenvalues is possible (quantum chaos). The MHD spectrum exhibits certain nongeneric properties, which we show, using a toy model, to be understandable from the number-theoretic properties of the asymptotic spectrum in the limit of large toroidal and poloidal mode (quantum) numbers when only a single radial mode number is retained. Much more realistically, using the ideal MHD code CAS3D, we have constructed a data set of several hundred growth-rate eigenvalues for an interchange-unstable three-dimensional stellarator equilibrium with a rather flat, nonmonotonic rotational transform profile. A statistical analysis of eigenvalue spacings shows evidence of generic quantum chaos, which we attribute to the mixing effect of having a large number of radial mode numbers.

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## I. INTRODUCTION

The tokamak and the stellarator fusion concepts both seek to contain a plasma in a toroidal magnetic field. To a good approximation, the tokamak field is axisymmetric, so the study of the spectrum of normal modes of small oscillations about equilibrium is simplified by the existence of an ignorable coordinate. A stellarator, on

the other hand, is inherently nonaxisymmetric, and the lack of a continuous symmetry means there are no “good quantum numbers” to characterize the spectrum.

This makes the numerical computation of the spectrum a challenging task, but numerical matrix eigenvalue programs, such as the three-dimensional TERP-SICHORE [1] and CAS3D [2] codes, are routinely used to assess the ideal magnetohydrodynamic (MHD) stability of proposed fusion-relevant experiments. An example is the design of the 5-fold-symmetric Wendelstein 7-X

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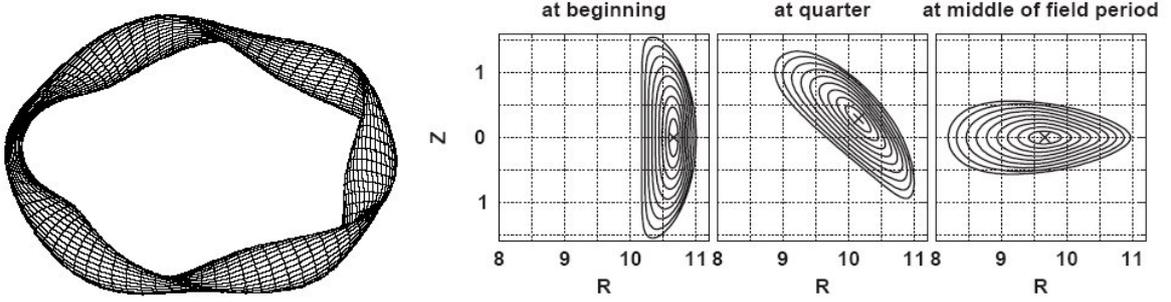


Fig. 1. Left: plasma boundary of a 5-periodic toroidal equilibrium geometrically related to W7-X configurations (at  $\langle\beta\rangle = 0.05$ , VMEC calculation). Right three frames: vertical cross-sections of the configuration at three different toroidal angles within a field period, with  $Z$  the vertical coordinate and  $R$  the distance from the  $Z$  axis. The plasma boundary has been scaled such that the minor radius of the torus  $a \approx 1$  and the major radius equals the aspect ratio  $R/a = A$ .

(W7-X) stellarator, where CAS3D was used [3] to study a number of different cases.

The configurations foreseen to be studied experimentally in W7-X are MHD stable, but in the present paper we are concerned with an unstable case from this study, a high-mirror-ratio, high-rotational-transform equilibrium (Fig. 1). Due to its less pronounced shaping, this case is quite unstable, which contrasts with the properties of genuine W7-X configurations. The three-dimensional nature of the equilibrium breaks all continuous symmetries, coupling both poloidal ( $m$ ) and toroidal ( $n$ ) Fourier harmonics and thus precluding separation of variables and simple classification of the eigenvalues.

These eigenvalues,  $\omega^2 \equiv -\gamma^2$ , are real due to the self-adjointness [4] of the force and kinetic energy operators in ideal MHD, linearized about a static equilibrium. This is analogous to the Hermitian nature of quantum mechanics, so we might *a priori* expect to be able to take over the mathematical techniques used in quantum mechanics. Thus, we study the W7-X Mercier (interchange)-unstable case mentioned above by using statistical techniques from the theory of quantum chaos [5, eg].

This is of practical importance for numerical analysis of the convergence of eigenvalue codes because, if the system is quantum-chaotic, convergence of individual eigenvalues cannot be expected, and a statistical description must be used. However, there is a fundamental question as to whether the ideal MHD spectrum lies in the same universality classes as the spectra of typical quantum mechanical systems.

This question has been addressed recently [6, 7] by studying the interchange unstable spectrum in an effectively cylindrical model of a stellarator. In the cylindrical case the eigenvalue problem is separable into three one-dimensional eigenvalue problems, with radial, poloidal, and toroidal (axial) quantum numbers  $l$ ,  $m$ , and  $n$ , respectively. If the spectrum falls within the generic quantum chaos theory universality class for integrable, non-chaotic systems [8], then the probability distribution function for the separation of neighboring eigenvalues is

a Poisson distribution. However, the work of Ref. 6 indicates that the universality class depends on the method of regularization (ie truncation of the countably infinite set of ideal-MHD interchange growth rates): a smooth, physically-motivated finite-Larmor-radius roll-off in the spectrum appears to give the generic Poisson statistics for separable systems, but a sharp truncation in  $m$  and  $n$  gives highly non-generic statistics. The latter case is less physical, but corresponds closely to the practice in MHD eigenvalue studies of using a restricted  $m, n$  basis set, but a relatively fine mesh in the radial direction.

A careful analysis of the spectrum of ideal-MHD interchange modes in a separable cylindrical approximation [7] revealed non-generic behavior of the spectral statistics—a bimodal probability distribution function (PDF), rather than the expected Poisson distribution. The non-genericity of this separable case indicates that caution must be applied in applying conventional quantum chaos theory in non-separable geometries. The study [7] indicated that the non-generic behavior of ideal-MHD interchange modes is due to the peculiar feature of the dispersion relation for these modes that the eigenvalues in the short-wavelength limit depend only on the *direction* of the wave vector, not on its magnitude. (This is unusual behavior, but it is shared with internal gravity waves in geophysical fluid dynamics.) It was suggested in Ref. 7 that the detailed features of the spectrum could be understood from the properties of Farey sequences.

In the present paper, we discuss fictitious eigenvalues from a toy model, used to elucidate the importance of number-theoretic effects, that illustrate how nongeneric the MHD eigenvalue spectrum can be if only one radial eigenmode is used. Then we present the results of a quantum chaos analysis for the W7-X case.

## II. TOY EIGENVALUE PROBLEM

To gain insight into the nongeneric behavior found in the separable case [7] with only one radial eigenmode, we study the energy spectrum  $\{\mathcal{E}_{n,m}\}$  for a “toy” quantum

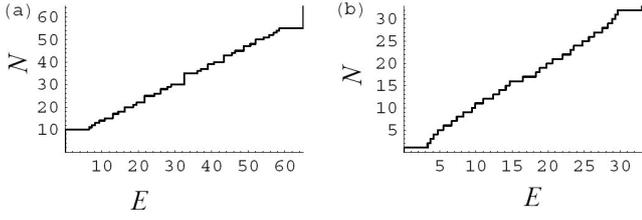


Fig. 2. Number  $N(E)$  of renormalized eigenvalues  $E_{m,n}$  below a given value  $E$  for  $m_{\max} = 10$ : (a) for the Hamiltonian  $H = p_\phi/p_\theta$ ; (b) for the Farey sequence.

mechanical Hamiltonian  $H = p_\phi/p_\theta$ , where the configuration space is the 2-torus  $\theta \in [0, 2\pi)$ ,  $\phi \in [0, 2\pi)$  with periodic boundary conditions. In the semiclassical approximation, we see that  $H$  depends only on the direction of  $\mathbf{p}$ , not its magnitude, as for MHD interchange modes and internal gravity waves.

The eigenvalue problem is the time-independent Schrödinger equation,  $H\psi = \mathcal{E}\psi$ , the eigenfunctions being  $\exp[i(m\theta + n\phi)]/4\pi^2$ , where  $m$  and  $n$  are integers. The eigenvalues are given by  $\mathcal{E}_{n,m} = n/m$  ( $m \neq 0$ ).

Note the singular nature of the spectrum—it is discrete, yet infinitely dense, the rationals being dense on the real line. Also, the spectrum is infinitely degenerate as eigenvalues are repeated whenever  $m$  and  $n$  have a common factor. Mathematically, such a spectrum, neither point nor continuous, belongs to the *essential spectrum* [9].

In order to analyze this spectrum by using standard quantum chaos techniques, we first regularize it by bounding the region of the  $m, n$  lattice studied, and then allowing the bound to increase indefinitely. Fortunately, the PDF,  $P(s)$ , is independent of the precise shape of the bounding line when we follow standard practice [5] in renormalizing (unfolding) the energy levels to make the average spacing unity. Thus, we adopt the simplest choice, taking the bounded region to be the triangle  $0 \leq n \leq m \leq m_{\max}$ . As the points  $(n, m)$  form a lattice in the plane with mean areal density of 1, we can estimate the asymptotic, large- $m_{\max}$  behavior of the number of levels,  $N_{\max}$ , from the area of the bounding triangle: the  $m$  axis, the line  $n = m$  and the line  $m = 1$ , which gives the “Weyl formula” [5]  $N_{\max} \sim m_{\max}^2/2$ .

The Farey sequence  $\mathcal{F}(Q)$  is the set of all rational numbers  $p/q$  between 0 and 1,  $1 < q \leq Q$ , arranged in order of increasing arithmetic size and keeping only mutually prime pairs of integers  $p$  and  $q$ . Farey sequences are important in number theory [10] and have application in various dynamical systems problems, such as the theory of mode locking in circle maps [11]. They even have a connection with the famous Riemann hypothesis [12]. They are conveniently generated using the *Mathematica* algorithm [13]

```
Farey[Q_] := Union[{0,1},
Flatten[Table[p/q,{q, Q},{p, q-1}]]].
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The list  $\mathcal{G}(m_{\max}) \equiv \{\mathcal{E}_{n,m}\}$ , sorted into a non-decreasing sequence  $\{\mathcal{E}_i | i = 1, 2, \dots, N_{\max}\}$ , is very similar to the

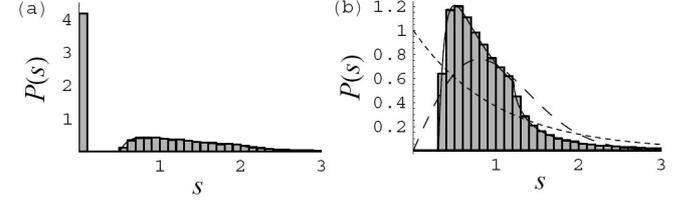


Fig. 3. Separation statistics: (a) for the model Hamiltonian; (b) for the Farey sequence. The solid curves are from the Farey spacing measure, Eq. (2). In (b) the short-dashed curve is for the Poisson Process of the generic integrable problem and the long-dashed curve is that for the Gaussian orthogonal ensemble of random matrices (quantum chaotic case).

Farey sequence  $\mathcal{F}(m_{\max})$  except for the high degeneracy (multiplicity) of numerically identical levels, especially when  $n/m$  is a low-order rational.

Define the renormalized (unfolded) energy as  $E_{n,m} \equiv N_{\max}\mathcal{E}_{n,m}$ . The normalization by  $N_{m_{\max}}$  ensures that  $E_{N_{\max}} = N_{\max}$ , so the mean slope of the Devil’s staircase shown in Fig. 2(a) is unity. The large vertical steps visible in Fig. 2(a) are due to the high degeneracy at low-order rational values of  $n/m$ . This high degeneracy is also the cause of the delta function spike at the origin visible in the level-separation probability distribution plot shown in Fig. 3(a), which is very similar to Fig. 9(a) in [7].

The tail in Fig. 3(a) is due to the non-degenerate component of the spectrum, obtained by reducing all fractions  $p/q$  to lowest terms and deleting duplications. Thus the eigenvalues in this set are  $N_Q$  times the terms of the Farey sequence  $\mathcal{F}(Q)$ .

To study the statistics of this non-degenerate component, one naturally defines the *Farey spectrum*  $\{E_i^F\}$  as  $N^F(Q)$  times the terms of the Farey sequence  $\mathcal{F}(Q)$ , where  $N^F(Q)$  is the number of terms in  $\mathcal{F}(Q)$ . The asymptotic behavior of  $N^F(Q)$  in the large- $n$  limit is given [14, p. 391] by  $N^F(Q) \sim 3Q^2/\pi^2 + O(Q \ln Q)$ . The staircase plot and the separation distribution  $P^F(s)$  for the Farey spectrum are given in Fig. 2(b) and Fig. 3(b), respectively.

It is a standard result in the theory of Farey sequences [10, p.301] that the smallest and the largest nearest-neighbor spacings in  $\mathcal{F}(Q)$  are given, respectively, by

$$\frac{1}{Q(Q-1)} \quad \text{and} \quad \frac{1}{Q}, \quad (1)$$

so that the support of the tail component of  $P(s)$  in Fig. 3(a) becomes  $[1/2, \infty)$  in the limit  $Q \rightarrow \infty$ , while that of  $P^F(s)$  in Fig. 3(b) is  $[3/\pi^2, \infty)$ . Augustin *et al.* [12], Eq. (1.9), derived the spacing density for the Farey sequence as

$$g_1(t) \equiv \begin{cases} 0, & \text{for } 0 \leq t \leq \frac{3}{\pi^2}, \\ \frac{6}{\pi^2 t^2} \ln\left(\frac{\pi^2 t}{3}\right), & \text{for } \frac{3}{\pi^2} \leq t \leq \frac{12}{\pi^2}, \\ \frac{12}{\pi^2 t^2} \ln\left[\frac{\pi^2 t}{6} \left(1 - \sqrt{1 - \frac{12}{\pi^2 t}}\right)\right] & \text{for } \frac{12}{\pi^2} \leq t. \end{cases} \quad (2)$$

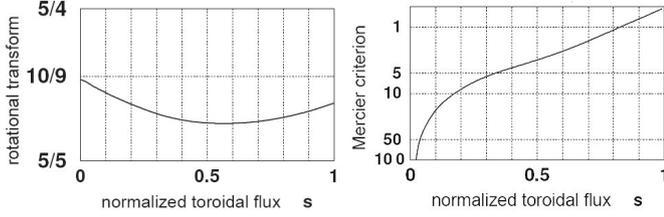


Fig. 4. Left: the rotational transform  $\iota = \iota(s)$  profile versus normalized toroidal flux  $s$  ( $s \propto r^2$  near the magnetic axis). Right: a measure of the Mercier stability versus normalized toroidal flux for the 3-dimensional W7-X-like case of Fig. 1. A negative value indicates instability.

The solid curve in Fig. 3(b) is obtained by setting  $P^F(s) = g_1(s)$  and is seen to agree well with the numerical results. The solid curve in Fig. 3(a) is obtained by setting  $P(s) = [N^F(Q)/N(Q)]^2 g_1(N^F s/N)$  and agrees well with the tail of the histogram. The ratio of the area of the tail in Fig. 3(a) to the strength of the delta function in Fig. 3(a) is  $N^F(Q)/[N(Q) - N^F(Q)] \approx 1.55$ . We have verified that the probability distributions remain unchanged if subranges of the spectra are used, in agreement with the result included in Theorem 1.1 of Ref. 12 that the convergence to a probability measure is independent of the interval chosen.

### III. W7-X RESULTS

The W7-X variant equilibrium studied was generated with the VMEC [15] code, which assumes the magnetic field to be integrable, so that all magnetic field lines lie on nested toroidal flux surfaces, which we label by  $s$ , the enclosed toroidal magnetic flux divided by the toroidal flux enclosed by the plasma boundary. The magnetic field is characterized on each flux surface by its winding number  $\iota(s)$ . (In tokamaks its inverse,  $q \equiv 1/\iota$ , is more commonly used.) As seen in Fig. 4, the rotational transform profile is nonmonotonic and has low shear ( $\iota_{\text{axis}} = 1.1066$ ,  $\iota_{\text{min}} = 1.0491$ ,  $\iota_{\text{edge}} = 1.0754$ ), so it is close to, but greater than, unity over the whole plasma. Figure 4 also shows the equilibrium to be interchange unstable, because the Mercier stability criterion is violated over the whole plasma.

The CAS3D code expands the eigenfunctions in a finite Fourier basis in the toroidal and the poloidal angles, selected so as to include all  $|n| \leq n_{\text{max}}$  and all  $m$  such that  $n/m$  lies in a band including the range of  $\iota$ . The Fourier tableau is depicted graphically in Fig. 5. In this code,

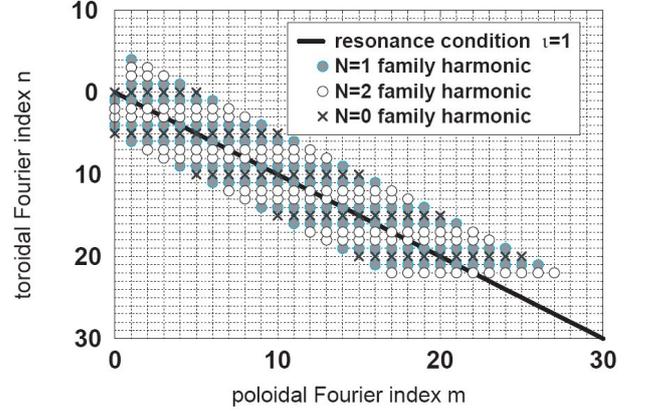


Fig. 5. Choice of basis set of toroidal and poloidal Fourier harmonics.

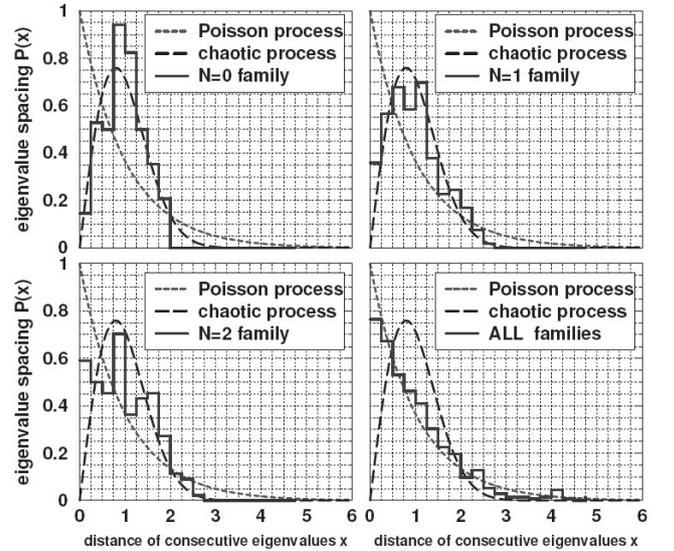


Fig. 6. Unfolded eigenvalue spacing distributions from mode-family datasets  $N = 0$  (137 values),  $N = 1$  (214 values) and  $N = 2$  (178 values) calculated by CAS3D for our W7-X-like equilibrium, and the distribution for the combined spectrum,  $N = 0, 1$  and 2.

the radial dependence of the perturbation functions is treated by using a hybrid Finite-Element approach, with a linear interpolation for the normal displacement and piecewise constant interpolations for the scalar components that describe the MHD displacement within the magnetic surfaces. In the calculations discussed here, 301 radial grid points were used. The kinetic energy was used as the normalization; therefore, the unstable eigen-

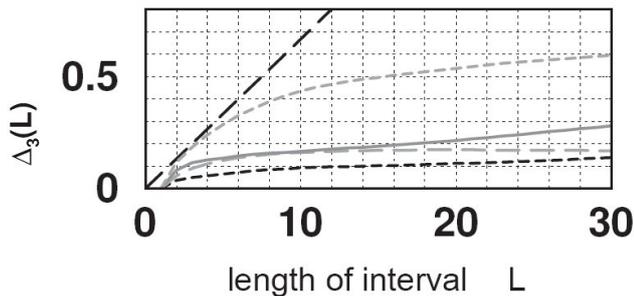


Fig. 7. Dyson-Mehta spectral rigidity as a function of subinterval length  $L$ .  $N = 0$  family: black short-dashed;  $N = 1$ : gray long-dashed;  $N = 2$ : gray solid; combined data set: gray short-dashed; Poisson-process: black long-dashed.

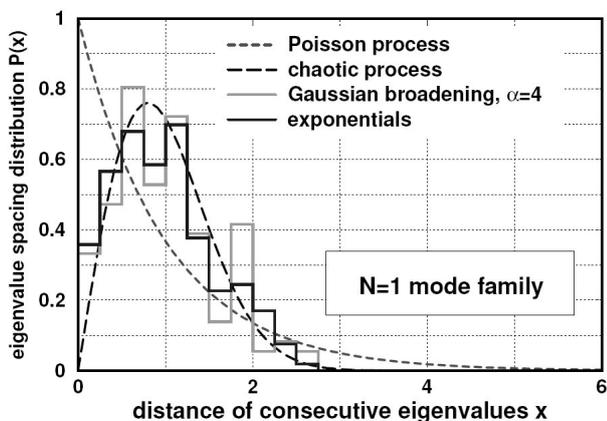


Fig. 8. Unfolded eigenvalue spacing distributions for mode-family  $N = 1$ , calculated using two different unfolding methods. The results are seen to be consistent to within statistical error.

values  $\lambda$  may be converted to a nondimensional growth rate  $\gamma$  viz.  $\gamma\tau_A = R_0(0)\sqrt{|\lambda|}/B_0(0)$ . Here,  $R_0(0)$  is the major radius, and  $B_0(0)$  the equilibrium magnetic field measured on the magnetic axis.

Because of the 5-fold symmetry of the equilibrium, any toroidal Fourier harmonic  $n$  in an eigenfunction is coupled to toroidal harmonics  $n \pm 5$ . With the poloidal harmonics chosen to be positive,  $m \geq 0$ , there are just three uncoupled mode-families  $N = 0, 1, 2$  (compare [2]).

We characterize the statistics of the ensembles of eigenvalues within the three mode-families using two standard measures from quantum chaos theory [5, 16, 17], first renormalizing (“unfolding”) the eigenvalues so their average separation is unity. The first measure, shown in Fig. 6, is the probability distribution function,  $P(x)$ , for the eigenvalue separation  $x$  (the  $s$  of Sec. II). The other, shown in Fig. 7, is the Dyson-Mehta rigidity  $\Delta_3(L)$ , where  $L$  is the subrange of unfolded eigenvalues used.

As seen from Fig. 6, when the statistics are analyzed within the three mode-families the eigenvalue spacing distribution function is closer to the Wigner conjecture

form found for generic chaotic systems [5] than to the Poisson distribution for separable systems, as might be expected from Ref. 18. However, when the spectra from the three uncoupled mode-families are combined, there are enough accidental degeneracies for the spacing distribution to become close to Poissonian.

To test the sensitivity to the precise method of unfolding chosen, we did the statistics using two different methods: the Gaussian unfolding [17] and a fit with exponentials. The results, shown in Fig. 8, indicate little sensitivity to the unfolding method.

#### IV. CONCLUSION

Although not presented here, when *all* unstable eigenmodes (*i.e.*, all  $l$ ,  $m$ , and  $n$ ) are included, the eigenvalue spacing statistics for the ideal-MHD interchange eigenvalue spectrum in a separable cylindrical approximation is close to that of generic separable wave equations [8], despite our earlier finding [6,7] that the spectrum in the subspace of the most unstable radial eigenmode  $l = 0$  is nongeneric, as explained by the model presented in Sec. II. In this paper we have shown that, in a strongly three-dimensional stellarator equilibrium related to W7-X, the unstable interchange (Mercier) mode spectrum has, to within statistical uncertainties, similar statistics to that of generic quantum chaotic systems. That is, the overwhelming majority of eigenvalues are not “good quantum numbers” and can, thus, be expected to display sensitivity to small perturbations. This needs to be borne in mind when doing convergence studies using stability codes such as CAS3D.

An interesting question for further work is whether other modes, such as drift waves, are quantum chaotic in stellarators, or if this is a peculiarity of MHD modes. There is already evidence that kinetic effects make the semiclassical (WKB) dynamics closer to integrable [19]. Another question is whether quantum chaos of Mercier modes occurs in machines with more field periods than the 5 of W7-X. Earlier work [20] suggested that, in a 10-field-period heliotron equilibrium related to the Large Helical Device (LHD), the spectrum is close to that of an equivalent axisymmetric torus, and thus not chaotic. However, a relatively few modes were studied and the spacing statistics were not calculated.

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