A CONVERGENT FEM-DG METHOD FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

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ABSTRACT. This paper presents a new numerical method for the compressible Navier-Stokes equations governing the flow of an ideal isentropic gas. To approximate the continuity equation, the method utilizes a discontinuous Galerkin discretization on piecewise constants and a basic upwind flux. For the momentum equation, the method is a new combined discontinuous Galerkin and finite element method approximating the velocity in the Crouzeix-Raviart finite element space. While the diffusion operator is discretized in a standard fashion, the convection and time-derivative are discretized using discontinuous Galerkin on the element average velocity and a Lax-Friedrich type flux. Our main result is convergence of the method to a global weak solution as discretization parameters go to zero. The convergence analysis constitutes a numerical version of the existence analysis of Lions and Feireisl.

CONTENTS

1. INTRODUCTION

In this paper, we will construct a convergent numerical method for the compressible Navier-Stokes equations:

$$\varrho_t + \operatorname{div}(\varrho u) = 0 \qquad \text{in } (0,T) \times \Omega, \qquad (1.1)$$

$$(\varrho u)_t + \operatorname{div}(\varrho u \otimes u) = -\nabla p(\varrho) + \Delta u, \quad \text{in } (0, T) \times \Omega, \qquad (1.2)$$

where $\Omega \subset \mathbb{R}^3$ is an open, bounded, domain with Lipschitz boundary $\partial\Omega$, and $T > 0$ is a given finite final time. The unknowns are the fluid density ϱ and vector velocity u, while the operator $\otimes$ denotes the tensor product of two vectors $((v \otimes v)_{i,j} = v_i v_j)$. The mechanism driving the flow is the pressure $p(\varrho)$ which is assumed to be that of an ideal isentropic gas (constant entropy);

$$p(\varrho) = a\varrho^\gamma.$$

To close the system of equations (1.1) - (1.2), standard no-slip boundary condition is assumed

$$u|_{\partial\Omega} = 0, \qquad (1.3)$$

together with initial data

$$\varrho(0, \cdot) = \varrho_0 \in L^{\gamma+1}(\Omega), \qquad \varrho u(0, \cdot) = m_0 \in L^{\frac{3}{2}}(\Omega). \qquad (1.4)$$

From the point of view of applications, the system (1.1) - (1.2) is the simplest form of the equations governing the flow of an ideal viscous and isentropic gas [1, 22]. In the available engineering literature, the reader can find a variaty of flows for which the assumption of constant entropy (reversibility) is a reasonable approximation. However, while viscosity in (1.2) is modeled by the Laplace operator (Δu), practical applications will make use of a more appropriate stress tensor, the simplest being that of a Newtonian fluid with constant coefficients

$$\operatorname{div} \mathbb{S} = \mu \operatorname{div} \left(\nabla u + \nabla u^T\right) + \lambda \nabla \operatorname{div} u.$$

Note that our diffusion is a special case of $\mathbb{S}$. Indeed, $\mathbb{S}$ reduces to the Laplace operator when $\mu = 1$ and $\lambda = \frac{1}{2}$. The analysis in this paper can be generalized to all relevant cases of constant non-vanishing μ and λ. However, this comes at the cost of more terms as the chosen finite element space (Crouzeix-Raviart) does not satisfy Korn's inequality [19], but with a negligible gain in terms of ideas and novelty. The more physically relevant case where μ and λ are functions of the density of the form $\varrho^{\frac{\gamma-1}{2}}$ is not included in the available existence theory (cf. [4, 13]) and there does not seem to be any obvious way of including it here either.

In terms of physical applicability of the forthcoming results, a more pressing issue is the equation of state for the pressure. For purely technical reasons, we will be forced to require that γ is greater than the spatial dimension;

$$\gamma > 3.$$

This is a severe restriction on γ with no physical applications (to the author's knowledge). Kinetic theory predicts a value of γ depending on the specific gas in question: $\sim \frac{5}{3}$ for monoatomic gas (e.g helium), $\sim \frac{7}{5}$ for diatomic gas (e.g air), and creeping towards one for more complex molecules and/or higher temperatures. It should be noted that global existence is only known for $\gamma > \frac{3}{2}$ and hence only in the case of monoatomic gas (see [10] and the references therein). In this paper, the restriction on γ seems absolutely necessary to prove convergence of the method, but is not required for stability. In fact, the strict condition on γ is related to the numerical

diffusion introduced by the method and is in perfect analogy to the problems encountered in a vanishing diffusion limit for the continuity equation (1.1) (see for instance [10]). That being said, it might be calming that the upcoming analysis can be easily extended to pressures of the form

$$p(\varrho) = a\varrho^{\gamma_1} + \kappa\varrho^{\gamma_2}, \quad \gamma_1 \in (1, 2), \quad \gamma_2 > 3,$$

where $\kappa > 0$ may be chosen arbitrarily small. Hence, for all practical purposes, the numerical method presented herein is convergent for cases very close to the physically relevant ones.

The numerical literature contains a vast body of methods for compressible fluid flows. Many of them are widely applied and constitute an indispensable tool in a variety of disciplines such as engineering, meteorology, or astrophysics. While the complexity and range of applicability of numerical methods for compressible viscous fluids is increasing, the convergence properties of any of these methods have thus far remained unknown. In fact, prior to this paper, there have not been reported any convergence results for numerical approximations of compressible viscous flow in more than one spatial dimension. In one dimension, the available results are due to David Hoff and collaborators [5, 26, 27, 28] (see also [13]) and concerns the equations posed in Lagrangian variables and relies on the 1D existence theory which have yet to be extended to multiple dimensions. That being said, in the recent years there have appeared a number of convergent methods for simplified versions of (1.1)-(1.2). In [8, 9, 11], convergent finite volume and finite element methods for the stationary compressible Stokes equations was developed. Simultaneously, in [14, 15, 16], K. Karlsen and the author developed convergent finite element methods for the non-stationary compressible Stokes equations. In the upcoming analysis, we will utilize ideas from all of these recent papers.

Let us now discuss the choice of numerical method. For the approximation of the continuity equation, we will use the standard upwind finite volume method. However, we will formulate this method as a discontinuous Galerkin method where the density ϱ is approximated by piecewise constants. For the velocity we will use the Crouzeix-Raviart finite element space. The method and some its properties was originally inspired by [25, 18] (cf. [14]), but variants can be found several places in the literature (see for instance [12]). For the momentum equation, we are leaning on the knowledge gained through [8, 9, 11, 14, 15, 16], from which it is evident that proving convergence for any numerical method is a hard task. In particular, to develop a numerical analogy of the continuous existence theory, it seems necessary that the numerical discretization of the diffusion and pressure respects orthogonal Hodge decompositions (see Section 8 for an explanation) in some form.

The distinctively new and completely original feature of the upcoming method is the discretization of the material derivative in the momentum equation. Our discretization will be of the form

$$(\varrho u)_t + \operatorname{div}(\varrho u \otimes u)$$
$$\approx \int_\Omega \partial_t(\varrho_h \widehat{u}_h) v_h \, dx - \sum_\Gamma \int_\Gamma \operatorname{Up}(\varrho u \otimes \widehat{u}) \, [\![\widehat{v}_h]\!]_\Gamma \, dS(x), \quad \forall v_h,$$

where $\widehat{v_h}$ denotes the L^2 projection of v_h onto piecewise constants. The operator $\mathrm{Up}(\varrho u \otimes u)$ is the specific upwind flux which we shall use (see Section 3 for precision). Hence, the material derivative is discretized using discontinuous Galerkin *with the same polynomial order* as the continuity discretization. This stands in contrast to the diffusion and pressure terms which are solved with the Crouzeix-Raviart element space and hence in particular with piecewise constant divergence div v_h matching the density (and pressure) space. In the language of finite differences, this means that the pressure and diffusion is solved using staggered grid while the material derivative are solved at the same nodal values as the density. The great benefit of this approach is that it solves the long-lasting problem of incorporating both the hyperbolic nature of the material derivative and the nature of the diffusion-pressure coupling. In particular, our main result yields stability and convergence for all Mach and Reynolds numbers.

By proving convergence of a numerical method we will in effect also give an alternative existence proof for global weak solutions to the equations (1.1)-(1.2). While the first global existence result for incompressible Navier-Stokes was achieved by Leray more than 80 years ago, a similar result for compressible flow was obtained by P-L. Lions in the mid 90s. In the celebrated book [17], Lions obtains weak solutions of (1.1)-(1.2) as the a.e weak limit of a sequence of approximate solutions. Consequently, from the point of view of analysis, we will in this paper perform similar analysis to that of [17], but for a numerical method. That is, we will develop a numerical analogy of the continuous existence theory. The key difficulty when passing to a limit is presented by the non-linear pressure $p(\varrho)$. Specifically, compactness of ϱ is needed, while the available estimates provides no form of continuity of ϱ. In all current proofs of existence, the necessary compactness is derived using renormalized solutions together with a remarkable sequential continuity result for the quantity $p(\varrho) - \mathrm{div}\, u$. The result provides a.e convergence of the density, but gives no insight into continuity properties of ϱ. In the original proof, Lions needed that $\gamma > \frac{9}{5}$. The existence theory was further developed by Feireisl and the requirement lowered to $\gamma > \frac{3}{2}$. This seems to be optimal in the absence of pointwise bounds on the density. However, it is interesting that this still does not include the case of air in three dimensions. The reader is strongly encouraged to consult [23] for a thorough and well-written introduction to the mathematical theory of solutions to (1.1)-(1.2).

Organization of the paper: In the next section, we will go through some preliminary knowledge where we attempt to make clear the solution concept, basic compactness results, and the finite element theory used in the analysis. Then, in Section 3, we present the numerical method, give some basic properties of the method, and state the main convergence result (Theorem 3.5). We will then move on to Section 4 in which we establish stability of the method and draw conclusions in terms of uniform integrability. The following section, Section 5, is a fundamental section where we will estimate the weak error (in a weak norm) of the transport operators in the discretization. The material contained in this section will be used ubiquitously in the convergence analysis. In Section 6 we prove higher integrability of the

density. That is, the density enjoys more integrability than the energy esti-
mate provides. Then, in Section 7, we will pass to the limit in the method
and conclude that the limit is almost a global weak solution. It will then
only remain to prove strong convergence of the density approximation. In
Section 9, we establish the fundamental ingredient in the proof of density
compactness; - weak sequential continuity of the effective viscous flux. Fi-
nally, in Section 10 we prove strong convergence of the density and conclude
the main result (Theorem 3.5). The paper ends with an appendix section
containing the proof of well-posedness for the numerical method.

2. Preliminary material

The purpose of this section is to state some results that will be needed in
the upcoming convergence analysis.

2.1. Weak and renormalized solutions.

Definition 2.1. We say that a pair (ϱ, u) is a weak solution of (1.1) - (1.2)
with initial condition (1.4) and boundary condition (1.3) provided:

(1) The continuity equation (1.1) holds in the sense of distributions

$$\int_0^T \int_\Omega \varrho(\phi_t + u \cdot \nabla\phi) \; dxdt = -\int_\Omega \varrho_0 \phi(0, \cdot) \; dx,$$

for all $\phi \in C_0^\infty([0,T] \times \overline{\Omega})$.

(2) The momentum equation (1.2) holds in the sense

$$\int_0^T \int_\Omega -\varrho u v_t - \varrho u \otimes u : \nabla v - p(\varrho) \operatorname{div} v + \nabla u \nabla v \; dxdt$$

$$= \int_\Omega m_0 v(0, \cdot) \; dx,$$

for all $v \in [C_0^\infty([0,T] \times \Omega)]^3$.

(3) The energy inequality holds

$$\sup_{t \in (0,T)} \int_\Omega \frac{\varrho u^2}{2} + \frac{1}{\gamma - 1} p(\varrho) \; dx$$

$$+ \int_0^T \int_\Omega |\nabla u|^2 \; dxdt \le \int_\Omega \frac{m_0^2}{2\varrho_0} + \frac{1}{\gamma - 1} p(\varrho_0) \; dx. \tag{2.1}$$

Definition 2.2 (Renormalized solutions)**.** Given $u \in L^2(0,T; W_0^{1,2}(\Omega))$, we
say that $\varrho \in L^\infty(0,T; L^\gamma(\Omega))$ is a renormalized solution of (1.1) if

$$B(\varrho)_t + \operatorname{div}(B(\varrho)u) + b(\varrho) \operatorname{div} u = 0,$$

in $\mathcal{D}'\left([0,T] \times \overline{\Omega}\right)$ for any $B \in C[0,\infty) \cap C^1(0,\infty)$ with $B(0) = 0$ and $b(\varrho) := B'(\varrho)\varrho - B(\varrho)$.

We shall need the following well-known lemma [17] stating that square-
integrable weak solutions ϱ are also renormalized solutions.

Lemma 2.3. *Suppose (ϱ, u) is a weak solution according to Definition 2.1. If
$\varrho \in L^2((0,T) \times \Omega)$, then ϱ is a renormalized solution according to Definition
2.2.*

2.2. **Compactness results.** In the analysis, we shall need a number of compactness results.

Lemma 2.4. *Let X be a separable Banach space, and suppose $v_n \colon [0, T] \to X^\star$, $n = 1, 2, \ldots$, is a sequence for which $\|v_n\|_{L^\infty([0,T];X^\star)} \leq C$, for some constant C independent of n. Suppose the sequence $[0, T] \ni t \mapsto \langle v_n(t), \Phi \rangle_{X^\star, X}$, $n = 1, 2, \ldots$, is equi-continuous for every Φ that belongs to a dense subset of X. Then v_n belongs to $C\left([0, T]; X^\star_{\mathrm{weak}}\right)$ for every n, and there exists a function $v \in C\left([0, T]; X^\star_{\mathrm{weak}}\right)$ such that along a subsequence as $n \to \infty$ there holds $v_n \to v$ in $C\left([0, T]; X^\star_{\mathrm{weak}}\right)$.*

To obtain strong compactness of the density approximation, we will utilize the following lemma.

Lemma 2.5. *Let O be a bounded open subset of $\mathbb{R}^M$, $M \geq 1$. Suppose $g \colon \mathbb{R} \to (-\infty, \infty]$ is a lower semicontinuous convex function and $\{v_n\}_{n \geq 1}$ is a sequence of functions on O for which $v_n \rightharpoonup v$ in $L^1(O)$, $g(v_n) \in L^1(O)$ for each n, $g(v_n) \rightharpoonup \overline{g(v)}$ in $L^1(O)$. Then $g(v) \leq \overline{g(v)}$ a.e. on O, $g(v) \in L^1(O)$, and $\int_O g(v)\, dy \leq \liminf_{n \to \infty} \int_O g(v_n)\, dy$. If, in addition, g is strictly convex on an open interval $(a, b) \subset \mathbb{R}$ and $g(v) = \overline{g(v)}$ a.e. on O, then, passing to a subsequence if necessary, $v_n(y) \to v(y)$ for a.e. $y \in \{y \in O \mid v(y) \in (a, b)\}$.*

In what follows, we will often obtain a priori estimates for a sequence $\{v_n\}_{n \geq 1}$ that we write as "$v_n \in_{\mathrm{b}} X$" for some functional space X. What this really means is that we have a bound on $\|v_n\|_X$ that is independent of n. The following lemma is taken from [14].

Lemma 2.6. *Given $T > 0$ and a small number $h > 0$, write $(0, T] = \cup_{k=1}^{M}(t_{k-1}, t_k]$ with $t_k = hk$ and $Mh = T$. Let $\{f_h\}_{h>0}^{\infty}$, $\{g_h\}_{h>0}^{\infty}$ be two sequences such that:*

(1) *the mappings $t \mapsto g_h(t, x)$ and $t \mapsto f_h(t, x)$ are constant on each interval $(t_{k-1}, t_k]$, $k = 1, \ldots, M$.*
(2) *$\{f_h\}_{h>0}$ and $\{g_h\}_{h>0}$ converge weakly to f and g in $L^{p_1}(0, T; L^{q_1}(\Omega))$ and $L^{p_2}(0, T; L^{q_2}(\Omega))$, respectively, where $1 < p_1, q_1 < \infty$ and*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1.$$

(3) *the discrete time derivative satisfies*

$$\frac{g_h(t, x) - g_h(t - h, x)}{h} \in_b L^1(0, T; W^{-1,1}(\Omega)).$$

(4) *$\|f_h(t, x) - f_h(t, x - \xi)\|_{L^{p_2}(0, T; L^{q_2}(\Omega))} \to 0$ as $|\xi| \to 0$, uniformly in h.*

Then, $g_h f_h \rightharpoonup gf$ in the sense of distributions on $(0, T) \times \Omega$.

2.3. **Finite element spaces and some basic properties.** Let E_h denote a shape regular tetrahedral mesh of Ω. Let Γ_h denote the set of faces in E_h. We will approximate the density in the space of piecewise constants on E_h and denote this space by $Q_h(\Omega)$. For the approximation of the velocity we

use the Crouzeix–Raviart element space [6]:

$$V_h(\Omega) = \left\{ v_h \in L^2(\Omega) : \ v_h|_E \in \mathbb{P}_1^3(E), \ \forall E \in E_h, \right.$$
$$\left. \int_\Gamma [\![v_h]\!]_\Gamma \ dS(x) = 0, \ \forall \Gamma \in \Gamma_h \right\}, \tag{2.2}$$

where $[\![\cdot]\!]_\Gamma$ denotes the jump across a face Γ. To incorporate the boundary condition, we let the degrees of freedom of $V_h(\Omega)$ vanish at the boundary. Consequently, the finite element method is nonconforming in the sense that the velocity approximation space is not a subspace of the corresponding continuous space, $W_0^{1,2}(\Omega)$.

For the purpose of analysis, we shall also need the div-conforming Nédélec finite element space of first order and kind [20, 21]

$$\mathcal{N}_h(\Omega) = \left\{ v_h, \ \mathrm{div}\, v_h \in L^2(\Omega) : v_h|_E \in \mathbb{P}_0^3 \oplus \mathbb{P}_0^1 \boldsymbol{x}, \ \forall E \in E_h, \right.$$
$$\left. \int_\Gamma [\![v \cdot \nu]\!] \ dS(x) = 0, \ \forall \Gamma \in \Gamma_h \right\}. \tag{2.3}$$

We introduce the canonical interpolation operators

$$\Pi_h^V : W_0^{1,2}(\Omega) \mapsto V_h(\Omega),$$
$$\Pi_h^Q : L^2(\Omega) \mapsto Q_h(\Omega), \tag{2.4}$$
$$\Pi_h^{\mathcal{N}} : W^{1,2}(\Omega) \mapsto \mathcal{N}_h(\Omega),$$

defined by

$$\int_\Gamma \Pi_h^V v_h \ dS(x) = \int_\Gamma v_h \ dS(x), \quad \forall \Gamma \in \Gamma_h,$$
$$\int_\Gamma \Pi_h^{\mathcal{N}} v_h \cdot \nu \ dS(x) = \int_\Gamma v_h \cdot \nu \ dS(x), \quad \forall \Gamma \in \Gamma_h, \tag{2.5}$$
$$\int_E \Pi_h^Q \phi \ dx = \int_E \phi \ dx, \quad \forall E \in E_h.$$

Then, by virtue of (2.5) and Stokes' theorem,

$$\mathrm{div}\, \Pi_h^{\mathcal{N}} v = \mathrm{div}_h \Pi_h^V v = \Pi_h^Q \,\mathrm{div}\, v, \qquad \mathrm{curl}_h \Pi_h^V v = \Pi_h^Q \,\mathrm{curl}\, v, \tag{2.6}$$

for all $v \in W_0^{1,2}(\Omega)$. Here, curl_h and div_h denote the curl and divergence operators, respectively, taken inside each element.

Let us now state some basic properties of the finite element spaces. We start by recalling from [3, 6, 20] a few interpolation error estimates.

Lemma 2.7. *There exists a constant $C > 0$, depending only on the shape regularity of E_h, such that for any $1 \le p \le \infty$,*

$$\|\Pi_h^Q \phi - \phi\|_{L^p(E)} \le Ch\|\nabla\phi\|_{L^p(E)},$$
$$\|\Pi_h^V v - v\|_{L^p(E)} + h\|\nabla(\Pi_h^V v - v)\|_{L^p(E)} \le ch^s\|\nabla^s v\|_{L^p(E)}, \quad s = 1, 2,$$
$$\|\Pi_h^{\mathcal{N}} v - v\|_{L^p(E)} + h\|\,\mathrm{div}(\Pi_h^{\mathcal{N}} v - v)\|_{L^p(E)} \le ch^s\|\nabla^s v\|_{L^p(E)}, \quad s = 1, 2,$$

for all $\phi \in W^{1,p}(E)$ and $v \in W^{s,p}(E)$.

By scaling arguments, the trace theorem, and the Poincaré inequality, we obtain

Lemma 2.8. *For any $E \in E_h$ and $\phi \in W^{1,2}(E)$, we have*

(1) *trace inequality,*

$$\|\phi\|_{L^2(\Gamma)} \leq ch_E^{-\frac{1}{2}} \left(\|\phi\|_{L^2(E)} + h_E\|\nabla\phi\|_{\boldsymbol{L}^2(E)} \right), \quad \forall \Gamma \in \Gamma_h \cap \partial E.$$

(2) *Poincaré inequality,*

$$\left\| \phi - \frac{1}{|E|} \int_E \phi \; dx \right\|_{L^2(E)} \leq Ch_E\|\nabla\phi\|_{\boldsymbol{L}^2(E)}.$$

In both estimates, h_E is the diameter of the element E.

Scaling arguments and the equivalence of finite dimensional norms yields the classical inverse estimate (cf. [2]):

Lemma 2.9. *There exists a positive constant C, depending only on the shape regularity of E_h, such that for $1 \leq q, p \leq \infty$ and $r = 0, 1$,*

$$\|\phi_h\|_{W^{r,p}(E)} \leq Ch^{-r+\min\{0,\frac{3}{p}-\frac{3}{q}\}} \|\phi_h\|_{L^q(E)},$$

for any $E \in E_h$ and all polynomial functions $\phi_h \in \mathbb{P}_k(E)$, $k = 0, 1, \dots$.

Since the Crouzeix-Raviart element space is not a subspace of $W^{1,2}$, it is not a priori clear that functions in this space are compact in L^p, $p < 6$. However, from the Sobolev inequality on each element and an interpolation argument we obtain the needed result.

Lemma 2.10. *For $u_h \in V_h(\Omega)$, let p satisfy $2 \leq p < 6$ and determine a such that*

$$\frac{1}{p} = \frac{a}{2} + \frac{(1-a)}{6}.$$

The following space translation estimate holds

$$\|u_h(\cdot) - u_h(\cdot - \xi)\|_{L^p(\Omega\setminus\{x:\text{dist}(x,\partial\Omega)\}<|\xi|)} \leq C \left(h^2 + |\xi|^2 \right)^{\frac{a}{2}} \|\nabla u_h\|_{L^2(\Omega)},$$

where C is independent of h and ξ.

Proof. From the work of Stummel [24], we have that

$$\|u_h(\cdot) - u_h(\cdot - \xi)\|_{L^2(\Omega)} \leq C \left(h^2 + |\xi|^2 \right)^{\frac{1}{2}} \|\nabla_h u_h\|_{L^2(\Omega)}. \tag{2.7}$$

The standard Sobolev inequality gives

$$\|u_h(\cdot) - u_h(\cdot - \xi)\|_{L^6(\Omega)} \leq C\|\nabla_h u_h\|_{L^2(\Omega)}. \tag{2.8}$$

Hence, the proof follows from interpolation between (2.7) and (2.8). $\square$

Finally, we recall the following well-known property of the Crouzeix-Raviart element space.

Lemma 2.11. *For any $u_h \in V_h(\Omega)$ and $v \in W_0^{1,2}(\Omega)$,*

$$\int_\Omega \nabla_h u_h \nabla_h \left(\Pi_h^V v - v \right) \; dx = 0. \tag{2.9}$$

Proof. By direct calculation, we obtain

$$\int_\Omega \nabla_h u_h \nabla_h (\Pi_h^V v - v) \ dx$$

$$= -\sum_E \int_E \Delta u_h (\Pi_h^V v - v) \ dx + \int_{\partial E} (\nabla u_h \cdot \nu)(\Pi_h^V v - v) \ dS(x).$$

Now, since u_h is linear on each element $\Delta u_h = 0$ and ∇u_h is constant. Moreover, since the normal vector ν is constant on each face of the element, we have that

$$\int_\Omega \nabla_h u_h \nabla_h (\Pi_h^V v - v) \ dx$$

$$= \sum_E \sum_{\Gamma \subset \partial E} \left(\int_\Gamma (\Pi_h^V v - v) \ dS(x) \right) (\nabla u_h \cdot \nu) = 0,$$

by definition of the interpolation operator Π_h^V. Hence, we have proved (2.9). $\square$

3. Numerical method and main result

For a given timestep $\Delta t > 0$, we divide the time interval $[0, T]$ in terms of the points $t^k = k\Delta t$, $k = 0, \ldots, M$, where we assume $M\Delta t = T$. To discretize space, we let $\{E_h\}_h$ be a shape-regular family of tetrahedral meshes of Ω, where h is the maximal diameter. It will be a standing assumption that h and Δt are related like $\Delta t = ch$ for some constant c. We also let Γ_h denote the set of faces in E_h.

The functions that are piecewise constant with respect to the elements of a mesh E_h are denoted by $Q_h(\Omega)$ and by $V_h(\Omega)$ we denote the Crouzeix–Raviart finite element space (2.2) formed on E_h. To incorporate the boundary condition, we let the degrees of freedom of $V_h(\Omega)$ vanish at the boundary:

$$\int_\Gamma v_h \ dS(x) = 0 \quad \forall \Gamma \in \Gamma_h \cap \partial\Omega, \quad \forall v_h \in V_h(\Omega).$$

We will need some additional notation to accommodate discontinuous Galerkin discretization. Related to the boundary ∂E of an element E, we write f_+ for the trace of the function f taken within the element E, and f_- for the trace of f from the outside. Related to a face Γ shared between two elements E_- and E_+, we will write f_+ for the trace of f within E_+, and f_- for the trace of f within E_-. Here E_- and E_+ are defined such that ν points from E_- to E_+, where ν is fixed as one of the two possible normal components on Γ. The jump of f across the face Γ is denoted $[\![f]\!]_\Gamma = f_+ - f_-$.

To pose the method, and in the convergence analysis, we shall need the canonical interpolation operators (2.4). In fact, we shall need the operators Π_h^Q and $\Pi_h^{\mathcal{N}}$ to such an extent that we introduce the convenient notation

$$\widetilde{v} = \Pi_h^{\mathcal{N}} v, \qquad \widehat{\phi} = \Pi_h^Q \phi. \tag{3.1}$$

To discretize the convective operator $\text{div}(\varrho u)$ in the continuity equation (1.1), we will utilize a standard upwind method in the degrees of freedom

E_h	-	the mesh.
E	-	an element in the mesh.
∂E	-	the boundary of E.
Γ	-	a face in the mesh.
Γ_h	-	all faces in the mesh.
$Q_h(\Omega)$	-	the space of piece constant scalars on E_h.
$V_h(\Omega)$	-	the Crouzeix-Raviart vector space on E_h.
$\mathcal{N}_h(\Omega)$	-	the div conforming Nédélec space
		of first order and kind.
Π_h^Q	-	the L^2 projection operator onto Q_h.
Π_h^V	-	the canonical interpolation operator onto V_h.
Π_h^N	-	the canonical interpolation operator onto $\mathcal{N}_h$.
$\widehat{f}$	-	$\Pi_h^Q f$ (the piecewise constant projection).
$\widetilde{v}$	-	$\Pi_h^N v$ (the Nédélec interpolation).
f^+	-	$\max\{f, 0\}$.
f^-	-	$\min\{f, 0\}$.
$f_+\|_{\partial E}$	-	the trace of f taken from within E.
$f_-\|_{\partial E}$	-	the trace of f taken from outside E.
$f_+\|_\Gamma$	-	the trace of f taken against the normal vector ν.
$f_-\|_\Gamma$	-	the trace of f taken in the direction of ν.
$[\![f]\!]_\Gamma$	-	$f_+ - f_-$.
$\mathrm{Up}(\varrho u)\|_{\partial E}$	-	$\varrho_+(\widetilde{u}_h \cdot \nu)^+ + \varrho_-(\widetilde{u}_h \cdot \nu)^-$.
$\mathrm{Up}(\varrho u)\|_\Gamma$	-	$\varrho_-(\widetilde{u}_h \cdot \nu)^+ + \varrho_+(\widetilde{u}_h \cdot \nu)^-$.
$\mathrm{Up}(\varrho u \otimes \widehat{u})_{\partial E}$	-	$\mathrm{Up}^+(\varrho u)\widehat{u}_+ + \mathrm{Up}^-(\varrho u)\widehat{u}_-$.
$\mathrm{Up}(\varrho u \otimes \widehat{u})_\Gamma$	-	$\mathrm{Up}^-(\varrho u)\widehat{u}_+ + \mathrm{Up}^+(\varrho u)\widehat{u}_-$.

TABLE 1. Notation

of u_h. The upwind discretization is defined as follows

$$
\begin{aligned}
\mathrm{Up}(\varrho u)\|_\Gamma &= \varrho_- \left(\frac{1}{|\Gamma|} \int_\Gamma u_h \cdot \nu \; dS(x) \right)^+ + \varrho_+ \left(\frac{1}{|\Gamma|} \int_\Gamma u_h \cdot \nu \; dS(x) \right)^-, \\
&= \varrho_-(\widetilde{u}_h \cdot \nu)^+ + \varrho_+(\widetilde{u}_h \cdot \nu)^-, \quad \forall \Gamma \in \Gamma_h,
\end{aligned}
\tag{3.2}
$$

where we have used the notation (3.1).

For the convective operator $\mathrm{div}(\varrho u \otimes u)$ in the momentum equation (1.2) we will use the following Lax-Friedrich type upwind flux

$$
\mathrm{Up}(\varrho u \otimes \widehat{u}) = \mathrm{Up}^+(\varrho u)\widehat{u}_+ + \mathrm{Up}^-(\varrho u)\widehat{u}_-.
\tag{3.3}
$$

Observe that this operator is posed for the average value of u_h over each element. This is non-standard and has to the author's knowledge not been studied previously. We are now ready to pose the method.

Definition 3.1 (Numerical method)**.** Let $\varrho_0 \in L^{\gamma+1}(\Omega)$ and $m_0 \in L^{\frac{3}{2}}(\Omega)$ be given initial data and assume that $T > 0$ is a given finite final time. Define

the numerical initial data

$$\varrho_h^0 = \Pi_h^Q(\varrho_0 + \kappa h), \qquad u_h^0 = \Pi_h^V\left[\frac{m_0}{\varrho_0 + \kappa h}\right], \qquad (3.4)$$

where κ is a small positive number. Determine sequentially

$$(\varrho_h^k, u_h^k) \in Q_h(\Omega) \times V_h(\Omega), \quad k = 1, \ldots, M,$$

satisfying, for all $q_h \in Q_h(\Omega)$,

$$
\int_\Omega \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} q_h \, dx - \sum_\Gamma \int_\Gamma \mathrm{Up}^k(\varrho u) \, [\![q_h]\!]_\Gamma \, dS(x) \\
+ h^{1-\epsilon} \sum_\Gamma \int_\Gamma \left[\!\left[\varrho_h^k\right]\!\right]_\Gamma [\![q_h]\!]_\Gamma \, dS(x) = 0,
\qquad (3.5)
$$

and for all $v_h \in V_h(\Omega)$,

$$
\int_\Omega \frac{\varrho_h^k \widehat{u}_h^k - \varrho_h^{k-1}\widehat{u}_h^{k-1}}{\Delta t} v_h \, dx - \sum_\Gamma \int_\Gamma \mathrm{Up}^k(\varrho u \otimes \widehat{u}) \, [\![\widehat{v}]\!]_\Gamma \, dS(x) \\
+ \int_\Omega \nabla_h u_h^k \nabla_h v_h - p(\varrho_h^k)\, \mathrm{div}\, v_h \, dx \\
+ h^{1-\epsilon} \sum_E \int_{\partial E} \left(\frac{\widehat{u}_-^k + \widehat{u}_+^k}{2}\right) \left[\!\left[\varrho_h^k\right]\!\right] [\![\widehat{v}_h]\!]_\Gamma \, dS(x) = 0,
\qquad (3.6)
$$

where $\epsilon > \frac{1}{6}$ should be chosen as small as possible.

For the purpose of analysis, we will need to extend the pointwise-in-time numerical solution (ϱ_h^k, u_h^k), $k = 0, \ldots, M$, to a piecewise constant in time. For this purpose, we will use the following definition

$$(\varrho_h, u_h)(t, \cdot) = (\varrho_h^k, u_h^k), \quad t \in [t^k, t^{k+1}), \quad k = 0, \ldots, M. \qquad (3.7)$$

Remark 3.2. The terms involving $h^{1-\epsilon}$ in (3.5) - (3.6) are needed for purely technical reasons. In particular, they are not needed to obtain stability of the method or to prove convergence of the continuity method (3.5). See Section 8 and [9] for more on why they are needed.

3.1. **The method is well-defined.** Since the numerical method is nonlinear and implicit, it is not trivial that it is actually well-defined. In addition, the transport operators in the momentum equation is posed for the element average velocity and hence does not provide a full set of equations in themselves. In fact, it is only due to the diffusion that the system has a sufficient number of equations for the degrees of freedom of u_h. We shall prove the existence of a numerical solution through a topological degree argument. The proof is very similar to that of [12, 14] and is for this reason deferred to the appendix.

Proposition 3.3. *For each fixed $h > 0$, there exists a solution*

$$(\varrho_h^k, u_h^k) \in Q_h(\Omega) \times V_h(\Omega), \quad \varrho_h^k(\cdot) > 0, \quad k = 1, \ldots, M,$$

to the numerical method posed in Definition 3.1.

In the upcoming analysis, we will need that the density solution is positive. For this purpose, we recall the following result from [14] (see also [12]):

Lemma 3.4. *Fix any* $k = 1, \ldots, M$ *and suppose* $\varrho_h^{k-1} \in Q_h(\Omega)$, $u_h^k \in V_h(\Omega)$ *are given bounded functions. Then the solution* $\varrho_h^k \in Q_h(\Omega)$ *of the discontinuous Galerkin method* (3.5) *satisfies*

$$\min_{x \in \Omega} \varrho_h^k \geq \min_{x \in \Omega} \varrho_h^{k-1} \left(\frac{1}{1 + \Delta t \| \operatorname{div}_h u_h^k \|_{L^\infty(\Omega)}} \right).$$

Consequently, if $\varrho_h^{k-1}(\cdot) > 0$, *then* $\varrho_h^k(\cdot) > 0$.

3.2. **Main result.** Our main result is that the numerical method converges to a weak solution of the compressible Navier-Stokes equations (1.1) - (1.4).

Theorem 3.5. *Suppose* $\gamma > 3$, $T > 0$ *is a given finite final time, and that the initial data* (ϱ_0, m_0) *satisfies*

$$\int_\Omega \frac{m_0^2}{2\varrho_0} + \frac{1}{\gamma - 1} p(\varrho_0) \ dx \leq C, \quad \varrho_0 \in L^\infty(0, T; L^{\gamma+1}(\Omega)).$$

Let $\{(\varrho_h, u_h)\}_{h>0}$ *be a sequence of numerical solutions constructed according to Definition 3.1 and* (3.7) *with* $\Delta t = ch$. *Along a subsequence as* $h \to 0$,

$$u_h \rightharpoonup u \ in \ L^2(0, T; L^6(\Omega)),$$

$$\nabla_h u_h \rightharpoonup \nabla u \ in \ L^2(0, T; L^2(\Omega))$$

$$\varrho_h u_h, \ \varrho_h \widetilde{u}_h \rightharpoonup \varrho u \ in \ L^2(0, T; L^{\frac{6\gamma}{3+\gamma}}(\Omega)),$$

$$\varrho_h \widehat{u}_h \rightharpoonup \varrho u \ in \ C(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega)).$$

$$\varrho_h \widetilde{u}_h \otimes \widehat{u}_h \rightharpoonup \varrho u \otimes u \ in \ L^2(0, T; L^{\frac{3\gamma}{3+\gamma}}(\Omega)),$$

$$\varrho_h \to \varrho \ a.e \ and \ in \ L^p_{loc}((0, T) \times \Omega), \ p < \gamma + 1,$$

where (ϱ, u) *is a weak solution of the isentropic compressible Navier-Stokes equations* (1.1) - (1.2) *in the sense of Definition 2.1.*

The proof of Theorem 3.5 will be developed in the remaining sections of this paper. The final conclusion will come in Section 10.1.

4. ENERGY AND STABILITY

In this section we will prove that our method is stable and satisfies a numerical analogy of the continuous energy inequality (2.1). Both the stability estimate and large parts of the subsequent convergence analysis relies on a renormalized type identity derived from the continuity scheme (3.5). The proof of this identity can be found in [14, 15]. We shall utilize the following form:

Lemma 4.1. *Let* (ϱ^h, u^h) *solve the continuity scheme* (3.5). *Then, for any* $B \in C^2(\mathbb{R}_+)$ *with* $B'' \geq 0$, *there holds*

$$\int_\Omega \frac{B(\varrho_h^k) - B(\varrho_h^{k-1})}{\Delta t} \ dx + (\Delta t)^{-1} \int_\Omega B''(\varrho_\ddagger)(\varrho_h^k - \varrho_h^{k-1})^2 \ dx$$

$$+ \sum_\Gamma \int_\Gamma B''(\varrho_\dagger) \left[\!\left[\varrho_h^k \right]\!\right]^2 \left(|\widetilde{u_h^k} \cdot \nu| + h^{1-\epsilon} \right) \ dS(x)$$

$$\leq - \int_\Omega (\varrho B'(\varrho) - B(\varrho)) \operatorname{div} u_h \ dx,$$

where $\varrho_{\ddagger}$ and $\varrho_{\dagger}$ are some numbers in the range $[\varrho^{k-1}, \varrho^k]$ and $[\varrho^k_-, \varrho^k_+]$, respectively.

We now prove our main stability result.

Proposition 4.2. *For given Δt, $h > 0$, let (u^k_h, ϱ^k_h), $k = 0, \ldots, M$ be the numerical approximation of (1.1)-(1.2) in the sense of Definition 3.1. Then,*

$$\max_{m=1,\ldots,M} \int_\Omega \frac{\varrho^m_h |\widehat{u}^m_h|^2}{2} + \frac{1}{\gamma - 1} p(\varrho^m_h) \ dx$$

$$+ \Delta t \sum_{k=1}^M \int_\Omega |\nabla_h u^k_h|^2 \ dx + \Delta t \sum_{i=1}^5 \sum_{k=1}^M D^k_i \qquad (4.1)$$

$$\leq \int_\Omega \frac{\varrho^0_h |\widehat{u}^0_h|^2}{2} + \frac{1}{\gamma - 1} p(\varrho^0_h) \ dx,$$

where the numerical diffusion terms are given by

$$D^k_1 = \sum_\Gamma \int_\Gamma P''(\varrho_\dagger) \left[\!\left[\varrho^k_h\right]\!\right]_\Gamma^2 |u^k_h \cdot \nu| \ dS(x),$$

$$D^k_2 = \sum_\Gamma \int_\Gamma \left| \mathrm{Up}^k(\varrho u) \right| \left[\!\left[\widehat{u}^k_h\right]\!\right]_\Gamma^2 \ dS(x),$$

$$D^k_3 = h^{1-\epsilon} \sum_\Gamma \int_0^T \int_\Gamma \varrho_{\dagger\dagger} [\![\varrho_h]\!]_\Gamma^2 \ dS(x) dt,$$

$$D^k_4 = (\Delta t)^{-1} \int_\Omega P''(\varrho_\ddagger)(\varrho^k_h - \varrho^{k-1}_h)^2 \ dx,$$

$$D^k_5 = (\Delta t)^{-1} \int_\Omega \varrho^{k-1}_h |u^k_h - u^{k-1}_h|^2 \ dx.$$

Proof. Let $v_h = u^k_h$ in the momentum scheme (3.6), to obtain

$$\int_\Omega \frac{\varrho^k_h \widehat{u}^k_h - \varrho^{k-1}_h \widehat{u}^{k-1}_h}{\Delta t} u^k_h \ dx + \int_\Omega |\nabla_h u^k_h|^2 \ dx$$

$$= -\sum_E \int_{\partial E} \mathrm{Up}^k(\varrho u \otimes \widehat{u}) \widehat{u}^k_+ \ dS(x) + \int_\Omega p(\varrho^k_h) \operatorname{div} u^k_h \ dx \qquad (4.2)$$

$$+ h^{1-\epsilon} \sum_E \int_{\partial E} \left(\frac{\widehat{u}^k_- + \widehat{u}^k_+}{2} \right) \left[\!\left[\varrho^k_h\right]\!\right] \widehat{u}^k_h \ dS(x).$$

From Lemma 4.1 with $B(z) = \frac{1}{\gamma-1} p(z)$, we see that the pressure term

$$\int_\Omega p(\varrho^k_h) \operatorname{div} u^k_h \ dx$$

$$\geq -\int_\Omega \frac{1}{\gamma - 1} \frac{p(\varrho^k_h) - p(\varrho^{k-1}_h)}{\Delta t} \ dx - D^k_1 - D^k_3 - D^k_4. \qquad (4.3)$$

Next, we turn our attention to the first term after the equality in (4.2). From the definition of $\mathrm{Up}(\varrho u \otimes \widehat{u})$, we have that

$$\sum_E \int_{\partial E} \mathrm{Up}^k(\varrho u \otimes \widehat{u}) \widehat{u}^k_+ \ dS(x)$$

$$= \sum_E \int_{\partial E} \left(\mathrm{Up}^+(\varrho u)\widehat{u}_+^k + \mathrm{Up}^-(\varrho u)\widehat{u}_-^k \right) \widehat{u}_+^k \, dS(x)$$

$$= \sum_E \int_{\partial E} \mathrm{Up}^k(\varrho u)\frac{(\widehat{u}_+^k)^2}{2} - \mathrm{Up}^-(\varrho u)\frac{(\widehat{u}_+^k)^2}{2} \tag{4.4}$$

$$+ \mathrm{Up}^+(\varrho u)\frac{(\widehat{u}_+^k)^2}{2} + \mathrm{Up}^-(\varrho u)\widehat{u}_-^k \widehat{u}_+^k dS(x).$$

By setting $q_h = (1/2)(\widehat{u}_h^k)^2$ in the continuity scheme (3.5), we see that the first term after the equality in (4.4) appears

$$\int_\Omega \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \frac{|\widehat{u}_h^k|^2}{2} \, dx$$

$$= -\sum_E \int_{\partial E} \mathrm{Up}^k(\varrho u)\frac{(\widehat{u}_+^k)^2}{2} \, dS(x) + h^{1-\epsilon} \sum_E \int_{\partial E} \left[\!\left[\varrho_h^k \right]\!\right] \frac{(\widehat{u}_+^k)^2}{2} \, dS(x)$$

$$= -\sum_E \int_{\partial E} \mathrm{Up}^k(\varrho u)\frac{(\widehat{u}_+^k)^2}{2} \, dS(x) \tag{4.5}$$

$$+ h^{1-\epsilon} \sum_E \int_{\partial E} \left[\!\left[\varrho_h^k \right]\!\right] \left(\frac{\widehat{u}_+^k + \widehat{u}_-^k}{2} \right) \widehat{u}_+^k \, dS(x).$$

To see the contribution of the second term in (4.4), we first recall that

$$\mathrm{Up}^+(\varrho u)\big|_{\partial E^+} = -\,\mathrm{Up}^-(\varrho u)\big|_{\partial E^-}, \qquad \partial E^+ \cap \partial E^- = \Gamma,$$

since the normal vector has opposite signs. Using this, we obtain

$$\sum_E \int_{\partial E} \mathrm{Up}^+(\varrho u)\frac{(\widehat{u}_+^k)^2}{2} - \mathrm{Up}^-(\varrho u)\frac{(\widehat{u}_+^k)^2}{2} + \mathrm{Up}^-(\varrho u)\widehat{u}_-^k \widehat{u}_+^k \, dS(x)$$

$$= \sum_\Gamma \int_\Gamma \mathrm{Up}^+(\varrho u)\frac{(\widehat{u}_+^k)^2}{2} - \mathrm{Up}^-(\varrho u)\frac{(\widehat{u}_-^k)^2}{2} - \mathrm{Up}^-(\varrho u)\frac{(\widehat{u}_+^k)^2}{2}$$

$$- \mathrm{Up}^+(\varrho u)\frac{(\widehat{u}_-^k)^2}{2} + \mathrm{Up}^-(\varrho u)\widehat{u}_-^k \widehat{u}_+^k - \mathrm{Up}^+(\varrho u)\widehat{u}_+^k \widehat{u}_-^k \, dS(x)$$

$$= \sum_\Gamma \int_\Gamma |\mathrm{Up}^k(\varrho u)| \left(\frac{(\widehat{u}_+^k)^2}{2} + \frac{(\widehat{u}_-^k)^2}{2} - \widehat{u}_+^k \widehat{u}_-^k \right) \, dS(x) \equiv D_2^k.$$

By applying this together with (4.5) in (4.4), we discover

$$-\sum_E \int_{\partial E} \mathrm{Up}^k(\varrho u \otimes \widehat{u})\widehat{u}_+^k \, dS(x)$$

$$+ h^{1-\epsilon} \sum_E \int_{\partial E} \left[\!\left[\varrho_h^k \right]\!\right] \left(\frac{\widehat{u}_+^k + \widehat{u}_-^k}{2} \right) \widehat{u}_+^k \, dS(x) \tag{4.6}$$

$$= \int_\Omega \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \frac{|\widehat{u}_h^k|^2}{2} \, dx - D_2^k.$$

Now, setting (4.6) and (4.3) into (4.2) reveals

$$\int_\Omega \frac{\varrho_h^k \widehat{u}_h^k - \varrho_h^{k-1} \widehat{u}_h^{k-1}}{\Delta t} u_h^k - \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \frac{|\widehat{u}_h^k|^2}{2} + \frac{1}{\gamma - 1} \frac{p(\varrho_h^k) - p(\varrho_h^{k-1})}{\Delta t} \; dx$$
$$+ \int_\Omega |\nabla_h u_h^k|^2 \; dx + D_1^k + D_2^k + D_3^k + D_4^k \leq 0.$$

A simple calculation gives

$$\int_\Omega \frac{\varrho_h^k \widehat{u}_h^k - \varrho_h^{k-1} \widehat{u}_h^{k-1}}{\Delta t} u_h^k - \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \frac{|\widehat{u}_h^k|^2}{2} \; dx$$
$$= \int_\Omega \frac{\varrho_h^k |u_h^k|^2 - \varrho_h^{k-1} |u_h^{k-1}|^2}{2\Delta t} + \frac{\varrho_h^{k-1} |u_h^k - u_h^{k-1}|^2}{\Delta t} \; dx$$
$$\equiv \int_\Omega \frac{\varrho_h^k |u_h^k|^2 - \varrho_h^{k-1} |u_h^{k-1}|^2}{2\Delta t} \; dx + D_5^k.$$

Consequently, by combining the two previous inequalities,

$$\int_\Omega \frac{\varrho_h^k |u_h^k|^2 - \varrho_h^{k-1} |u_h^{k-1}|^2}{2\Delta t} \; dx + \frac{p(\varrho_h^k) - p(\varrho_h^{k-1})}{(\gamma - 1)\Delta t} \; dx$$
$$+ \int_\Omega |\nabla_h u_h^k|^2 \; dx + \sum_{i=1}^{5} D_i^k \leq 0.$$

We conclude by multiplying with Δt and summing over $k = 1, \ldots, M$.   □

Observe that the energy estimate does not provide L^∞ control in time on $\varrho_h |u_h|^2$. Instead, we only gain this control on the projection $\varrho_h |\widehat{u}_h|^2$. The following corollary is an immediate consequence of the energy estimate (Proposition 4.2) and the Hölder inequality (cf. [10]).

Corollary 4.3. *Under the conditions of Proposition 4.2,*

$$\varrho_h \in_b L^\infty(0, T; L^\gamma(\Omega)), \quad p(\varrho_h) \in_b L^\infty(0, T; L^1(\Omega))$$
$$u_h \in_b L^2(0, T; L^6(\Omega)), \quad \nabla_h u_h \in_b L^2(0, T; L^2(\Omega)),$$

$$\varrho_h \widehat{u}_h \in_b L^\infty(0, T; L^{m_\infty}(\Omega)), \quad \varrho_h u_h \in_b L^2(0, T; L^{m_2}(\Omega)),$$
$$\varrho_h |\widehat{u}_h|^2 \in L^\infty(0, T; L^1(\Omega)), \quad \varrho_h |u_h|^2 \in_b L^2(0, T; L^{c_2}(\Omega)),$$

where the exponents are given by (since $\gamma > 3$)

$$m_\infty = \frac{2\gamma}{\gamma + 1} > \frac{3}{2}, \qquad m_2 = \frac{6\gamma}{3 + \gamma} > 3, \qquad c_2 = \frac{3\gamma}{3 + \gamma} > \frac{3}{2}.$$

5. Estimates on the numerical operators

To prove convergence of the numerical method, our strategy will be to adapt the continuous existence theory to the numerical setting. We will succeed with this by controlling the weak error of the numerical operators relative to their continuous counterparts. The purpose of this section, is to derive the needed error estimates.

For notational convenience, let us define

$$D_t^h f = \frac{f(t) - f(t - \Delta t)}{\Delta t},$$

and observe that this satisfies

$$D_t^h \varrho_h(t) = \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \quad t \in [t^k, t^{k+1}).$$

5.1. **The convective discretizations.** We begin by deriving identities for the distributional error of the numerical convection operators.

Lemma 5.1. *Fix two functions* $\phi \in C_0^\infty(\Omega)$, $v \in [C_0^\infty(\Omega)]^d$. *The numerical transport operators in* (3.5) *and* (3.6) *satisfies the following identities*

$$\sum_\Gamma \int_\Gamma \mathrm{Up}(\varrho u) \; \left[\!\left[\Pi_h^Q \phi \right]\!\right]_\Gamma \; dS(x) = \int_\Omega \varrho_h \widetilde{u}_h \nabla \phi \; dx + P_1(\phi), \qquad (5.1)$$

$$\sum_\Gamma \int_\Gamma \mathrm{Up}(\varrho u \otimes \widehat{u}) \; \left[\!\left[\widehat{\Pi_h^V v} \right]\!\right]_\Gamma \; dS(x) = \int_\Omega \varrho_h \widetilde{u}_h \otimes \widehat{u}_h : \nabla v \; dx + \sum_{i=2}^4 P_i(v), \qquad (5.2)$$

where the error functionals P_i, $i = 1, \dots, 4$, *are*

$$P_1(\phi) = \sum_E \int_{\partial E} [\![\varrho_h]\!]_{\partial E} \, (\widetilde{u}_h \cdot \nu)^- (\Pi_h^Q \phi - \phi) \; dS(x),$$

$$P_2(v) = \sum_E \int_{\partial E} [\![\varrho_h]\!]_{\partial E} \, (\widetilde{u}_h \cdot \nu)^- \widehat{u}_h \left(\widehat{\Pi_h^V v} - v \right) \; dS(x),$$

$$P_3(v) = \sum_E \int_{\partial E} \mathrm{Up}_-(\varrho u) \, [\![\widehat{u}_h]\!]_{\partial E} \left(\widehat{\Pi_h^V v} - v \right) \; dS(x),$$

$$P_4(v) = \sum_E \int_E \varrho_h \, \mathrm{div} \, u_h \widehat{u}_h \left(\Pi_h^V v - v \right) \; dx.$$

Proof. Using the continuity of $\mathrm{Up}(\varrho u)$ and ϕ across edges, we calculate

$$\sum_\Gamma \int_\Gamma \mathrm{Up}(\varrho u) \; \left[\!\left[\Pi_h^Q \phi \right]\!\right]_\Gamma \; dS(x)$$

$$= -\sum_E \int_{\partial E} \mathrm{Up}(\varrho u) \; \Pi_h^Q \phi \; dS(x)$$

$$= -\sum_E \int_{\partial E} \mathrm{Up}(\varrho u) \left(\Pi_h^Q \phi - \phi \right) \; dS(x)$$

$$= -\sum_E \int_{\partial E} \left(\varrho_+(\widetilde{u}_h \cdot \nu)^+ + \varrho_-(\widetilde{u}_h \cdot \nu)^- \right) \left(\Pi_h^Q \phi - \phi \right) \; dS(x),$$

where the last identity is the definition of $\mathrm{Up}(\varrho u)$. Next, we add and subtract to deduce

$$
\sum_\Gamma \int_\Gamma \mathrm{Up}(\varrho u) \; \left[\!\!\left[\Pi_h^Q \phi \right]\!\!\right]_\Gamma \; dS(x)
$$

$$
= -\sum_E \int_{\partial E} \varrho_+ (\widetilde{u}_h \cdot \nu) \left(\Pi_h^Q \phi - \phi \right)
$$

$$
+ (\varrho_- - \varrho_+)(\widetilde{u}_h \cdot \nu)^- \left(\Pi_h^Q \phi - \phi \right) \; dS(x)
$$

$$
= -\sum_E \int_E \mathrm{div} \left(\varrho_h \widetilde{u}_h \left(\Pi_h^Q \phi - \phi \right) \right) \; dS(x) + P_1(\phi)
$$

$$
= \int_\Omega \varrho_h \widetilde{u}_h \nabla \phi \; dx - \sum_E \int_E \varrho_h \, \mathrm{div}\, u_h \left(\Pi_h^Q \phi - \phi \right) \; dx + P_1(\phi).
$$

We conclude (5.1) by recalling that $\mathrm{div}_h\, u_h$ is constant on each element and hence the second term is zero.

To derive (5.2), we apply the definition (3.3) of $\mathrm{Up}(\varrho u \otimes \widehat{u})$ and add to subtract to obtain

$$
\sum_\Gamma \int_\Gamma \mathrm{Up}(\varrho u \otimes \widehat{u}) \left[\!\!\left[\widehat{\Pi_h^V v} \right]\!\!\right]_\Gamma \; dS(x)
$$

$$
= -\sum_E \int_{\partial E} \mathrm{Up}(\varrho u \otimes \widehat{u}) \left(\widehat{\Pi_h^V v} - v \right) \; dS(x)
$$

$$
= -\sum_E \int_{\partial E} \left(\mathrm{Up}_+(\varrho u)\widehat{u}_+ + \mathrm{Up}_-(\varrho u)\widehat{u}_- \right) \left(\widehat{\Pi_h^V v} - v \right) \; dS(x)
$$

$$
= -\sum_E \int_{\partial E} \left(\mathrm{Up}(\varrho u)\widehat{u}_+ + \mathrm{Up}_-(\varrho u)(\widehat{u}_- - \widehat{u}_+) \right) \left(\widehat{\Pi_h^V v} - v \right) \; dS(x)
$$

$$
= -\sum_E \int_{\partial E} \mathrm{Up}(\varrho u)\widehat{u}_+ \left(\widehat{\Pi_h^V v} - v \right) \; dS(x) + P_3(v).
$$

To proceed, we apply the definition of $\mathrm{Up}(\varrho u)$ (3.2) and add and subtract to obtain

$$
\sum_\Gamma \int_\Gamma \mathrm{Up}(\varrho u \otimes \widehat{u}) \left[\!\!\left[\widehat{\Pi_h^V v} \right]\!\!\right]_\Gamma \; dS(x)
$$

$$
= -\sum_E \int_{\partial E} \left(\varrho_+ (\widetilde{u}_h \cdot \nu)^+ + \varrho_- (\widetilde{u}_h \cdot \nu)^- \right) \widehat{u}_+ \left(\widehat{\Pi_h^V v} - v \right) \; dS(x) + P_3(v)
$$

$$
= -\sum_E \int_{\partial E} \varrho_+ (\widetilde{u}_h \cdot \nu)\widehat{u}_+ \left(\widehat{\Pi_h^V v} - v \right)
$$

$$
+ (\varrho_- - \varrho_+)(\widetilde{u}_h \cdot \nu)^- \widehat{u}_+ \left(\widehat{\Pi_h^V v} - v \right) \; dS(x) + P_3(v)
$$

$$
= -\sum_E \int_{\partial E} \varrho_+ (\widetilde{u}_h \cdot \nu)\widehat{u}_+ \left(\widehat{\Pi_h^V v} - v \right) \; dS(x) + P_2(v) + P_3(v)
$$

We can then apply the divergence theorem to the first term to obtain

$$\sum_\Gamma \int_\Gamma \mathrm{Up}(\varrho u \otimes \widehat{u}) \left[\!\!\left[\widehat{\Pi_h^V v} \right]\!\!\right]_\Gamma \, dS(x)$$

$$= \int_\Omega \varrho_h \widetilde{u}_h \otimes \widehat{u}_h : \nabla v \, dx - \sum_E \int_E \varrho_h \, \mathrm{div} \, u_h \widehat{u}_h \left(\Pi_h^V v - v \right) \, dx$$

$$+ P_2(v) + P_3(v)$$

$$= \int_\Omega \varrho_h \widetilde{u}_h \otimes \widehat{u}_h : \nabla v \, dx + P_4(v) + P_2(v) + P_3(v),$$

which is (5.2).   □

Identities like (5.1) and (5.2) can be derived for any numerical method. The difficult part is to control the error terms, in our case is given by P_i, $i = 1, \ldots, 4$. The following proposition provides sufficient control on the error terms. Note in particular the integrability required of the test-functions.

Proposition 5.2. *Let* (ϱ_h^k, u_h^k), $k = 1, \ldots, M$ *be the numerical solution obtained using the scheme* (3.5) *-* (3.6). *Let* (ϱ_h, u_h) *be the piecewise constant extension of* (ϱ_h^k, u_h^k), $k = 1, \ldots, M$ *in time to all of* $[0, M\Delta t]$ *(i.e* (3.7)*). Then, the* P_i, $i = 1, \ldots, 4$ *in Proposition* 5.1 *are also piecewise constant in time and there exists a constant* $C > 0$, *independent of* h *and* Δt, *such that*

$$\int_0^T |P_1(\phi)| \, dt \le h^{\frac{1}{2} - \min\{3\frac{4-\gamma}{4\gamma}, 0\}} C \|\nabla \phi\|_{L^4(0,T;L^{\frac{12}{5}}(\Omega)}, \tag{5.3}$$

$$\int_0^T |P_2(v)| \, dt \le h^{\frac{1}{2} - \min\{3\frac{4-\gamma}{4\gamma}, 0\}} C \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))}, \tag{5.4}$$

$$\int_0^T |P_3(v)| \, dt \le h^{\frac{1}{2}} C \|\nabla v\|_{L^4(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\Omega))}, \tag{5.5}$$

$$\int_0^T |P_4(v)| \, dt \le h^{\frac{2(\gamma-9/4)}{\gamma}} C \|\nabla v\|_{L^\infty(0,T;L^\gamma(\Omega))}. \tag{5.6}$$

In particular, for $\gamma > 3$, *we have that*

$$\int_0^T |P_1(\phi)| \, dt \le h^{\frac{1}{4}} C \|\nabla \phi\|_{L^4(0,T;L^{\frac{12}{5}}(\Omega)},$$

$$\int_0^T |P_2(v)| + |P_3(v)| \, dt \le C h^{\frac{1}{4}} \|\nabla v\|_{L^\infty(0,T;L^\gamma(\Omega))}. \tag{5.7}$$

Proof. We will prove one inequality at the time.

1. Bound on P_1^h: An application of the Cauchy-Schwartz inequality yields

$$
\int_0^T |P_1^h|\, dt := \int_0^T \left| \sum_E \int_{\partial E} [\![\varrho_h]\!]_{\partial E}\, (u_h \cdot \nu)^- (\Pi_h^Q \phi - \phi)\, dS(x) \right|\, dt
$$

$$
\leq \left(\int_0^T \sum_E \int_{\partial E} [\![\varrho_h]\!]_{\partial E}^2\, |u_h \cdot \nu|\, dS(x) dt \right)^{\frac{1}{2}}
\tag{5.8}
$$

$$
\times \left(\int_0^T \sum_E \int_{\partial E} |u_h| (\Pi_h^Q \phi - \phi)^2\, dS(x) dt \right)^{\frac{1}{2}}
$$

$$
:= \sqrt{I_1} \times \sqrt{I_2}.
$$

By setting $B(z) = \frac{1}{2}z^2$ in Lemma 4.1 and integrating in time, we obtain

$$
\begin{aligned}
I_1 &\leq \frac{1}{2} \int_\Omega (\varrho_h)^2\, dx + \int_0^T \int_\Omega (\varrho_h)^2 \operatorname{div}_h u_h\, dx dt \\
&\leq C \|\varrho_h\|_{L^\infty(0,T;L^4(\Omega))}^2 + \sqrt{T} \|\varrho_h\|_{L^\infty(0,T;L^4(\Omega))}^2 \|\operatorname{div}_h u_h\|_{L^2(0,T;L^2(\Omega))} \\
&\leq C \|\varrho_h\|_{L^\infty(0,T;L^4(\Omega))}^2 (1 + \sqrt{T}).
\end{aligned}
\tag{5.9}
$$

By applying the Trace Lemma 2.8 and the Hölder inequality, we deduce

$$
\begin{aligned}
I_2 &\leq h^{-1} \int_0^T \int_\Omega |u_h| (\Pi_h^Q \phi - \phi)^2\, dx dt \\
&\leq h^{-1} C \int_0^T \|u_h\|_{L^6(\Omega)} \\
&\quad \times \left(\|\Pi_h^Q \phi - \phi\|_{L^{\frac{12}{5}}(\Omega)}^2 + h^2 \|\nabla \phi\|_{L^4(0,T;L^{\frac{12}{5}}(\Omega))}^2 \right) dt \\
&\leq h C \|u_h\|_{L^2(0,T;L^6(\Omega))} \|\nabla \phi\|_{L^4(0,T;L^{\frac{12}{5}}(\Omega))}^2,
\end{aligned}
\tag{5.10}
$$

where the last inequality is the error estimate on the interpolation error of Π_h^Q in L^p (see Lemma 2.7).

Consequently, by applying (5.9) and (5.10) in (5.8), we see that

$$
\int_0^T |P_1|\, dt \leq h^{\frac{1}{2}} C \|\varrho_h\|_{L^\infty(0,T;L^4(\Omega))} \|\nabla \phi\|_{L^4(0,T;L^{\frac{12}{5}}(\Omega))}.
\tag{5.11}
$$

From the standard inverse inequality (Lemma 2.9), we have that

$$
\|\varrho_h\|_{L^\infty(0,T;L^4(\Omega))} \leq h^{\max\{-3\frac{4-\gamma}{4\gamma},0\}} C \|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))},
\tag{5.12}
$$

where the last inequality follows from $\gamma > 3$. Applying this in (5.11) finally yields

$$
\int_0^T |P_1|\, dt \leq h^{\frac{1}{2} - \min\{3\frac{4-\gamma}{4\gamma},0\}} C \|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))} \|\nabla \phi\|_{L^4(0,T;L^{\frac{12}{5}}(\Omega))}.
$$

Since Corollary 4.3 provides $\varrho_h \in_{\mathrm{b}} L^\infty(0,T;L^\gamma(\Omega))$, we can conclude (5.3).

2. Bound on P_2^h: An application of the Cauchy-Schwartz inequality yields

$$
\begin{aligned}
\int_0^T |P_2|\ dt := \int_0^T &\left| \sum_E \int_{\partial E} [\![\varrho_h]\!]_{\partial E}\, (u_h \cdot \nu)^- \widehat{u}_h (\Pi_h^V v - v)\ dS(x) \right|\ dt \\
\leq &\left(\int_0^T \sum_E \int_{\partial E} [\![\varrho_h]\!]_{\partial E}^2\, |u_h \cdot \nu|\ dS(x)dt \right)^{\frac{1}{2}} \\
&\times \left(\int_0^T \sum_E \int_{\partial E} |u_h||\widehat{u}_h|(\Pi_h^V v - v)^2\ dS(x)dt \right)^{\frac{1}{2}} \\
=: &\sqrt{I_1} \times \sqrt{I_2}.
\end{aligned}
\tag{5.13}
$$

From (5.9) and (5.12), we have that

$$
\sqrt{I_1} \leq h^{\max\{-3\frac{4-\gamma}{4\gamma},0\}} C \|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))}.
\tag{5.14}
$$

Using the Trace inequality in Lemma 2.8, the Hölder inequality, the stability of Π_h^Q in L^6, and the interpolation error estimate for Π_h^V in L^3 (Lemma 2.7), we deduce

$$
\begin{aligned}
I_2 \leq h^{-1} \int_0^T &\left(\int_\Omega |u_h|^3 \left| \Pi_h^Q u_h \right|^3\ dx \right)^{\frac{1}{3}} \\
&\times \left(\|\Pi_h^V v - v\|_{L^3(\Omega)}^2 + h^2 C \|\nabla v\|_{L^3(\Omega)}^2 \right)\ dt \\
\leq h^{-1} C \int_0^T &\|u_h\|_{L^6(\Omega)} \|u_h\|_{L^6(\Omega)} \\
&\times \left(\|\Pi_h^V v - v\|_{L^3(\Omega)}^2 + h^2 C \|\nabla v\|_{L^3(\Omega)}^2 \right)\ dt \\
\leq hC \|u_h\|_{L^2(0,T;L^6(\Omega))}^2 &\|\nabla v\|_{L^\infty(0,T;L^3(\Omega))}^2,
\end{aligned}
\tag{5.15}
$$

where the term involving u_h is bounded by Corollary 4.3.

Now, setting (5.14) and (5.15) in (5.13) yields

$$
\int_0^T |P_2|\ dt \leq h^{\frac{1}{2} - \min\{3\frac{4-\gamma}{4\gamma},0\}} C \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))},
$$

which concludes our proof of (5.4).

3. *Bound on P_3^h:* An application of the Cauchy-Schwartz inequality yields

$$\int_0^T |P_3(v)| \, dt$$

$$= \int_0^T \left| \sum_E \int_{\partial E} \mathrm{Up}_-(\varrho u) [\![\widehat{u}_h]\!]_{\partial E} \left(\Pi_h^Q \Pi_h^V v - v \right) \, dS(x) \right| \, dt$$

$$\leq \left(2 \sum_\Gamma \int_0^T \int_\Gamma |\mathrm{Up}(\varrho u)| [\![\widehat{u}_h]\!]^2 \, dS(x) dt \right)^{\frac{1}{2}}$$

$$\times \left(\sum_E \int_0^T \int_{\partial E} |\mathrm{Up}(\varrho u)| \left(\Pi_h^Q \Pi_h^V v - v \right)^2 \, dS(x) dt \right)^{\frac{1}{2}}$$

$$:= I_1 \times I_2.$$

From the energy estimate (Proposition 4.2), we have that

$$I_1 = \left(\sum_{k=1}^M \Delta t D_2^k \right)^{\frac{1}{2}} \leq C,$$

and hence it only remains to bound I_2.

Using the definition (3.2) of $\mathrm{Up}(\varrho u)$ and the Hölder inequality

$$I_2^2 = \sum_E \int_0^T \int_{\partial E} |\mathrm{Up}(\varrho u)| \left(\Pi_h^Q \Pi_h^V v - v \right)^2 \, dS(x) dt$$

$$\leq \sum_E \int_0^T \int_{\partial E} (\varrho_+ + \varrho_-) |u_h \cdot \nu| \left([\Pi_h^Q \Pi_h^V v]_+ - v \right)^2 \, dS(x) dt$$

$$\leq \sum_E \int_0^T \left(\|\varrho_+ u_h\|_{L^{\frac{6\gamma}{6+\gamma}}(\partial E)} + \|\varrho_- u_h\|_{L^{\frac{6\gamma}{6+\gamma}}(\partial E)} \right)$$

$$\times \left\| [\Pi_h^Q \Pi_h^V v] - v \right\|_{L^{\frac{6\gamma}{5\gamma-6}}(\partial E)}^2 \, dt.$$

To proceed, we apply the trace Lemma 2.8 followed by the inverse estimate (Lemma 2.9)

$$I_2^2 \leq 2h^{-1} C \|\varrho_h u_h\|_{L^2(0,T;L^{\frac{6\gamma}{6+\gamma}}(\Omega))}$$

$$\times \left(\left\| \Pi_h^Q \Pi_h^V v - v \right\|_{L^4(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\Omega))}^2 + h^2 \|\nabla v\|_{L^4(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\Omega))}^2 \right)$$

$$\leq 2h^{-1} C \|\varrho_h u_h\|_{L^2(0,T;L^{\frac{6\gamma}{6+\gamma}}(\Omega))}$$

$$\times \left(\left\| \Pi_h^Q \Pi_h^V v - \Pi_h^V v \right\|_{L^4(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\Omega))}^2 \right.$$

$$\left. + \left\| \Pi_h^V v - v \right\|_{L^4(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\Omega))}^2 + h^2 \|\nabla v\|_{L^4(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\Omega))}^2 \right)$$

$$\leq hC4 \|\varrho_h u_h\|_{L^2(0,T;L^{\frac{6\gamma}{6+\gamma}}(\Omega))} \|\nabla v\|_{L^4(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\Omega))}^2,$$

where we in the last inequality have used both the interpolation error of Π_h^Q and Π_h^V (see Lemma 2.7). In particular, we have used the following estimate

$$\left\| \Pi_h^Q \Pi_h^V v - \Pi_h^V v \right\|^2_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega)} \le h^2 C \sum_E \|\nabla \Pi_h^V v\|^2_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega)}$$

$$\le h^2 C \|\nabla v\|^2_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega)}.$$

Now, from Corollary 4.3 we have that $\|\varrho_h u_h\|_{L^2(0,T;L^{\frac{6\gamma}{6+\gamma}}(\Omega))} \le C$, and hence we can actually conclude that

$$\int_0^T |P_3(v)| \, dt \le h^{\frac{1}{2}} C \|\nabla v\|_{L^4(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\Omega))},$$

which is (5.5).

4. Bound on P_4^h: By direct calculation using the Hölder inequality

$$\int_0^T |P_4(v)| \, dt$$

$$\le \| \operatorname{div}_h u_h \|_{L^2(0,T;L^2(\Omega))} \left(\int_0^T \int_\Omega |\varrho_h \widehat{u}_h|^2 \left(\Pi_h^V v - v \right)^2 \, dx dt \right)^{\frac{1}{2}}$$

$$\le h C \|\varrho_h \widehat{u}_h\|_{L^2(0,T;L^{\frac{2\gamma}{\gamma-2}}(\Omega))} \|\nabla v\|_{L^\infty(0,T;L^\gamma(\Omega))},$$

where we have used that $\operatorname{div}_h u_h \in_{\mathrm{b}} L^2(0,T;L^2(\Omega))$ and the interpolation error Lemma 2.7. Now, we apply the inverse estimate in Lemma 2.9 to obtain

$$h \|\varrho_h \widehat{u}_h\|_{L^2(0,T;L^{\frac{2\gamma}{\gamma-2}}(\Omega))} \le h^{\frac{2(\gamma-9/4)}{\gamma}} C \|\varrho_h \widehat{u}_h\|_{L^2(0,T;L^{\frac{6\gamma}{3+\gamma}}(\Omega))}.$$

We conclude (5.6) by recalling that the last term is bounded by Corollary 4.3. □

5.2. **The material momentum transport operator.** In our proof of convergence we will need an identity like (5.2) for the discretization of both terms in the discrete material transport operator $((\varrho u)_t + \operatorname{div}(\varrho u \otimes u))$ combined. To obtain the desired estimates, we will rely on the following weak time-continuity result:

Lemma 5.3. *Let (ϱ_h, u_h) satisfy the energy estimate (4.1). Then,*

$$\|\varrho_h u_h - (\varrho_h u_h)(-\Delta t)\|_{L^{\frac{6}{5}}(\Delta t, T; L^{\frac{3}{2}}(\Omega))} \le (\Delta t)^{\frac{1}{\gamma}} C$$

As a consequence,

$$\|D_t(\varrho_h u_h)\|_{L^{\frac{6}{5}}(\Delta t, T; L^{\frac{3}{2}}(\Omega))} \le (\Delta t)^{\frac{1-\gamma}{\gamma}} C.$$

Proof. From the energy estimate (4.1), we have that

$$\sum_k \int_\Omega P''(\varrho_\ddagger)[\varrho_h^k - \varrho_h^{k-1}] \, dx \le C,$$

where $P''(\varrho_{\ddagger})$ is determined as the remainder in a Taylor expansion and the mean value theorem. In particular, a simple calculation gives

$$\int_\Omega P''(\varrho_{\ddagger})[\varrho_h^k - \varrho_h^{k-1}]^2 \, dx \geq \nu(\gamma) \int_\Omega [\varrho_h^k - \varrho_h^{k-1}]^\gamma \, dx,$$

where $\nu(\gamma)$ only depend on γ. Hence,

$$\sum_k \|\varrho_h^k - \varrho_h^{k-1}\|_{L^\gamma(\Omega)}^\gamma \leq C. \tag{5.16}$$

By adding and subtracting, we have that

$$\|\varrho_h^k \widehat{u}_h^k - \varrho_h^{k-1} \widehat{u}_h^{k-1}\|_{L^{\frac{3}{2}}(\Omega)}$$
$$= \|\varrho_h^{k-1}[\widehat{u}_h^k - \widehat{u}_h^{k-1}] + \widehat{u}_h^k[\varrho_h^k - \varrho_h^{k-1}]\|_{L^{\frac{3}{2}}(\Omega)}$$
$$\leq \|\varrho_h^{k-1}\|_{L^3(\Omega)}^{\frac{1}{2}} \left\| \sqrt{\varrho_h^{k-1}}[\widehat{u}_h^k - \widehat{u}_h^{k-1}] \right\|_{L^2(\Omega)}$$
$$C\|\widehat{u}_h^k\|_{L^6(\Omega)}\|\varrho_h^k - \varrho_h^{k-1}\|_{L^\gamma(\Omega)}$$

We then integrate in time, apply several applications of the Hölder inequality, and utilize (5.16) to deduce

$$\sum_k \Delta t \|\varrho_h^k \widehat{u}_h^k - \varrho_h^{k-1} \widehat{u}_h^{k-1}\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{6}{5}}$$

$$\leq \Delta t \sum_k \|\varrho_h^{k-1}\|_{L^3(\Omega)}^{\frac{6}{10}} \left\| \sqrt{\varrho_h^{k-1}}[\widehat{u}_h^k - \widehat{u}_h^{k-1}] \right\|_{L^2(\Omega)}^{\frac{6}{5}}$$
$$+ C\Delta t \sum_k \|\widehat{u}_h^k\|_{L^6(\Omega)}^{\frac{6}{5}} \|\varrho_h^k - \varrho_h^{k-1}\|_{L^\gamma(\Omega)}^{\frac{6}{5}}$$

$$\leq (\Delta t)^{\frac{3}{5}} \|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))}^{\frac{2}{5}} \left(\sum_k \Delta t \right)^{\frac{2}{5}}$$

$$\times \left(\sum_k \left\| \sqrt{\varrho_h^{k-1}}[\widehat{u}_h^k - \widehat{u}_h^{k-1}] \right\|_{L^2(\Omega)}^2 \right)^{\frac{3}{5}}$$

$$+ \Delta t C \left(\sum_k \|u_h^k\|_{L^6(\Omega)}^2 \right)^{\frac{3}{5}} \left(\sum_k \|\varrho_h^k - \varrho_h^{k-1}\|_{L^\gamma(\Omega)}^3 \right)^{\frac{2}{5}}$$

$$\leq (\Delta t)^{\frac{3}{5}} C T^{\frac{2}{5}} + (\Delta t)^{\frac{6}{5\gamma}} C \left(\sum_k \Delta t \right)^{\frac{2(\gamma-3)}{5\gamma}} \left(\sum_k \|\varrho_h^k - \varrho_h^{k-1}\|_{L^\gamma(\Omega)}^\gamma \right)^{\frac{6}{5\gamma}}$$

$$\leq C \left((\Delta t)^{\frac{3}{5}} + (\Delta t)^{\frac{6}{5\gamma}} \right).$$

This concludes the proof. $\qquad \square$

Using the previous lemma, we are now ready to prove the following result and error bound.

Lemma 5.4. *Let (ϱ_h, u_h) be the numerical solution obtained through Definition [3.1](#) and ([3.7](#)). Then, for all $v \in L^\infty(0, T; W^{1,6}(\Omega))$,*

$$\int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h) \Pi_h^V v \; dx + \sum_E \int_0^T \int_{\partial E} \mathrm{Up}(\varrho u \otimes \widehat{u}) \widehat{\Pi_h^V v_+} \; dS(x) dt$$

$$= \int_0^T D_t^h(\varrho_h \widehat{u}_h) v - \varrho_h \widetilde{u}_h \otimes \widehat{u}_h : \nabla v \; dx dt + F(v),$$

where the reminder is bounded as

$$|F(v)| \leq h^{\frac{1}{\gamma}} C \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))}, \tag{5.17}$$

with constant independent of h and Δt.

Proof. Using ([5.2](#)), we have the identity

$$\int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h) \Pi_h^V v \; dx + \sum_E \int_0^T \int_{\partial E} \mathrm{Up}(\varrho u \otimes \widehat{u}) \widehat{\Pi_h^V v_+} \; dS(x) dt$$

$$= \int_0^T D_t^h(\varrho_h \widehat{u}_h) \Pi_h^V v \; - \varrho_h u_h \otimes \widehat{u}_h : \nabla v \; dx dt + \sum_{i=2}^4 \int_0^T P_i(v) \; dt$$

$$= \int_0^T D_t^h(\varrho_h \widehat{u}_h) v - \varrho_h u_h \otimes \widehat{u}_h : \nabla v \; dx dt$$

$$+ \sum_{i=2}^4 \int_0^T P_i(v) \; dt + \int_0^T D_t^h(\varrho_h \widehat{u}_h) \left(\Pi_h^V v - v\right) \; dx dt.$$

We thus define $F(v)$ by

$$|F(v)| := \left| \sum_{i=2}^4 \int_0^T P_i \; dt + \int_0^T D_t^h(\varrho_h \widehat{u}_h) \left(\Pi_h^V v - v\right) \; dx dt \right|$$

$$\leq \sum_{i=2}^4 \int_0^T |P_i| \; dt + h \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))} \int_0^T \left\| D_t^h(\varrho_h \widehat{u}_h) \right\|_{L^{\frac{3}{2}}(\Omega)} \; dt$$

$$\leq \sum_{i=2}^4 \int_0^T |P_i| \; dt + h(\Delta t)^{-1} (\Delta t)^{\frac{1}{\gamma}} \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))}$$

$$= \sum_{i=2}^4 \int_0^T |P_i| \; dt + h^{\frac{1}{\gamma}} \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))},$$

where we have used Lemma [5.3](#) and $h = a\Delta t$.

Finally, by applying ([5.7](#)) to bound the P_i terms, we obtain the desired ([5.17](#)). $\square$

5.3. The artificial stabilization terms.

To prove convergence of the numerical method we will need to prove that the artificial stabilization terms converges to zero. Moreover, we will need that these terms are small in a suitable Lebesgue space.

Lemma 5.5. *If (ϱ_h, u_h) is the numerical solution obtained by Definition 3.1 and (3.7), the artificial stabilization terms satisfies*

$$\left| h^{1-\epsilon} \sum_\Gamma \int_0^T \int_\Gamma [\![\varrho_h]\!]_\Gamma \left[\!\!\left[\Pi_h^Q \phi \right]\!\!\right]_\Gamma \, dS(x)dt \right|$$
$$\leq h^{\frac{11-6\epsilon}{12}} C \|\nabla\phi\|_{L^2(0,T;L^2(\Omega))}, \tag{5.18}$$

$$\left| h^{1-\epsilon} \sum_\Gamma \int_0^T \int_\Gamma \left(\frac{\widehat{u}_- + \widehat{u}_+}{2} \right) [\![\varrho_h]\!]_\Gamma \left[\!\!\left[\widehat{\Pi_h^V v} \right]\!\!\right]_\Gamma \, dS(x)dt \right|$$
$$\leq h^{\frac{13-6\epsilon}{12}} C \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))}, \tag{5.19}$$

for all sufficiently smooth ϕ and v.

Proof. We will begin by proving (5.18). By direct calculation, using the Cauchy-Schwartz inequality, we deduce

$$h^{1-\epsilon} \sum_\Gamma \int_0^T \int_\Gamma [\![\varrho_h]\!]_\Gamma \left[\!\!\left[\Pi_h^Q \phi \right]\!\!\right]_\Gamma \, dS(x)dt$$

$$= -h^{1-\epsilon} \sum_E \int_0^T \int_{\partial E} [\![\varrho_h]\!]_\Gamma \left(\Pi_h^Q \phi \right) \, dS(x)dt$$

$$= -h^{1-\epsilon} \sum_E \int_0^T \int_{\partial E} [\![\varrho_h]\!]_\Gamma \left(\Pi_h^Q \phi - \phi \right) \, dS(x)dt$$

$$\leq h^{1-\epsilon} \left(\sum_\Gamma \int_0^T \int_\Gamma [\![\varrho_h]\!]_\Gamma^2 \, dS(x)dt \right)^{\frac{1}{2}} \tag{5.20}$$

$$\times \left(\sum_E \int_{\partial E} (\Pi_h^Q \phi - \phi)^2 \, dS(x)dt \right)^{\frac{1}{2}}$$

$$\leq h^{\frac{1}{2}} h^{1-\epsilon} \left(\sum_\Gamma \int_0^T \int_\Gamma [\![\varrho_h]\!]_\Gamma^2 \, dS(x)dt \right)^{\frac{1}{2}} \|\nabla\phi\|_{L^2(0,T;L^2(\Omega))},$$

where the last inequality comes from the trace theorem (Lemma 2.8) and the interpolation error estimate for Π_h^Q. Now, to bound the jump term, we apply Lemma 4.1 with $B(\varrho) = \varrho^2$ to obtain

$$h^{1-\epsilon} \sum_\Gamma \int_0^T \int_\Gamma [\![\varrho_h]\!]_\Gamma^2 \, dS(x)dt$$

$$\leq \int_\Omega \varrho_0^2 \, dx + \int_0^T \int_\Omega \varrho_h^2 \operatorname{div} u_h \, dxdt$$
$$\leq \|\varrho_0\|_{L^2(\Omega)}^2 + C\|\varrho_h\|_{L^\infty(0,T;L^4(\Omega))}^2 \|\operatorname{div} u_h\|_{L^2(0,T;L^2(\Omega))} \tag{5.21}$$
$$\leq \|\varrho_0\|_{L^2(\Omega)}^2 + h^{\max\{-\frac{3(\gamma-4)}{2\gamma},0\}} C\|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))}^2$$
$$\leq \|\varrho_0\|_{L^2(\Omega)}^2 + h^{-\frac{1}{6}} C\|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))}^2 \leq h^{-\frac{1}{6}} C,$$

where we have applied the inverse estimate (Lemma 2.9), the energy bound, and that $\gamma > 3$. Finally, we apply (5.21) in (5.20) to discover

$$h^{1-\epsilon} \sum_{\Gamma} \int_0^T \int_{\Gamma} [\![\varrho_h]\!]_{\Gamma} \left[\!\left[\Pi_h^Q \phi\right]\!\right]_{\Gamma} dS(x) dt$$

$$\leq h^{\frac{1}{2}} h^{\frac{1-\epsilon}{2}} \left(h^{1-\epsilon} \sum_{\Gamma} \int_0^T \int_{\Gamma} [\![\varrho_h]\!]_{\Gamma}^2 dS(x) dt \right)^{\frac{1}{2}} \|\nabla \phi\|_{L^2(0,T;L^2(\Omega))}$$

$$\leq h^{1-\frac{\epsilon}{2}-\frac{1}{12}} C \|\nabla \phi\|_{L^2(0,T;L^2(\Omega))},$$

which is (5.18).

Next, we prove (5.19). Since $\int_{\Gamma} \left[\!\left[\Pi_h^V v\right]\!\right]_{\Gamma} dS(x) = 0$, we have the identity

$$- h^{1-\epsilon} \sum_{\Gamma} \int_0^T \int_{\Gamma} \left(\frac{\widehat{u}_- + \widehat{u}_+}{2} \right) [\![\varrho_h]\!]_{\Gamma} \left[\!\left[\widehat{\Pi_h^V v}\right]\!\right]_{\Gamma} dS(x) dt$$

$$= h^{1-\epsilon} \sum_{E} \int_0^T \int_{\partial E} \left(\frac{\widehat{u}_- + \widehat{u}_+}{2} \right) [\![\varrho_h]\!]_{\Gamma} \left(\widehat{\Pi_h^V v} - \Pi_h^V v \right) dS(x).$$

An application of the Hölder inequality and several applications of the trace theorem (Lemma 2.8) allow us to deduce

$$\left| h^{1-\epsilon} \sum_{\Gamma} \int_0^T \int_{\Gamma} \left(\frac{\widehat{u}_- + \widehat{u}_+}{2} \right) [\![\varrho_h]\!]_{\Gamma} \left[\!\left[\widehat{\Pi_h^V v}\right]\!\right]_{\Gamma} dS(x) dt \right|$$

$$\leq h^{1-\epsilon} \left(\sum_{\Gamma} \int_0^T \int_{\Gamma} [\![\varrho_h]\!]_{\Gamma}^2 dS(x) \right)^{\frac{1}{2}}$$

$$\times \sup_t \left(\sum_{E} \int_{\partial E} \left| \widehat{\Pi_h^V v} - \Pi_h^V v \right|^3 dS(x) \right)^{\frac{1}{3}} \tag{5.22}$$

$$\times \left(\int_0^T \left(\sum_{E} \int_{\partial E} \left| \frac{\widehat{u}_- + \widehat{u}_+}{2} \right|^6 dS(x) \right)^{\frac{2}{6}} dt \right)^{\frac{1}{2}}$$

$$\leq h^{1-\epsilon} C \left(\sum_{\Gamma} \int_0^T \int_{\Gamma} [\![\varrho_h]\!]_{\Gamma}^2 dS(x) \right)^{\frac{1}{2}}$$

$$\times \left(h^{-\frac{1}{3}} \|\widehat{\Pi_h^V v} - \Pi_h^V v\|_{L^\infty(0,T;L^3(\Omega))} + h^{\frac{2}{3}} \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))} \right)$$

$$\times C h^{-\frac{1}{6}} \|\widehat{u}_h\|_{L^2(0,T;L^6(\Omega))}.$$

Next, we apply (5.21) and the fact that $\gamma > 3$ to (5.22) to get

$$\left| h^{1-\epsilon} \sum_{\Gamma} \int_0^T \int_{\Gamma} \left(\frac{\widehat{u}_- + \widehat{u}_+}{2} \right) [\![\varrho_h]\!]_{\Gamma} \left[\!\left[\widehat{\Pi_h^V v}\right]\!\right]_{\Gamma} dS(x) dt \right|$$

$$\leq h^{\frac{1-\epsilon}{2}+\frac{2}{3}-\frac{1}{6}-\frac{1}{12}} C \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))}$$

$$= h^{\frac{13-6\epsilon}{12}} C \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))},$$

which is (5.19). $\qquad \square$

5.4. **Weak time control.** We conclude this section by proving h-independent bounds on the discrete time derivates in the numerical method.

Lemma 5.6. *Under the conditions of the previous lemma,*

$$D_t \varrho_h \in_b L^{\frac{4}{3}}(0, T; W^{-1, \frac{3}{2}}(\Omega)), \tag{5.23}$$

$$D_t (\varrho_h u_h) \in_b L^1(0, T; W^{-1, \frac{3}{2}}(\Omega)). \tag{5.24}$$

Proof. Let $\phi \in C_0^\infty([0, T) \times \Omega)$ be arbitrary and set $\Pi_h^Q \phi$ as test-function in the continuity scheme (3.5) to obtain

$$
\begin{aligned}
&\int_0^T \int_\Omega D_t^h(\varrho_h) \phi \ dxdt \\
&= \sum_\Gamma \int_0^T \int_\Gamma \mathrm{Up}(\varrho u) \left[\!\left[\Pi_h^Q \phi \right]\!\right]_\Gamma + h^{1-\epsilon} \left[\!\left[\varrho_h \right]\!\right]_\Gamma \left[\!\left[\Pi_h^Q \phi \right]\!\right]_\Gamma \ dS(x)dt \\
&= \int_0^T \int_\Omega \varrho_h \tilde{u}_h \nabla \phi \ dxdt + P_1(\phi) \\
&\qquad + h^{1-\epsilon} \sum_\Gamma \int_0^T \int_\Gamma \left[\!\left[\varrho_h \right]\!\right]_\Gamma \left[\!\left[\Pi_h^Q \phi \right]\!\right]_\Gamma \ dS(x)dt,
\end{aligned}
\tag{5.25}
$$

where the last equality is (5.1). From Proposition 5.2, have that

$$|P_1(v)| \le h^{\frac{1}{4}} C \|\nabla \phi\|_{L^4(0, T; L^{\frac{12}{5}}(\Omega))}. \tag{5.26}$$

Moreover, from Lemma 5.5, we have that

$$h^{1-\epsilon} \sum_\Gamma \int_0^T \int_\Gamma \left[\!\left[\varrho_h \right]\!\right]_\Gamma \left[\!\left[\Pi_h^Q \phi \right]\!\right]_\Gamma \ dS(x)dt \le h^{\frac{11-6\epsilon}{12}} \|\nabla \phi\|_{L^2(0, T; L^2(\Omega))}. \tag{5.27}$$

Hence, by applying (5.26), (5.27), and the Hölder inequality to (5.25), we conclude

$$
\begin{aligned}
&\left| \int_0^T \int_\Omega D_t^h(\varrho_h) \phi \ dxdt \right| \\
&\qquad \le \|\varrho_h u_h\|_{L^2(0, T; L^3(\Omega))} \|\nabla \phi\|_{L^2(0, T; L^{\frac{3}{2}}(\Omega))} \\
&\qquad\qquad + h^{\frac{1}{4}} C \|\nabla \phi\|_{L^4(0, T; L^{\frac{12}{5}}(\Omega))} + h^{\frac{11-6\epsilon}{12}} \|\nabla \phi\|_{L^2(0, T; L^2(\Omega))}.
\end{aligned}
$$

We conclude (5.23) by recalling that ϕ was chosen arbitrarily.

Next, let $v \in [C_0^\infty([0, T) \times \Omega)]^d$ be an arbitrary vector and set $v_h = \Pi_h^V v$ as test-function in the momentum scheme (3.6) to obtain

$$
\begin{aligned}
&\int_0^T \int_\Omega D_t^h(\varrho_h u_h) v_h \ dxdt + \sum_E \int_0^T \int_{\partial E} \mathrm{Up}(\varrho u \otimes u) \widehat{v}_h dS(x)dt \\
&\qquad + \int_0^T \int_\Omega \nabla_h u_h \nabla_h v_h - p(\varrho_h) \operatorname{div}_h v_h \ dxdt \\
&\qquad\qquad + h^{1-\epsilon} \sum_E \int_{\partial E} \left(\frac{\widehat{u}_- + \widehat{u}_+}{2} \right) \left[\!\left[\varrho_h \right]\!\right]_{\partial E} \widehat{v}_h \ dS(x) = 0.
\end{aligned}
$$

Now, to the first two terms we apply the identity in Lemma 5.4 and reorder terms to obtain

$$
\begin{aligned}
\int_0^T &\int_\Omega D_t^h(\varrho_h u_h) v \; dxdt \\
&= \int_0^T \int_\Omega \varrho_h \tilde{u}_h \otimes \widehat{u}_h : \nabla v \; dxdt + F(v) \\
&\quad + \int_0^T \int_\Omega \nabla_h u_h \nabla_h v_h - p(\varrho_h) \operatorname{div}_h v_h \; dxdt \\
&\quad + h^{1-\epsilon} \sum_E \int_0^T \int_{\partial E} \left(\frac{\widehat{u}_- + \widehat{u}_+}{2} \right) [\![\varrho_h]\!]_{\partial E} \, \widehat{v}_h \; dS(x)dt.
\end{aligned}
\tag{5.28}
$$

From Lemma 5.4, we also have the bound

$$
|F(v)| \le h^{\frac{1}{4}} C \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))}.
\tag{5.29}
$$

Next, we invoke Lemma 5.5 to obtain the bound

$$
\begin{aligned}
\left| h^{1-\epsilon} \sum_E \int_0^T \int_{\partial E} \left(\frac{\widehat{u}_- + \widehat{u}_+}{2} \right) [\![\varrho_h]\!]_{\partial E} \, \widehat{v}_h \; dS(x)dt \right| \\
\le h^{\frac{13-6\epsilon}{12}} C \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))}.
\end{aligned}
\tag{5.30}
$$

By applying (5.29) and (5.30), together with the Hölder inequality in (5.28), we deduce

$$
\begin{aligned}
\left| \int_0^T \int_\Omega D_t^h(\varrho_h u_h) v \; dxdt \right| &\le \| \varrho_h |u_h|^2 \|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} \|\nabla \phi\|_{L^2(0,T;L^3(\Omega))} \\
&\quad + h^{\frac{1}{4}} C \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))} \\
&\quad + \|p(\varrho_h)\|_{L^\infty(0,T;L^\gamma(\Omega))} \|\operatorname{div} v\|_{L^1(0,T;L^{\frac{3}{2}}(\Omega))} \\
&\quad + h^{\frac{13-6\epsilon}{12}} C \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))}.
\end{aligned}
$$

From this we easily conclude (5.24).

$\square$

6. Higher integrability on the density

In order to pass to the limit in the pressure, we will need higher integrability on the density. That is, from the energy estimate (Corollary 4.3) we only know that

$$
p(\varrho_h) \in_b L^\infty(0,T;L^1(\Omega)),
$$

and L^1 is not weakly closed. Hence, it is not clear that $p(\varrho_h)$ converges to an integrable function. To prove higher integrability, we shall make use of the operator $\mathcal{A}^i[\cdot] : L^p(\Omega) \mapsto W^{1,p}(\Omega)$

$$
\mathcal{A}^i[q] = \frac{d}{dx_i} \Delta^{-1} \left[\Pi^E q \right] \Big|_\Omega, \quad i = 1, \ldots, 3.
$$

Here, Π_E is the extension by zero operator to all of $\mathbb{R}^n$ and $|_\Omega$ denotes the restriction to Ω. The Δ^{-1} operator is the usual convolution with the

Newtonian potential

$$\Delta^{-1}\phi = -\lambda \int_{\mathbb{R}^3} \frac{\phi(y)}{|x-y|}\ dy, \quad \lambda > 0.$$

Using $\mathcal{A}_i$, we define two operators $\mathcal{A}^\nabla$ and $\mathcal{A}^{\mathrm{div}}$ acting on scalars and vectors, respectively. For a scalar q and a vector $v = [v_1, v_2, v_3]^T$ they are defined

$$\mathcal{A}^\nabla[q] = \begin{pmatrix} \mathcal{A}^1[q] \\ \mathcal{A}^2[q] \\ \mathcal{A}^3[q] \end{pmatrix}, \qquad \mathcal{A}^{\mathrm{div}}[v] = \mathcal{A}^1[v_1] + \mathcal{A}^2[v_2] + \mathcal{A}^3[v_3]. \qquad (6.1)$$

By direct calculation, one easily verifies the following lemma.

Lemma 6.1. *For any two $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p, q < \infty$, the following identity holds*

$$\int_\Omega v \mathcal{A}^\nabla[g\phi]\ dx = -\int_\Omega \mathcal{A}^{div}[f]\phi g\ dx, \quad \forall \phi \in C_0^\infty(\Omega).$$

Moreover, there is a constant C such that,

$$\|A^i[f]\|_{L^p(\Omega)} + \|\nabla A^i[f]\|_{L^q(\Omega)} \le C\|f\|_{L^q(\Omega)}, \quad p < \frac{3q}{3-q}.$$

We are now ready to prove higher integrability of the numerical density.

Proposition 6.2. *Let (ϱ_h, u_h) be the numerical approximation constructed through Definition 3.1 and (3.7). The following integrability estimate holds*

$$\varrho_h \in_b L_{loc}^{\gamma+1}([0,T] \times \Omega).$$

Proof. Let $\phi \in C_0^\infty(\Omega)$ be arbitrary and define the test-functions

$$v = \phi \mathcal{A}^\nabla[\varrho_h\phi], \qquad v_h = \Pi_h^V v.$$

Since ϱ_h is piecewise constant in time, so is v and hence also v_h. Moreover, since ϕ vanishes at the boundary, the degrees of freedom of v_h is zero at the boundary. As a consequence, v_h is a valid test-function in the momentum scheme (3.6).

Note that the energy estimate, Lemma 6.1, and the Hölder inequality provides the bound

$$\|\nabla v\|_{L^\infty(0,T;L^p(\Omega))} \le C\|\phi\|_{W^{1,\infty}(\Omega)}^2 \|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))} \le C\|\phi\|_{W^{1,\infty}(\Omega)}^2, \quad (6.2)$$

for any $p \le \gamma$.

Now, by applying v_h as test-function in (3.6), integrating in time, and reordering terms, we obtain

$$\int_0^T \int_\Omega p(\varrho_h)\operatorname{div}_h v_h\ dxdt = I_1 + I_2 + I_3, \qquad (6.3)$$

where

$$I_1 = \int_0^T \int_\Omega \nabla_h u_h \nabla_h v_h\ dxdt,$$

$$I_2 = h^{1-\epsilon} \sum_\Gamma \int_0^T \int_\Gamma \left(\frac{\widehat{u}_+ + \widehat{u}_-}{2} \right) [\![\varrho_h]\!]_\Gamma\ [\![\widehat{v}_h]\!]_\Gamma\ dS(x)dt,$$

$$I_3 = \int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h) v_h \ dxdt - \sum_\Gamma \int_0^T \int_\Gamma \mathrm{Up}(\varrho u \otimes \widehat{u}) \ [\![\widehat{v}_h]\!]_\Gamma \ dS(x)dt,$$

Before we start deriving bounds for I_1, I_2, and I_3, let us first consider the term on the left-hand side of (6.3). For this purpose, recall from Section 2 that the finite element spaces are chosen such that

$$\mathrm{div}_h \, \Pi_h^V v = \Pi_h^Q \, \mathrm{div} \, v.$$

Hence, we have that

$$
\begin{aligned}
\int_\Omega p(\varrho_h) \, \mathrm{div}_h \, v_h \ dx &= \int_\Omega p(\varrho_h) \Pi_h^Q \left[\mathrm{div} \left(\phi \mathcal{A}^\nabla[\phi \varrho_h] \right) \right] \ dx \\
&= \int_\Omega p(\varrho_h) \, \mathrm{div} \left(\phi \mathcal{A}^\nabla[\phi \varrho_h] \right) \ dx \qquad (6.4) \\
&= \int_\Omega p(\varrho_h) \nabla \phi \cdot \mathcal{A}^\nabla[\phi \varrho_h] + \phi^2 p(\varrho_h) \varrho_h \ dx.
\end{aligned}
$$

By setting this expression in (6.3), we see that

$$\int_0^T \int_\Omega \phi^2 p(\varrho_h) \varrho_h \ dx = I_1 + I_2 + I_3 + I_4, \qquad (6.5)$$

where now

$$I_4 = - \int_\Omega p(\varrho_h) \nabla \phi \cdot \mathcal{A}^\nabla[\phi \varrho_h] \ dx.$$

Thus, the proof follows provided we can bound I_1, I_2, I_3, and I_4.

1. Bounds on I_1 and I_2: Using the Cauchy-Schwartz inequality and the energy estimate (Corollary 4.3), we have that

$$
\begin{aligned}
|I_1| &\leq 2\|\nabla_h u_h\|_{L^2(0,T,L^2(\Omega))} \|\nabla_h v_h\|_{L^2(0,T;L^2(\Omega))} \\
&\leq C\|\nabla v\|_{L^2(0,T;L^2(\Omega))} \leq C\|\phi\|^2_{W^{1,\infty}(\Omega)},
\end{aligned}
\qquad (6.6)
$$

where the last inequality is (6.2).

To bound the I_2 term, we apply Lemma 5.5 to obtain

$$|I_2| \leq h^{\frac{1}{2}} \|\nabla v\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq h^{\frac{1}{2}} C\|\phi\|^2_{W^{1,\infty}(\Omega)}, \qquad (6.7)$$

where again the last inequality is (6.2).

2. Bound on I_3: From Lemma 5.4, we have the identity

$$I_3 = \int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h) v - \varrho_h \widetilde{u}_h \otimes \widehat{u}_h : \nabla v \ dxdt + F(v), \qquad (6.8)$$

where $F(v)$ is bounded by (5.17) and (6.2) as

$$|F(v)| \leq h^{\frac{1}{4}} C\|\nabla v\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq h^{\frac{1}{4}} C\|\phi\|^2_{W^{1,\infty}(\Omega)}.$$

It remains to bound the two other terms in I_3. Let us begin with the easiest term. For this purpose, we apply the Hölder inequality and (6.2) to deduce

$$\left| \int_0^T \int_\Omega \varrho_h \widetilde{u}_h \otimes \widehat{u}_h : \nabla v \ dxdt \right|$$

$$\leq \| \varrho_h \widetilde{u}_h \widehat{u}_h \|_{L^2(0,T;L^{\frac{\gamma}{\gamma-1}}(\Omega))} \| \nabla v \|_{L^\infty(0,T;L^\gamma(\Omega))} \tag{6.9}$$

$$\leq C \| \phi \|^2_{W^{1,\infty}(\Omega)} \| \varrho_h \widetilde{u}_h \widehat{u}_h \|_{L^2(0,T;L^{\frac{\gamma}{\gamma-1}}(\Omega))} \leq C \| \phi \|^2_{W^{1,\infty}(\Omega)},$$

where the last bound comes from Corollary 4.3 using that

$$\frac{3\gamma}{3+\gamma} \geq \frac{\gamma}{\gamma-1} \quad \text{for} \quad \gamma \geq 3.$$

The remaining term in I_3 is more complicated. By summation by parts in time followed by Lemma 6.1, we calculate

$$\int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h) v \ dxdt$$

$$= \Delta t \sum_{k=1}^M \int_\Omega \frac{\varrho_h^k \widehat{u}_h^k - \varrho_h^{k-1} \widehat{u}_h^{k-1}}{\Delta t} v^k \ dx$$

$$= -\Delta t \sum_{k=1}^M \int_\Omega \varrho_h^{k-1} \widehat{u}_h^{k-1} \frac{v^k - v^{k-1}}{\Delta t} \ dx$$

$$- \int_\Omega \varrho_h^0 \widehat{u}_h^0 v^1 \ dx + \int_\Omega \varrho_h^M \widehat{u}_h^M v^M \ dx.$$

To this identity, we apply the definition of v^k and Lemma 6.1 and write

$$\int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h) v \ dxdt$$

$$= -\Delta t \sum_{k=1}^M \int_\Omega \varrho_h^{k-1} \widehat{u}_h^{k-1} \phi \mathcal{A}^\nabla \left[\phi \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \right] \ dx$$

$$- \int_\Omega \varrho_h^0 \widehat{u}_h^0 v^1 \ dx + \int_\Omega \varrho_h^M \widehat{u}_h^M v^M \ dx$$

$$= -\Delta t \sum_{k=1}^M \int_\Omega \mathcal{A}^{\text{div}} \left[\phi \varrho_h^{k-1} \widehat{u}_h^{k-1} \right] \phi \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \ dx \tag{6.10}$$

$$- \int_\Omega \varrho_h^0 \widehat{u}_h^0 v^1 \ dx + \int_\Omega \varrho_h^M \widehat{u}_h^M v^M \ dx.$$

The last two terms are easily bounded since (6.2) and $\gamma > 3$ gives that $v \in_\text{b} L^\infty(0,T;L^\infty(\Omega))$ and hence

$$\left| - \int_\Omega \varrho_h^0 \widehat{u}_h^0 v^1 \ dx + \int_\Omega \varrho_h^M \widehat{u}_h^M v^M \ dx \right| \leq C \| \phi \|^2_{L^\infty(\Omega)} \| \varrho_h u_h \|_{L^\infty(0,T;L^1(\Omega))},$$

where the last term is bounded by Corollary 4.3. To bound the other term in (6.10) we apply the continuity scheme (3.5) with

$$q_h = \Pi_h^Q \left[\mathcal{A}^{\text{div}} \left[\phi \varrho_h^{k-1} \widehat{u}_h^{k-1} \right] \phi \right]$$

which gives

$$- \Delta t \sum_{k=1}^{M} \int_{\Omega} \mathcal{A}^{\mathrm{div}} \left[\phi \varrho_h^{k-1} \widehat{u}_h^{k-1} \right] \phi \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \ dx$$

$$= \Delta t \sum_{k=1}^{M} \sum_{\Gamma} \int_{\Gamma} \mathrm{Up}^k(\varrho u) \left[\!\!\left[\Pi_h^Q \left[\mathcal{A}^{\mathrm{div}} \left[\phi \varrho_h^{k-1} \widehat{u}_h^{k-1} \right] \phi \right] \right]\!\!\right]_{\Gamma} \ dS(x)$$

$$= \Delta t \sum_{k=1}^{M} \int_{\Omega} \varrho_h^k \widetilde{u}_h^k \nabla \left(\mathcal{A}^{\mathrm{div}} \left[\phi \varrho_h^{k-1} \widehat{u}_h^{k-1} \right] \phi \right) \ dx$$

$$+ P_1 \left(\mathcal{A}^{\mathrm{div}} \left[\phi \varrho_h^{k-1} \widehat{u}_h^{k-1} \right] \phi \right),$$

where the last equality is (5.1) in Lemma 5.1. Now, by applying Proposition 5.2 (i.e (5.7)) and the Hölder inequality in the previous identity, we obtain

$$\left| -\Delta t \sum_{k=1}^{M} \int_{\Omega} \mathcal{A}^{\mathrm{div}} \left[\phi \varrho_h^{k-1} \widehat{u}_h^{k-1} \right] \phi \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \ dx \right|$$

$$\leq 2 \| \varrho_h \widetilde{u}_h \|_{L^2(0,T;L^2(\Omega))} \| \phi \|_{W^{1,\infty}(\Omega)}^2 \left\| \mathcal{A}^{\mathrm{div}} [\varrho_h \widehat{u}_h] \right\|_{L^2(0,T;W^{1,2}(\Omega))} \quad (6.11)$$

$$+ h^{\frac{1}{4}} C \| \phi \|_{W^{1,\infty}}^2 \left\| \mathcal{A}^{\mathrm{div}} [\varrho_h \widehat{u}_h] \right\|_{L^4(0,T;W^{1,3}(\Omega))}$$

$$\leq C \| \phi \|_{W^{1,\infty}(\Omega)}^2 \left(1 + h^{\frac{1}{4}} \| \varrho_h \widehat{u}_h \|_{L^4(0,T;L^3(\Omega))} \right),$$

where the last inequality follows from the properties of $\mathcal{A}^i$ (Lemma 6.1) together with Corollary 4.3 giving $\varrho_h \widetilde{u}_h \in_{\mathrm{b}} L^2(0,T;L^2(\Omega))$. Next, we apply the inverse estimate in Lemma 2.9 to bound the last term

$$h^{\frac{1}{4}} \| \varrho_h \widehat{u}_h \|_{L^4(0,T;L^3(\Omega))} \leq h^{\frac{1}{4}} (\Delta t)^{-\frac{1}{4}} C \| \varrho_h \widehat{u}_h \|_{L^2(0,T;L^3(\Omega))},$$

which is bounded by Corollary 4.3. By applying this in (6.11), using that $h = a\Delta t$, and setting the result into (6.10) we obtain

$$\left| \int_0^T \int_{\Omega} D_t^h (\varrho_h \widehat{u}_h) v \ dx dt \right| \leq C \| \phi \|_{W^{1,\infty}(\Omega)}^2.$$

This, together with (6.9) and (6.8), yields

$$|I_3| \leq C \| \phi \|_{W^{1,\infty}(\Omega)}^2. \qquad (6.12)$$

3. *Bound on I_4:* To bound the I_4, we will use that $\gamma > 3$ yielding $W^{1,\gamma}$ embedded in L^{∞}. The resulting calculation is

$$|I_4| = \left| \int_{\Omega} p(\varrho_h) \nabla \phi \cdot \mathcal{A}^{\nabla} [\phi \varrho_h] \ dx \right|$$

$$\leq C \| p(\varrho_h) \|_{L^{\infty}(0,T;L^1(\Omega))} \| \nabla \phi \|_{L^{\infty}(\Omega)} \| \mathcal{A}^{\nabla} [\phi \varrho_h] \|_{L^{\infty}(0,T;L^{\infty}(\Omega))} \qquad (6.13)$$

$$\leq C \| \nabla \phi \|_{L^{\infty}(\Omega)} \| \mathcal{A}^{\nabla} [\phi \varrho_h] \|_{L^{\infty}(0,T;W^{1,\gamma}(\Omega))} \leq C \| \nabla \phi \|_{L^{\infty}(\Omega)}^2,$$

where the last inequality is derived as in (6.2). Here, we have also used Corollary 4.3 which tell us that $p(\varrho_h) \in_{\mathrm{b}} L^{\infty}(0,T;L^1(\Omega))$.

4. Conclusion: By applying (6.6), (6.7), (6.12), and (6.13) in (6.5), we obtain

$$\int_0^T \int_\Omega \phi^2 p(\varrho_h)\varrho_h \ dxdt \leq C\|\phi\|_{W^{1,\infty}(\Omega)}^2.$$

Since $p(\varrho_h) = a\varrho_h^\gamma$ and $\phi \in C_0^\infty(\Omega)$ can be chosen arbitrary, this concludes the proof.

$\square$

7. Weak convergence

In this section, we will pass to the limit in the numerical method and conclude that the limit of (ϱ_h, u_h) is almost a weak solution of the compressible Navier-Stokes equations. What remains in order to prove the main theorem is that $p(\varrho_h) \rightharpoonup p(\varrho)$, which will be the topic of the ensuing sections.

Our starting point is that Corollary 4.3 and Proposition 6.2 allow us to assert the existence of functions

$$\varrho \in L^\infty(0, T; L^\gamma(\Omega)), \quad u \in L^2(0, T; W_0^{1,2}(\Omega)),$$

and a subsequence $h_j \to 0$ such that

$$\varrho_h \overset{\star}{\rightharpoonup} \varrho \text{ in } L^\infty(0, T; L^\gamma(\Omega)),$$
$$u_h \rightharpoonup u \text{ in } L^2(0, T; L^2(\Omega)), \tag{7.1}$$
$$\nabla_h u_h \rightharpoonup \nabla u \text{ in } L^2(0, T; L^2(\Omega)).$$

From Corollary 4.3, we immediately obtain the integrability

$$\varrho u \in L^\infty(0, T; L^{m_\infty}(\Omega)) \cap L^2(0, T; L^{m_2}(\Omega)),$$
$$\varrho u \otimes u \in L^\infty(0, T; L^1(\Omega)) \cap L^2(0, T; L^{c_2}(\Omega)),$$

with exponents

$$m_\infty = \frac{2\gamma}{\gamma+1} > \frac{3}{2}, \quad m_2 = \frac{6\gamma}{3+\gamma} > 3, \quad c_2 = \frac{3\gamma}{3+\gamma} > \frac{3}{2}.$$

Moreover, Lemma 2.4 together with Lemma 5.6 provides the bounds

$$\varrho_h \in_{\mathrm{b}} C(0, T; L^\gamma(\Omega)), \qquad \varrho_h u_h \in_{\mathrm{b}} C(0, T; L^{m_\infty}(\Omega)).$$

In the following lemma, we prove that the above convergences are sufficient to pass to the limit in both nonlinear terms involving u.

Lemma 7.1. *Given the convergences* (7.1),

$$\varrho_h u_h, \ \varrho_h \tilde{u}_h \rightharpoonup \varrho u \text{ in } L^2(0, T; L^{m_2}(\Omega)),$$
$$\varrho_h \widehat{u}_h \rightharpoonup \varrho u \text{ in } C(0, T; L^{m_\infty}(\Omega)). \tag{7.2}$$
$$\varrho_h \tilde{u}_h \otimes \widehat{u}_h \rightharpoonup \varrho u \otimes u \text{ in } L^2(0, T; L^{c_2}(\Omega)),$$

Proof. Lemma 2.10 tell us that u_h is spatially compact in $L^2(0, T; L^p(\Omega))$ for any $p < 6$. From Lemma 5.6 we have that $D_t^h \varrho_h \in_{\mathrm{b}} L^{\frac{4}{3}}(0, T; W^{-1, \frac{3}{2}}(\Omega))$. As a consequence, we can apply Lemma 2.6, with $g_h = \varrho_h$ and $f_h = u_h$, to obtain

$$\varrho_h u_h \rightharpoonup \varrho u \text{ in } L^2(0, T; L^{m_2}(\Omega)).$$

To conclude weak convergence of $\varrho_h \tilde{u}_h$ and $\varrho_h \widehat{u}_h$, we write

$$\varrho_h \tilde{u}_h = \varrho_h u_h + \varrho_h(\Pi_h^{\mathcal{N}} u_h - u_h), \qquad \varrho_h \widehat{u}_h = \varrho_h u_h + \varrho_h(\Pi_h^Q u_h - u_h).$$

The interpolation error estimate (Lemma 2.7) tell us that

$$\|\Pi_h^{\mathcal{N}} u_h - u_h\|_{L^2(0,T;L^2(\Omega))} + \|\Pi_h^Q u_h - u_h\|_{L^2(0,T;L^2(\Omega))} \leq C\|\nabla_h u_h\|_{L^2(0,T;L^2(\Omega))},$$

and hence $\varrho_h \widetilde{u}_h \rightharpoonup \varrho u$, $\varrho_h \widehat{u}_h \rightharpoonup \varrho u$ in the sense of distributions on $(0,T) \times \Omega$.

Next, since $D_t^h(\varrho_h \widehat{u}_h) \in_{\mathrm{b}} L^1(0,T;W^{-1,\frac{3}{2}}(\Omega))$ (from Lemma 5.6), another application of Lemma 2.6, this time with $g_h = \varrho_h \widehat{u}_h$ and $f_h = u_h$, renders

$$\varrho_h \widehat{u}_h \otimes u_h \rightharpoonup \varrho u \otimes u \text{ in } L^2(0,T;L^{c_2}(\Omega)).$$

Clearly, this also implies that $\varrho_h u_h \otimes \widehat{u}_h \rightharpoonup \varrho u \otimes u$. The convergence of $\varrho_h \widetilde{u}_h \otimes \widehat{u}_h$ then follows by writing

$$\varrho_h \widetilde{u}_h \otimes \widehat{u}_h = \varrho_h u_h \otimes \widehat{u}_h + \varrho_h (\Pi_h^{\mathcal{N}} u_h - u_h) \otimes \widehat{u}_h,$$

and applying the interpolation error estimate on the remainder. $\qquad\square$

Using the convergences we have derived thus far, we are able to pass to the limit in the continuity approximation (3.5).

Lemma 7.2. *Let (ϱ_h, u_h) be the numerical solution obtained by Definition 3.1 and (3.7). The limit (ϱ, u) is a weak solution of continuity equation:*

$$\varrho_t + \mathrm{div}(\varrho u) = 0 \quad \text{in } \mathcal{D}'([0,T] \times \overline{\Omega}).$$

Proof. Let $\phi \in C_0^\infty([0,T] \times \overline{\Omega})$ be arbitrary and set $\Pi_h^Q \phi$ as test-function in the continuity scheme (3.5) to obtain

$$
\begin{aligned}
\int_0^T & \int_\Omega D_t^h(\varrho_h)\phi \; dxdt \\
&= \sum_\Gamma \int_0^T \int_\Gamma \mathrm{Up}(\varrho u) \left[\!\left[\Pi_h^Q \phi \right]\!\right] - h^{1-\epsilon} [\![\varrho_h]\!]_\Gamma \left[\!\left[\Pi_h^Q \phi \right]\!\right] \; dS(x)dt \\
&= \int_0^T \int_\Omega \varrho_h \widetilde{u}_h \nabla\phi \; dxdt + P_1(\phi) \\
&\quad - h^{1-\epsilon} \sum_\Gamma \int_0^T \int_\Gamma [\![\varrho_h]\!]_\Gamma \left[\!\left[\Pi_h^Q \phi \right]\!\right] \; dS(x)dt,
\end{aligned}
\tag{7.3}
$$

where the last equality is Lemma 5.1.

From Proposition 5.2, we have that

$$|P_1(\phi)| \leq h^{\frac{1}{4}} C \|\phi\|_{L^4(0,T;L^{\frac{12}{5}}(\Omega))}. \tag{7.4}$$

From Lemma 5.5, we have the bound

$$\left| \sum_\Gamma \int_0^T \int_\Gamma h^{1-\epsilon} [\![\varrho_h]\!]_\Gamma \left[\!\left[\Pi_h^Q \phi \right]\!\right] \; dS(x)dt \right| \leq h^{\frac{11-6\epsilon}{12}} C \|\nabla\phi\|_{L^2(0,T;L^2(\Omega))}. \tag{7.5}$$

Summation by parts provides the identity

$$
\begin{aligned}
\int_0^T & \int_\Omega D_t^h(\varrho_h)\phi \; dxdt \\
&= -\int_0^{T-\Delta t} \int_\Omega \varrho_h D_t^h(\phi(+\Delta t)) \; dxdt - \int_\Omega \varrho_h^0 \phi(\Delta t) \; dx.
\end{aligned}
\tag{7.6}
$$

Now, we apply (7.6) in (7.3) and send $h \to 0$ along the subsequence where $\varrho_h \tilde{u}_h \rightharpoonup \varrho u$ and apply (7.4) and (7.5) to obtain

$$\int_0^T \int_\Omega \varrho(\phi_t + u \cdot \nabla\phi) \, dxdt = -\int_\Omega \varrho_0 \phi(0, \cdot) \, dx.$$

This concludes the proof.

$\square$

Next, we prove that the limit (ϱ, u) is almost a weak solution of the momentum equation (1.2) and hence that it only remains to prove strong convergence of the density.

Lemma 7.3. *Let (ϱ_h, u_h) be as in the previous lemma. The limit (ϱ, u) satisfies*

$$(\varrho u)_t + \mathrm{div}(\varrho u \otimes u) + \nabla\overline{p(\varrho)} - \Delta u = 0 \quad in \ \mathcal{D}'([0,T) \times \Omega), \qquad (7.7)$$

where $\overline{p(\varrho)}$ is the weak limit of $p(\varrho_h)$.

Proof. Let $v \in [C_0^\infty([0,T) \times \Omega)]^3$ be arbitrary and set $v_h = \Pi_h^V v$ as test-function in the momentum scheme (3.6). After an application of Lemma 5.4, we obtain the identity

$$\int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h)v - \varrho_h \tilde{u}_h \otimes \widehat{u}_h : \nabla v \, dxdt + F(v)$$
$$+ \int_0^T \int_\Omega \nabla_h u_h \nabla_h v_h - p(\varrho_h)\,\mathrm{div}_h v_h \, dxdt \qquad (7.8)$$
$$- h^{1-\epsilon} \sum_\Gamma \int_0^T \int_\Gamma [\![\varrho_h]\!]_\Gamma \left(\frac{\widehat{u}_+ + \widehat{u}_-}{2}\right) [\![\widehat{v}_h]\!]_\Gamma \, dS(x)dt = 0$$

Lemma 5.4 also provides the bound

$$|F(v)| \le h^{\frac{1}{4}} C \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))}. \qquad (7.9)$$

Moreover, from Lemma 5.5, we have the bound

$$\left| h^{1-\epsilon} \sum_\Gamma \int_0^T \int_\Gamma [\![\varrho_h]\!]_\Gamma \left(\frac{\widehat{u}_+ + \widehat{u}_-}{2}\right) [\![\widehat{v}_h]\!]_\Gamma \, dS(x)dt \right|$$
$$\le h^{\frac{13-6\epsilon}{12}} C \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))}. \qquad (7.10)$$

By applying summation by parts to the first term in (7.8), sending $h \to 0$ along the subsequence for which $\varrho_h \tilde{u}_h \otimes \widehat{u}_h \rightharpoonup \varrho u \otimes u$ (Lemma 7.1), and applying (7.9) and (7.10) we obtain

$$\int_0^T \int_\Omega -\varrho uv_t - \varrho u \otimes u : \nabla v + \nabla u \nabla v - \overline{p(\varrho)}\,\mathrm{div}\,v \, dxdt = \int_\Omega m_0 v(0, \cdot) \, dx,$$

which concludes our proof.

$\square$

8. The discrete Laplace operator

In the next section we establish the most important ingredient in our proof of compactness of the density approximation, namely weak continuity of the effective viscous flux. However, before we can embark on this proof, we will need to establish some properties related to our numerical Laplace operator.

In contrast to a standard continuous approximation scheme, our discrete Laplace operator does not respect Hodge decompositions. More specifically, to prove the upcoming Proposition 9.1 we shall need to use test-functions of the form

$$v = \mathcal{A}^\nabla[\varrho],$$

where we for the purpose of this discussion does not require v to satisfy boundary conditions. At the continuous level, testing with this v is equivalent to applying $\mathcal{A}^{\mathrm{div}}[\cdot]$ to the equation. In particular, since $\operatorname{curl} \mathcal{A}^\nabla = 0$, v satisfies

$$\int_\Omega \nabla u \nabla v \ dx = \int_\Omega \operatorname{div} u \varrho \ dx.$$

This property is not true for our discrete Laplace operator. However, in the upcoming analysis it is essential that, at least,

$$\int_\Omega \nabla_h u_h \nabla_h \left(\Pi_h^V \mathcal{A}^\nabla[\varrho_h] \right) \ dx = \int_\Omega \operatorname{div}_h u_h \varrho_h \ dx + \mathcal{O}(h^\alpha), \quad \alpha > 0. \quad (8.1)$$

This is the result that we will prove in this section.

As will become evident, the property (8.1) is not trivially satisfied for our discretization. It is the extra "artifical" stabilization terms in the scheme (adding diffusion in all directions) that will provide the needed ingredient. In fact, the property (8.1) is the only reason for the presence of these terms. This discretization strategy was devised by Eymard et. al for the stationary compressible Stokes equations [9]. In fact, most of the material contained in this section can be found there with only slight modifications to fit the present case.

In our proof, we shall need the operator $\Pi_h^L : Q_h(\Omega) \mapsto \mathcal{P}_h(\Omega)$ interpolating piecewise constant functions in the space of continuous piecewise linears $\mathcal{P}_h$ (Lagrange element space). This operator is defined by

$$\left(\Pi_h^L q_h \right)(\tau) = \frac{1}{\operatorname{card}(N_\tau)} \sum_{E \in N_E} q_h|_E,$$

for all vertices τ in the discretization, where N_τ is the set of elements having τ as a vertex. Note that shape regularity of E_h renders the cardinality of N_τ bounded. The following result can be found in [9, Lemma 5.8]:

Lemma 8.1. *Let $q_h \in Q_h(\Omega)$. There exists a constant C, depending only on the shape-regularity of E_h such that*

$$\| \nabla \left(\Pi_h^L q_h \right) \|_{L^2(\Omega)} \leq C \left(\sum_\Gamma \int_\Gamma \frac{[\![q_h]\!]_\Gamma^2}{h} \ dS(x) \right)^{\frac{1}{2}}, \qquad (8.2)$$

$$\| \Pi_h^L q_h - q_h \|_{L^2(\Omega)} \leq hC \left(\sum_\Gamma \int_\Gamma \frac{[\![q_h]\!]_\Gamma^2}{h} \ dS(x) \right)^{\frac{1}{2}}. \qquad (8.3)$$

We shall need the following auxiliary result.

Lemma 8.2. *Let $u_h \in V_h(\Omega)$ and $v \in W^{2,2}(\Omega)$ be arbitrary. Then,*

$$\int_\Omega \nabla_h u_h \nabla v \ dx = \int_\Omega \operatorname{curl}_h u_h \operatorname{curl}_h v + \operatorname{div}_h u_h \operatorname{div} v \ dx + E(v, u_h). \quad (8.4)$$

Furthermore, there is a constant $C > 0$, depending only on the shape-regularity of E_h such that

$$|E(v, u_h)| \leq hC \|\nabla_h u_h\|_{L^2(\Omega)} \|\nabla^2 v\|_{L^2(\Omega)}.$$

Proof. Using the Stoke's theorem and the identity $-\Delta = \operatorname{curl}\operatorname{curl} -\nabla\operatorname{div}$,

$$\int_\Omega \nabla_h u_h \nabla v \ dx$$

$$= \sum_E \int_E -u_h \Delta v \ dxdt + \int_0^T \int_{\partial E} (\nabla v \cdot \nu) u_h \ dS(x)$$

$$= \sum_E \int_E u_h \operatorname{curl}\operatorname{curl} v - u_h \nabla \operatorname{div} v \ dxdt + \int_0^T \int_{\partial E} (\nabla v \cdot \nu) u_h \ dS(x)$$

$$= \int_\Omega \operatorname{curl}_h u_h \operatorname{curl} v + \operatorname{div} u_h \operatorname{div} v \ dx$$

$$+ \sum_E \int_{\partial E} (\operatorname{curl} v \times \nu) u_h \times \nu - (\operatorname{div} v) u_h \cdot \nu + (\nabla v \cdot \nu) u_h \ dS(x),$$

which is (8.4) with

$$E(v) = \sum_E \int_{\partial E} (\operatorname{curl} v \times \nu) u_h \times \nu - (\operatorname{div} v) u_h \cdot \nu + (\nabla v \cdot \nu) u_h \ dS(x).$$

.

Next, we use that the trace of ∇v is continuous across edges to deduce

$$|E(v)| \leq \sum_E \int_{\partial E} |\nabla v| u_h \ dS(x)$$

$$= \sum_E \int_{\partial E} |\nabla v| (u_h - \Pi_h^Q u_h) \ dS(x)$$

$$= \sum_E \int_{\partial E} |\nabla v - \Pi_h^Q(\nabla v)| (u_h - \Pi_h^Q u_h) \ dS(x)$$

$$\leq h^{-\frac{1}{2}} C \left(\|\nabla v - \Pi_h^Q(\nabla v)\|_{L^2(\Omega)} + h\|\nabla^2 v\|_{L^2(\Omega)} \right)$$

$$\times h^{-\frac{1}{2}} \left(\|u_h - \Pi_h^Q u_h\|_{L^2(\Omega)} + h\|\nabla_h u_h\|_{L^2(\Omega)} \right)$$

$$\leq hC \|\nabla^2 v\|_{L^2(\Omega)} \|\nabla u_h\|_{L^2(\Omega)},$$

where we have applied the trace theorem (Lemma 2.8) and the interpolation error estimate (Lemma 2.7). This concludes the proof. □

As in [9], we obtain the following identity.

Lemma 8.3. *Let (ϱ_h, u_h) be the numerical solution obtained by Definition 3.1 and let $\mathcal{A}^\nabla[\cdot]$ be given by (6.1). For any $\psi \in C^\infty([0,T] \times \Omega)$ and $\phi \in C_0^\infty(\Omega)$,*

$$\int_0^T \int_\Omega \nabla_h u_h \nabla_h \Pi_h^V \left(\psi \mathcal{A}^\nabla[\phi \varrho_h] \right) \, dxdt$$

$$= \int_0^T \int_\Omega \operatorname{curl}_h u_h \nabla\psi \times \mathcal{A}^\nabla[\phi \varrho_h] + \operatorname{div}_h u_h \nabla\psi \cdot \mathcal{A}^\nabla[\phi \varrho_h] \, dxdt$$

$$+ \int_0^T \int_\Omega \phi\psi \operatorname{div}_h u_h \varrho_h \, dxdt + \mathcal{F}(\varrho_h),$$

where the $\mathcal{F}$ term is bounded as

$$|\mathcal{F}(\varrho_h)| \le h^{\frac{1}{2}(\epsilon - \frac{1}{6})} C, , \tag{8.5}$$

with C independent of h and where we recall the requirement $\epsilon > \frac{1}{6}$.

Proof. To simplify notation, let

$$v = \psi \mathcal{A}^\nabla[\phi \varrho_h], \qquad v_L = \psi \mathcal{A}^\nabla[\phi \Pi_h^L \varrho_h],$$

and observe that linearity of $\mathcal{A}^\nabla$ provides the identity

$$v - v_L = \psi \mathcal{A}^\nabla[\phi(\varrho_h - \Pi_h^L \varrho_h)].$$

By using Lemma 2.11 and adding and subtracting v_L, we deduce

$$\int_0^T \int_\Omega \nabla_h u_h \nabla_h \Pi_h^V v \, dxdt$$

$$= \int_0^T \int_\Omega \nabla_h u_h \nabla v \, dxdt$$

$$= \int_0^T \int_\Omega \nabla_h u_h \nabla v_L \, dxdt + \int_0^T \int_\Omega \nabla_h u_h \nabla(v - v_L) \, dxdt$$

$$= \int_0^T \int_\Omega \operatorname{curl}_h u_h \operatorname{curl} v_L + \operatorname{div}_h u_h \operatorname{div} v_L \, dxdt$$

$$+ \int_0^T \int_\Omega \nabla_h u_h \nabla(v - v_L) \, dxdt + E(v_L, u_h),$$

where the last equality is Lemma 8.2. Next, we once more add and subtract v_l to obtain

$$
\begin{aligned}
\int_0^T & \int_\Omega \nabla_h u_h \nabla_h \Pi_h^V v \ dxdt \\
&= \int_0^T \int_\Omega \operatorname{curl}_h u_h \operatorname{curl} v + \operatorname{div}_h u_h \operatorname{div} v \ dxdt \\
&\quad + \int_0^T \int_\Omega \operatorname{curl}_h u_h \operatorname{curl}(v_L - v) + \operatorname{div}_h u_h \operatorname{div}(v_L - v) \ dxdt \\
&\quad + \int_0^T \int_\Omega \nabla_h u_h \nabla(v - v_L) \ dxdt + E(v_L, u_h) \\
&= \int_0^T \int_\Omega \operatorname{curl}_h u_h \nabla\psi \times \mathcal{A}^\nabla[\phi \varrho_h] + \operatorname{div}_h u_h \nabla\psi \cdot \mathcal{A}^\nabla[\phi \varrho_h] \ dxdt \\
&\quad + \int_0^T \int_\Omega \phi\psi \operatorname{div}_h u_h \varrho_h \ dxdt + \mathcal{F}(\varrho_h),
\end{aligned}
\tag{8.6}
$$

where we have used the definition of v_l and introduced the quantity

$$
\begin{aligned}
\mathcal{F}(\varrho_h) := & \int_0^T \int_\Omega \operatorname{curl}_h u_h \operatorname{curl}(v_L - v) + \operatorname{div}_h u_h \operatorname{div}(v_L - v) \ dxdt \\
& \int_0^T \int_\Omega \nabla_h u_h \nabla(v - v_L) \ dxdt + E(v_L, u_h).
\end{aligned}
$$

In view of (8.6), it only remains to prove (8.5).

Some applications of the Cauchy-Schwartz inequality together with the properties of $\mathcal{A}^\nabla[\cdot]$ gives

$$
\begin{aligned}
|\mathcal{F}(\varrho_h)| &\leq C \|\nabla_h u_h\|_{L^2(\Omega)} \|\phi(\varrho_h - \Pi_h^L \varrho_h)\|_{L^2(0,T;L^2(B))} + |E(v_L, u_h)| \\
&\leq hC \left(\sum_{\Gamma \subset B} \int_0^T \int_\Gamma \frac{[\![\varrho_h]\!]_\Gamma^2}{h} \ dS(x) dt \right)^{\frac{1}{2}} + h \|\nabla^2 v\|_{L^2(0,T;L^2(\Omega))} \\
&\leq \sqrt{h} 2C \left(\sum_{\Gamma \subset B} \int_0^T \int_\Gamma [\![\varrho_h]\!]_\Gamma^2 \ dS(x) dt \right)^{\frac{1}{2}}.
\end{aligned}
$$

where the second last inequality is (8.3) and Lemma 8.2 and the last inequality is and application of (8.3). To bound the last term, we invoke (5.21) yielding

$$
|\mathcal{F}(\varrho_h)| \leq h^{\frac{1}{2}(\epsilon - \frac{1}{6})} C,
$$

which is (8.5)

$\square$

9. The effective viscous flux

The main tool that will allow us to conclude strong convergence of the density is a remarkable result discovered by P. L. Lions [17] for a continuous approximation scheme. The result states that the effective viscous flux

$$
\operatorname{div} u - p(\varrho),
$$

behaves as if it is converging strongly. More specific, in our numerical setting, the result goes as follows:

Proposition 9.1. *Let (ϱ_h, u_h) be the numerical solution obtained using Definition 3.1 and (3.7). Moreover, let (ϱ, u) be given through the convergences (7.1). Then,*

$$\lim_{h \to 0} \int_0^T \int_\Omega \phi\psi \left(\operatorname{div}_h u_h - p(\varrho_h)\right) \varrho_h \, dxdt$$
$$= \int_0^T \int_\Omega \phi\psi \left(\operatorname{div} u - \overline{p(\varrho)}\right) \varrho \, dxdt,$$

for all $\phi \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty([0,T) \times \Omega)$.

Hence, the product of the weakly converging effective viscous flux and the weakly converging density converges to the product of the weak limits. Our proof of this proposition will rely on a number of auxiliary results which we will prove first. The proof is concluded in Section 9.2.

9.1. **The numerical commutator estimate.** In the upcoming analysis, the following lemma will be essential. A proof based on the div-curl lemma can be found in [10].

Lemma 9.2. *Let v_n and w_n be sequences of vector valued functions such that $v_n \rightharpoonup v$ in $L^p(\Omega)$ and $w_n \rightharpoonup w$ in $L^q(\Omega)$, respectively, where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} \leq 1$. Moreover, let $B_n \rightharpoonup B$ in $L^p(\Omega)$. Then,*

$$(1) \quad v_n \nabla \mathcal{A}^{div}[w_n] - w_n \nabla \mathcal{A}^{div}[v_n] \quad \longrightarrow \quad v \nabla \mathcal{A}^{div}[w] - w \nabla \mathcal{A}^{div}[v]$$

$$(2) \quad B_n \nabla \mathcal{A}^{div}[w_n] - w_n \nabla \mathcal{A}^{\nabla}[B_n] \quad \longrightarrow \quad B \nabla \mathcal{A}^{div}[w] - w \nabla \mathcal{A}^{\nabla}[B]$$

in the sense of distributions on Ω.

Lemma 9.3. *Given the convergences (7.1) - (7.2),*

$$\lim_{h \to 0} \int_0^T \int_\Omega \phi\varrho_h \tilde{u}_h \nabla \left(\mathcal{A}^{div}[\psi\varrho_h \widehat{u}_h]\right) - \psi\varrho_h \tilde{u}_h \otimes \widehat{u}_h : \nabla \left(\mathcal{A}^{\nabla}[\phi\varrho_h]\right) \, dxdt$$
$$= \int_0^T \int_\Omega \phi\varrho u \nabla \left(\mathcal{A}^{div}[\psi\varrho u]\right) - \psi\varrho u \otimes u : \nabla \left(\mathcal{A}^{\nabla}[\phi\varrho]\right) \, dxdt.$$

Proof. We begin by observing the following identity

$$\int_0^T \int_\Omega \phi\varrho_h \tilde{u}_h \nabla \left(\mathcal{A}^{\mathrm{div}}[\psi\varrho_h \widehat{u}_h]\right) - \psi\varrho_h \tilde{u}_h \otimes \widehat{u}_h : \nabla \left(\mathcal{A}^{\nabla}[\phi\varrho_h]\right) \, dxdt$$
$$= \int_0^T \int_\Omega \tilde{u}_h \cdot \left[\nabla \left(\mathcal{A}^{\mathrm{div}}[\psi\varrho_h \widehat{u}_h]\right) \phi\varrho_h - \nabla \left(\mathcal{A}^{\nabla}[\phi\varrho_h]\right) \cdot \psi\varrho_h \widehat{u}_h\right] \, dxdt \quad (9.1)$$
$$=: \int_0^T \int_\Omega \tilde{u}_h \mathcal{H}^h \, dxdt,$$

where we have introduced $\mathcal{H}^h$ given by

$$\mathcal{H}^h = \nabla \left(\mathcal{A}^{\mathrm{div}}[\psi\varrho_h \widehat{u}_h]\right) \phi\varrho_h - \nabla \left(\mathcal{A}^{\nabla}[\phi\varrho_h]\right) \cdot \psi\varrho_h \widehat{u}_h.$$

From Lemma 7.1, we have that

$$\varrho_h \rightharpoonup \varrho \text{ in } C(0,T;L^\gamma(\Omega)) \cap L^2(0,TL^\gamma(\Omega)),$$

$$\varrho_h \widehat{u}_h \rightharpoonup \varrho u \text{ in } C(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \cap L^2(0,T;L^3(\Omega)). \qquad (9.2)$$

Since in addition $\gamma > 3$, the Hölder inequality, Lemma 6.1, and Corollary 4.3, provides the estimate

$$\|\mathcal{H}^h\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} \leq C\|\varrho_h \widehat{u}_h\|_{L^2(0,T;L^3(\Omega))}\|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq C, \qquad (9.3)$$

and similarly,

$$\|\mathcal{H}^h\|_{L^\infty(0,T;L^{\frac{2\gamma}{3+\gamma}}(\Omega))} \leq C\|\varrho_h \widehat{u}_h\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))}\|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq C, \qquad (9.4)$$

where $\frac{2\gamma}{\gamma+1} > 1$ since $\gamma > 3$.

By virtue of (9.2)-(9.4), we can apply Lemma 9.2, with $v_h = \psi\varrho_h\widehat{u}_h$ and $B_h = \phi\varrho_h$, to conclude the weak convergence

$$\mathcal{H}^h = \nabla\left(\mathcal{A}^{\text{div}}[\psi\varrho_h\widehat{u}_h]\right)\phi\varrho_h - \nabla\left(\mathcal{A}^\nabla[\phi\varrho_h]\right)\cdot\psi\varrho_h\widehat{u}_h$$

$$\rightharpoonup \nabla\left(\mathcal{A}^{\text{div}}[\psi\varrho u]\right)\phi\varrho - \nabla\left(\mathcal{A}^\nabla[\phi\varrho]\right)\cdot\psi\varrho u =: \mathcal{H}, \qquad (9.5)$$

in $L^2(0,T;L^{\frac{3}{2}}(\Omega)) \cap C(0,T;L^{\frac{2\gamma}{3+\gamma}}(\Omega))$ as $h \to 0$.

To proceed we will need the standard mollifier which we will denote by R^δ. It will be a standing assumption throughout that δ is sufficiently small.

Now, by adding and subtracting we write

$$\int_0^T \int_\Omega \tilde{u}_h\mathcal{H}^h \, dxdt$$

$$= \int_0^T \int_\Omega (\tilde{u}_h - u_h)\mathcal{H}^h + (u_h - R^\delta \star u_h)\mathcal{H}^h + (R^\delta \star u_h)\mathcal{H}^h \, dxdt \qquad (9.6)$$

The first term in (9.6) converges to zero as

$$\int_0^T \int_\Omega (\tilde{u}_h - u_h)\mathcal{H}^h \, dxdt$$

$$\leq \|\tilde{u}_h - u_h\|_{L^2(0,T;L^3(\Omega))}\|\mathcal{H}^h\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))}$$

$$\leq h\|\nabla_h u_h\|_{L^2(0,T;L^3(\Omega))}\|\mathcal{H}^h\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} \qquad (9.7)$$

$$\leq h^{\frac{1}{2}}\|\nabla_h u_h\|_{L^2(0,T;L^2(\Omega))}\|\mathcal{H}^h\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} \leq h^{\frac{1}{2}}C.$$

To bound the second term in (9.6), we apply the Hölder inequality and the space translation estimate of Lemma 2.10,

$$\int_0^T \int_\Omega (u_h - R^\delta \star u_h)\mathcal{H}^h \, dxdt$$

$$\leq \left\|u_h - R^\delta \star u_h\right\|_{L^2(0,T;L^3(\Omega))}\|\mathcal{H}^h\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} \qquad (9.8)$$

$$\leq C\left(h^2 + \delta^2\right)^{\frac{1}{4}}\|\nabla_h u_h\|_{L^2(0,T;L^2(\Omega))} \leq C\left(h^2 + \delta^2\right)^{\frac{1}{4}}.$$

Next, since $\mathcal{H}^h \rightharpoonup \mathcal{H}$ in $C(0, T; L^{\frac{2\gamma}{3+\gamma}}(\Omega))$, we have in particular that

$$\mathcal{H}^h \to \mathcal{H} \text{ in } L^2(0, T; W^{-1,p}(\Omega)), \quad p < \frac{3}{2}.$$

Hence, for each fixed δ, we can conclude that

$$\lim_{h \to 0} \int_0^T \int_\Omega (R^\delta \star u_h) \mathcal{H}^h \ dxdt = \int_0^T \int_\Omega (R^\delta \star u) \mathcal{H} \ dxdt.$$

Finally, we pass to the limit in (9.6) and apply (9.7) and (9.8) to obtain

$$\lim_{h \to 0} \int_0^T \int_\Omega \tilde{u}_h \mathcal{H}^h \ dxdt = \int_0^T \int_\Omega (R^\delta \star u) \mathcal{H} \ dxdt + \mathcal{O}(\sqrt{\delta}).$$

We conclude the proof by sending $\delta \to 0$ and recalling (9.1) and (9.5).

$\square$

Lemma 9.4. *Let (ϱ_h, u_h) be the numerical solution obtained through Definition 3.1 and (3.7). Let (ϱ, u) be the corresponding weak limit pair obtained from the convergences (7.1). Then, for any $\phi \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty([0, T] \times \Omega)$,*

$$\lim_{h \to 0} \left(\sum_\Gamma \int_0^T \int_\Gamma \mathrm{Up}(\varrho u) \left[\!\!\left[\Pi_h^Q \left[\phi \mathcal{A}^{div}[\psi \varrho_h \widehat{u}_h] \right] \right]\!\!\right]_\Gamma \right.$$
$$\left. - \mathrm{Up}(\varrho u \otimes u) \left[\!\!\left[\Pi_h^Q \Pi_h^V \left[\psi \mathcal{A}^\nabla[\phi \varrho_h] \right] \right]\!\!\right]_\Gamma \ dS(x)dt \right)$$
$$= \int_0^T \int_\Omega \varrho u \nabla(\phi \mathcal{A}^{div}(\psi \varrho u)) - \varrho u \otimes u : \nabla\left(\psi \mathcal{A}^\nabla[\phi \varrho] \right) \ dxdt,$$

where the operators $\mathcal{A}^\nabla[\cdot]$ and $\mathcal{A}^{div}[\cdot]$ are given by (6.1).

Proof. Our starting point is provided by Lemma 5.1 which in this case gives

$$\sum_\Gamma \int_0^T \int_\Gamma \mathrm{Up}(\varrho u) \left[\!\!\left[\Pi_h^Q \left[\mathcal{A}^{\mathrm{div}}[\psi \varrho_h \widehat{u}_h] \right] \right]\!\!\right]_\Gamma$$
$$- \mathrm{Up}(\varrho u \otimes u) \left[\!\!\left[\Pi_h^Q \Pi_h^V \left[\psi \mathcal{A}^\nabla[\phi \varrho_h] \right] \right]\!\!\right]_\Gamma \ dS(x)dt$$
$$= \int_0^T \int_\Omega \varrho_h \tilde{u}_h \nabla\left(\phi \mathcal{A}^{\mathrm{div}}[\psi \varrho_h \widehat{u}_h] \right) - \varrho_h \tilde{u}_h \otimes \widehat{u}_h : \nabla\left(\psi \mathcal{A}^\nabla[\phi \varrho_h] \right) \ dxdt \qquad (9.9)$$
$$+ P_1\left(\phi \mathcal{A}^{\mathrm{div}}[\psi \varrho_h \widehat{u}_h] \right) + \sum_{i=2}^4 P_i\left(\psi \mathcal{A}^\nabla[\phi \varrho_h] \right).$$

Let us first prove that the P_i terms are converging to zero as $h \to 0$.

1. Convergence of P_i to zero: From (5.3), we have the following bound

$$\left| P_1\left(\phi \mathcal{A}^{\mathrm{div}}[\psi \varrho_h \widehat{u}_h] \right) \right|$$
$$\leq h^{\frac{1}{2} - 3\frac{4-\gamma}{4\gamma}} C \left\| \nabla\left(\phi \mathcal{A}^{\mathrm{div}}[\psi \varrho_h \widehat{u}_h] \right) \right\|_{L^4(0,T;L^3(\Omega))} \qquad (9.10)$$
$$\leq h^{\frac{1}{2} - 3\frac{4-\gamma}{4\gamma}} C \|\phi\|_{W^{1,\infty}(\Omega)} \left\| |\mathcal{A}^{\mathrm{div}}[\psi \varrho_h \widehat{u}_h] \right\|_{L^4(0,T;W^{1,3}(\Omega))}.$$

From the regularization property of $\mathcal{A}^{\mathrm{div}}$ (Lemma 6.1) together with an inverse estimate in time (Lemma 2.9) we deduce

$$
\begin{aligned}
\left\| \left| \mathcal{A}^{\mathrm{div}}[\psi \varrho_h \widehat{u}_h] \right| \right\|_{L^4(0,T;W^{1,3}(\Omega))} & \\
& \leq C \| \psi \varrho_h \widehat{u}_h \|_{L^4(0,T;L^3(\Omega))} \\
& \leq C \| \psi \|_{L^\infty((0,T)\times\Omega)} \| \varrho_h \widehat{u}_h \|_{L^4(0,T;L^3(\Omega))} \\
& \leq (\Delta)^{-\frac{1}{4}} C \| \psi \|_{L^\infty((0,T)\times\Omega)} \| \varrho_h \widehat{u}_h \|_{L^2(0,T;L^3(\Omega))},
\end{aligned}
\tag{9.11}
$$

where the last term is bounded by the energy (Corollary 4.3). By setting (9.11) in (9.10) and using that $\Delta t = ah$, we obtain

$$
\left| P_1 \left(\phi \mathcal{A}^{\mathrm{div}}[\psi \varrho_h \widehat{u}_h] \right) \right| \leq h^{\frac{1}{4} - 3\frac{4-\gamma}{4\gamma}} C,
\tag{9.12}
$$

where the exponent for h is strictly positive as $\gamma > 3$.

For the remaining P^i terms, we apply (5.7) in Lemma 5.1 to obtain

$$
\begin{aligned}
\sum_{i=2}^{4} & \left| P_i \left(\psi \mathcal{A}^{\nabla}[\phi \varrho_h]] \right) \right| \\
& \leq h^{\frac{1}{4}} C \left\| \nabla \left(\psi \mathcal{A}^{\nabla}[\phi \varrho_h]] \right) \right\|_{L^\infty(0,T;L^\gamma(\Omega))} \\
& \leq h^{\frac{1}{4}} C \| \psi \|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \| \phi \|_{L^\infty(\Omega)} \| \varrho_h \|_{L^\infty(0,T;L^\gamma(\Omega))} \leq h^{\frac{1}{4}} C,
\end{aligned}
\tag{9.13}
$$

where we in the second last inequality have applied Lemma 6.1. The last inequality follows from $p(\varrho_h) \in_{\mathrm{b}} L^\infty(0,T;L^1(\Omega))$.

Consequently, (9.12) and (9.13) allow us to conclude that

$$
\lim_{h\to 0} \left(\left| P_1 \left(\phi \mathcal{A}^{\mathrm{div}}[\psi \varrho_h \widehat{u}_h] \right) \right| + \sum_{i=2}^{4} \left| P_i \left(\psi \mathcal{A}^{\nabla}[\phi \varrho_h]] \right) \right| \right) = 0.
\tag{9.14}
$$

2. Convergence of the commutator term: To prove convergence of the first term after the equality in (9.9) we will need to rewrite this term on a form for which Lemma 9.3 is applicable:

$$
\begin{aligned}
& \int_0^T \int_\Omega \varrho_h \tilde{u}_h \nabla \left(\phi \mathcal{A}^{\mathrm{div}}[\psi \varrho_h \widehat{u}_h] \right) - \varrho_h \tilde{u}_h \otimes \widehat{u}_h : \nabla \left(\psi \mathcal{A}^{\nabla}[\phi \varrho_h] \right) \; dx dt \\
& = \int_0^T \int_\Omega \phi \varrho_h \tilde{u}_h \nabla \left(\mathcal{A}^{\mathrm{div}}[\psi \varrho_h \widehat{u}_h] \right) - \psi \varrho_h \tilde{u}_h \otimes \widehat{u}_h : \nabla \left(\mathcal{A}^{\nabla}[\phi \varrho_h] \right) \; dx dt
\end{aligned}
\tag{9.15}
$$

$$
+ \int_0^T \int_\Omega \varrho_h \tilde{u}_h \nabla \phi \cdot \mathcal{A}^{\mathrm{div}}[\psi \varrho_h \widehat{u}_h] - \varrho_h \tilde{u}_h \otimes \widehat{u}_h : \nabla \psi \otimes \mathcal{A}^{\nabla}[\phi \varrho_h] \; dx dt.
$$

Now, observe that the first term after the equality is precisely the one covered by Lemma 9.3. Hence,

$$
\begin{aligned}
\lim_{h\to 0} & \int_0^T \int_\Omega \phi \varrho_h \widetilde{u}_h \nabla \left(\mathcal{A}^{\mathrm{div}}[\psi \varrho_h \widehat{u}_h] \right) - \psi \varrho_h \widetilde{u}_h \otimes \widehat{u}_h : \nabla \left(\mathcal{A}^{\nabla}[\phi \varrho_h] \right) \; dx dt \\
& = \int_0^T \int_\Omega \phi \varrho u \nabla \left(\mathcal{A}^{\mathrm{div}}[\psi \varrho u] \right) - \psi \varrho u \otimes u : \nabla \left(\mathcal{A}^{\nabla}[\phi \varrho] \right) \; dx dt.
\end{aligned}
\tag{9.16}
$$

Moreover, in view of the convergences in Lemma 7.1, we have that

$$\mathcal{A}^\nabla[\phi\varrho_h] \to \mathcal{A}^\nabla[\phi\varrho] \text{ in } C(0,T;L^p(\Omega)) \text{ for any } p < \infty$$

$$\mathcal{A}^{\mathrm{div}}[\psi\varrho_h\widehat{u}_h] \to \mathcal{A}^{\mathrm{div}}(\psi\varrho u) \text{ in } C(0,T;L^3(\Omega)).$$

As a consequence, there is no problems with concluding that

$$\lim_{h\to 0} \int_0^T \int_\Omega \varrho_h\widetilde{u}_h\nabla\phi \cdot \mathcal{A}^{\mathrm{div}}[\psi\varrho_h\widehat{u}_h] - \varrho_h\widetilde{u}_h \otimes \widehat{u}_h : \nabla\psi \otimes \mathcal{A}^\nabla[\phi\varrho_h] \; dxdt$$

$$= \int_0^T \int_\Omega \varrho u\nabla\phi \cdot \mathcal{A}^{\mathrm{div}}[\psi\varrho u] - \varrho\widetilde{u} \otimes u : \nabla\psi \otimes \mathcal{A}^\nabla[\phi\varrho] \; dxdt.$$

$$(9.17)$$

Finally, we send $h \to 0$ in (9.15) and apply (9.16) and (9.17) to conclude

$$\lim_{h\to 0} \int_0^T \int_\Omega \varrho_h\widetilde{u}_h\nabla\left(\phi\mathcal{A}^{\mathrm{div}}[\psi\varrho_h\widehat{u}_h]\right) - \varrho_h\widetilde{u}_h \otimes \widehat{u}_h : \nabla\left(\psi\mathcal{A}^\nabla[\phi\varrho_h]\right) \; dxdt$$

$$= \int_0^T \int_\Omega \phi\varrho u\nabla\left(\mathcal{A}^{\mathrm{div}}[\psi\varrho u]\right) - \psi\varrho u \otimes u : \nabla\left(\mathcal{A}^\nabla[\phi\varrho]\right) \; dxdt$$

$$\int_0^T \int_\Omega \varrho u\nabla\phi \cdot \mathcal{A}^{\mathrm{div}}[\psi\varrho u] - \varrho\widetilde{u} \otimes u : \nabla\psi \otimes \mathcal{A}^\nabla[\phi\varrho] \; dxdt \qquad (9.18)$$

$$= \int_0^T \int_\Omega \varrho u\nabla(\psi\mathcal{A}^{\mathrm{div}}(\phi\varrho u)) - \varrho u \otimes u : \nabla\left(\psi\mathcal{A}^\nabla[\phi\varrho]\right) \; dxdt.$$

3. Conclusion: We conclude the proof by sending $h \to 0$ in (9.9) and applying (9.14) and (9.18).

$\square$

9.2. **Proof of Proposition 9.1.** Define the test-function

$$v = \psi\mathcal{A}^\nabla[\phi\varrho_h], \qquad v_h = \Pi_h^V v.$$

By setting v_h as test-function in the momentum scheme (3.6), integrating in time, applying Lemma 8.3 for the term involving $\nabla_h u_h \nabla_h v_h$, the calculation (6.4) for the term involving the pressure, and reordering terms, we obtain

$$\int_0^T \int_\Omega \phi\psi(p(\varrho_h) - \mathrm{div}_h u_h)\varrho_h \; dxdt = \sum_{i=1}^4 J_i^h + \mathcal{F}(\varrho_h), \qquad (9.19)$$

where $\mathcal{F}(\varrho_h)$ is given by (8.5) and

$$J_1^h = -\int_0^T \int_\Omega p(\varrho_h)\nabla\phi \cdot \mathcal{A}^\nabla[\phi\varrho_h] \; dxdt,$$

$$J_2^h = \int_0^T \int_\Omega \mathrm{curl}_h u_h \nabla\psi \times \mathcal{A}^\nabla[\phi\varrho_h] + \mathrm{div}_h u_h \nabla\psi \cdot \mathcal{A}^\nabla[\phi\varrho_h] \; dxdt,$$

$$J_3^h = \int_0^T \int_\Omega D_t^h(\varrho_h\widehat{u}_h)v_h \; dxdt - \sum_\Gamma \int_0^T \int_\Gamma \mathrm{Up}(\varrho u \otimes \widehat{u}) [\![\widehat{v}_h]\!]_\Gamma \; dS(x)dt,$$

$$J_4^h = h^{1-\epsilon}\sum_\Gamma \int_0^T \int_\Gamma \left(\frac{\widehat{u}_+ + \widehat{u}_-}{2}\right) [\![\varrho_h]\!]_\Gamma [\![\widehat{v}_h]\!]_\Gamma \; dS(x)dt,$$

Next, define the test-function

$$w = \psi \mathcal{A}^{\nabla}[\phi \varrho].$$

Observe that this test-function is the limit of v. That is,

$$v \to w \quad \text{a.e as } h \to 0.$$

Setting w as test-function in the weak limit of the momentum scheme (7.7), and reordering terms, yields

$$\int_0^T \int_\Omega \psi \phi (\overline{p(\varrho)} - \operatorname{div} u) \varrho \; dx dt = J_1 + J_2 + J_3. \tag{9.20}$$

$$J_1 = -\int_0^T \int_\Omega \overline{p(\varrho)} \nabla \phi \cdot \mathcal{A}^{\nabla}[\phi \varrho] \; dx dt,$$

$$J_2 = \int_0^T \int_\Omega \operatorname{curl} u \nabla \psi \times \mathcal{A}^{\nabla}[\phi \varrho] + \operatorname{div} u \nabla \psi \cdot \mathcal{A}^{\nabla}[\phi \varrho] \; dx dt$$

$$J_3 = -\int_0^T \int_\Omega \varrho u w_t + \varrho u \otimes u : \nabla w \; dx dt - \int_\Omega m_0 w(0, \cdot) \; dx.$$

Now, observe that the proof of Proposition 9.1 is complete once we prove

$$J_1^h \to J_1, \quad J_2^h \to J_2, \quad J_3^h \to J_3, \quad J_4^h \to 0.$$

Since we have already established the convergences

$$\mathcal{A}^{\nabla}[\phi \varrho_h] \to \mathcal{A}^{\nabla}[\phi \varrho] \text{ in } C(0, T; L^p(\Omega)) \text{ for any } p < \infty$$
$$\mathcal{A}^{\operatorname{div}}[\psi \varrho_h \widehat{u}_h] \to \mathcal{A}^{\operatorname{div}}(\psi \varrho u) \text{ in } C(0, T; L^3(\Omega)),$$

we can immediately conclude that

$$J_1^h \to J_1 \qquad J_2^h \to J_2. \tag{9.21}$$

For the J_4^h term, the calculation (6.7) gives

$$|J_4^h| \le h^{\frac{1}{2}} C \to 0. \tag{9.22}$$

Hence, the only remaining ingredient in the proof of Proposition 9.1, is to establish $J_3^h \to J_3$. Indeed, let us for the moment take this result as granted (Lemma 9.5 below). Then, we can send $h \to 0$ in (9.19), using (8.5), (9.21), (9.22), to discover

$$\lim_{h \to 0} \int_0^T \int_\Omega \phi \psi (p(\varrho_h) - \operatorname{div}_h u_h) \varrho_h \; dx dt$$

$$= \lim_{h \to 0} \sum_{i=1}^4 J_i^h + \mathcal{F}(\varrho_h) = J_1 + J_2 + J_3$$

$$= \int_0^T \int_\Omega \phi \psi (\overline{p(\varrho)} - \operatorname{div}_h u) \varrho \; dx dt,$$

where the last equality is (9.20). This is precisely Proposition 9.1.

Hence, the proof is complete once we establish the following lemma.

Lemma 9.5. *Under the conditions of Proposition 9.1,*

$$\lim_{h \to 0} J_3^h = J_3,$$

where J_3^h and J_3 are given by (9.19) *and* (9.20)*, respectively.*

Proof of Lemma 9.5. To prove convergence of J_3^h, we shall need to rewrite the time derivative term in J_3^h using the continuity scheme. By adding and subtracting, and applying summation by parts, we deduce

$$\int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h) v_h \, dxdt$$

$$= -\int_0^{T-\Delta t} \int_\Omega \varrho_h \widehat{u}_h D_t^h(v(+\Delta t)) \, dxdt$$

$$- \int_\Omega \varrho_h^0 \widehat{u}_h^0 v(\Delta t) \, dx + \int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h)(v_h - v) \, dxdt$$

$$= -\int_{\Delta t}^T \int_\Omega (\varrho_h \widehat{u}_h)(\cdot - \Delta t) \, D_t^h \left(\psi \mathcal{A}^\nabla \left[\phi \varrho_h \right] \right) \, dxdt$$

$$- \int_\Omega \varrho_h^0 \widehat{u}_h^0 v(\Delta t) \, dx + \int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h)(v_h - v) \, dxdt$$

$$= -\int_{\Delta t}^T \int_\Omega (\varrho_h \widehat{u}_h \psi)(\cdot - \Delta t) \, \mathcal{A}^\nabla \left[\phi D_t^h \varrho_h \right] \, dxdt$$

$$- \int_{\Delta t}^T \int_\Omega (\varrho_h \widehat{u}_h)(\cdot - \Delta t) \, \mathcal{A}^\nabla \left[\phi \varrho_h \right] D_t^h \psi \, dxdt$$

$$- \int_\Omega \varrho_h^0 \widehat{u}_h^0 v(\Delta t) \, dx + \int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h)(v_h - v) \, dxdt$$

Next, we apply the integration by parts formula for $\mathcal{A}^\nabla[\cdot]$ (Lemma 6.1)

$$\int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h) v_h \, dxdt$$

$$= \int_{\Delta t}^T \int_\Omega \mathcal{A}^{\mathrm{div}}[(\varrho_h \widehat{u}_h \psi)(\cdot - \Delta t)] \phi D_t^h \varrho_h$$

$$- \int_{\Delta t}^T \int_\Omega (\varrho_h \widehat{u}_h)(\cdot - \Delta t) \, \mathcal{A}^\nabla \left[\phi \varrho_h \right] D_t^h \psi \, dxdt$$

$$- \int_\Omega \varrho_h^0 \widehat{u}_h^0 v(\Delta t) \, dx + \int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h)(v_h - v) \, dxdt.$$

We then rewrite the first term using the continuity scheme (3.5) with

$$q_h = \Pi_h^Q \left[\phi \mathcal{A}^{\mathrm{div}}[(\varrho_h \widehat{u}_h \psi)(\cdot - \Delta t)] \right].$$

The resulting expression reads

$$\int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h) v_h \, dxdt$$

$$= \sum_\Gamma \int_{\Delta t}^T \int_\Gamma \mathrm{Up}(\varrho u) \left[\!\!\left[\Pi_h^Q \left[\phi \mathcal{A}^{\mathrm{div}}[(\varrho_h \widehat{u}_h \psi)(\cdot - \Delta t)] \right] \right]\!\!\right]_\Gamma \, dS(x)dt$$

$$- h^{1-\epsilon} \sum_\Gamma \int_0^T \int_\Gamma [\![\varrho_h]\!] \left[\!\!\left[\Pi_h^Q \left[\phi \mathcal{A}^{\mathrm{div}}[(\varrho_h \widehat{u}_h \psi)(\cdot - \Delta t)] \right] \right]\!\!\right]_\Gamma dS(x)dt$$

$$- \int_{\Delta t}^T \int_\Omega (\varrho_h \widehat{u}_h)(\cdot - \Delta t) \, \mathcal{A}^\nabla [\phi \varrho_h] \, D_t^h \psi \; dxdt \qquad (9.23)$$

$$- \int_\Omega \varrho_h^0 \widehat{u}_h^0 v(\Delta t) \; dx + \int_0^T \int_\Omega D_t^h (\varrho_h \widehat{u}_h)(v_h - v) \; dxdt,$$

By applying (9.23) in the expression for J_3^h, we obtain

$$J_3^h = \sum_\Gamma \int_{\Delta t}^T \int_\Gamma \mathrm{Up}(\varrho u) \left[\!\!\left[\Pi_h^Q \left[\phi \mathcal{A}^{\mathrm{div}}[(\varrho_h \widehat{u}_h \psi)(\cdot - \Delta t)] \right] \right]\!\!\right]_\Gamma dS(x)dt$$

$$- \sum_\Gamma \int_0^T \int_\Gamma \mathrm{Up}(\varrho u \otimes \widehat{u}) \left[\!\!\left[\Pi_h^Q \Pi_h^V \left[\psi \mathcal{A}^\nabla [\varrho_h \phi] \right] \right]\!\!\right]_\Gamma dS(x)dt$$

$$- h^{1-\epsilon} \sum_\Gamma \int_0^T \int_\Gamma [\![\varrho_h]\!]_\Gamma \left[\!\!\left[\Pi_h^Q \left[\phi \mathcal{A}^{\mathrm{div}}[(\varrho_h \widehat{u}_h \psi)(\cdot - \Delta t)] \right] \right]\!\!\right]_\Gamma dS(x)dt$$

$$- \int_{\Delta t}^T \int_\Omega (\varrho_h \widehat{u}_h)(\cdot - \Delta t) \, \mathcal{A}^\nabla [\phi \varrho_h] \, D_t^h \psi \; dxdt$$

$$- \int_\Omega \varrho_h^0 \widehat{u}_h^0 v(\Delta t) \; dx + \int_0^T \int_\Omega D_t^h (\varrho_h \widehat{u}_h)(v_h - v) \; dxdt.$$

In order to apply Lemma 9.4 we need to remove the time shift $-\Delta t$ in the first term. By adding and subtracting, we obtain

$$J_3^h = \sum_\Gamma \int_{\Delta t}^T \int_\Gamma \mathrm{Up}(\varrho u) \left[\!\!\left[\Pi_h^Q \left[\phi \mathcal{A}^{\mathrm{div}}[\varrho_h \widehat{u}_h \psi] \right] \right]\!\!\right]_\Gamma dS(x)dt$$

$$- \sum_\Gamma \int_0^T \int_\Gamma \mathrm{Up}(\varrho u \otimes \widehat{u}) \left[\!\!\left[\Pi_h^Q \Pi_h^V \left[\psi \mathcal{A}^\nabla [\varrho_h \phi] \right] \right]\!\!\right]_\Gamma dS(x)dt$$

$$- \sum_\Gamma \int_{\Delta t}^T \int_\Gamma \mathrm{Up}(\varrho u) \Delta t \left[\!\!\left[\Pi_h^Q \left[\phi \mathcal{A}^{\mathrm{div}}[D_t^h (\varrho_h \widehat{u}_h \psi)] \right] \right]\!\!\right]_\Gamma dS(x)dt$$

$$- h^{1-\epsilon} \sum_\Gamma \int_{\Delta t}^T \int_\Gamma [\![\varrho_h]\!] \left[\!\!\left[\Pi_h^Q \left[\phi \mathcal{A}^{\mathrm{div}}[(\varrho_h \widehat{u}_h \psi)(\cdot - \Delta t)] \right] \right]\!\!\right]_\Gamma dS(x)dt$$

$$- \int_{\Delta t}^T \int_\Omega (\varrho_h \widehat{u}_h)(\cdot - \Delta t) \, \mathcal{A}^\nabla [\phi \varrho_h] \, D_t^h \psi \; dxdt \qquad (9.24)$$

$$- \int_\Omega \varrho_h^0 \widehat{u}_h^0 v(\Delta t) \; dx + \int_0^T \int_\Omega D_t^h (\varrho_h \widehat{u}_h)(v_h - v) \; dxdt.$$

$$:= \sum_{i=1}^7 K_i,$$

where the third term contains the time difference.

Now, observe that $K_1 + K_2$ is exactly the terms covered by Lemma 9.4. In particular,

$$
\begin{aligned}
\lim_{h \to 0} K_1 + K_2 \\
= \int_0^T \int_\Omega \varrho u \nabla \left(\phi \mathcal{A}^{\mathrm{div}}[\psi \varrho u] \right) - \varrho u \otimes u : \nabla(\psi \mathcal{A}^\nabla[\phi \varrho]) \; dx dt.
\end{aligned}
\tag{9.25}
$$

To bound the K_3 term (the new term), we apply Lemma 5.1 and Proposition 5.2 to deduce

$$
\begin{aligned}
|K_3| &= \left| \sum_\Gamma \int_0^T \int_\Gamma \mathrm{Up}(\varrho u) \Delta t \left[\!\left[\Pi_h^Q \left[\phi \mathcal{A}^{\mathrm{div}}[D_t^h(\varrho_h \widehat{u}_h \psi)] \right] \right]\!\right]_\Gamma \; dS(x) dt \right| \\
&= \Delta t \int_{\Delta t}^T \int_\Omega \varrho_h \tilde{u}_h : \nabla \left(\phi \mathcal{A}^{\mathrm{div}}[D_t^h(\varrho_h \widehat{u}_h \psi)] \right) \; dx dt \\
&\qquad + (\Delta t) P_1 \left(\phi \mathcal{A}^{\mathrm{div}}[D_t^h(\varrho_h \widehat{u}_h \psi)] \right) \\
&\leq \|\varrho_h \tilde{u}_h\|_{L^2(0,T;L^{\frac{6\gamma}{3+\gamma}}(\Omega))} \|\varrho_h \widehat{u}_h(t) - \varrho_h \widehat{u}_h(t - \Delta t)\|_{L^2(\Delta t, T; L^{\frac{6\gamma}{5\gamma-3}}(\Omega))} \\
&\qquad + h^{\frac{1}{2} - \max\{3\frac{4-\gamma}{4\gamma}, 0\}} C \|\varrho_h \widehat{u}_h(t) - \varrho_h \widehat{u}_h(t - \Delta t)\|_{L^4(\Delta t, T; L^{\frac{12}{5}}(\Omega))} \\
&\leq C \|\varrho_h \widehat{u}_h(t) - \varrho_h \widehat{u}_h(t - \Delta t)\|_{L^2(\Delta t, T; L^{\frac{6\gamma}{5\gamma-3}}(\Omega))} \\
&\qquad + h^{\alpha(\delta,\gamma)} C \|\varrho_h \widehat{u}_h(t) - \varrho_h \widehat{u}_h(t - \Delta t)\|_{L^{2-\delta}(\Delta t, T; L^{\frac{12}{5}}(\Omega))}
\end{aligned}
\tag{9.26}
$$

where $\alpha(\gamma, \delta) = \frac{1}{4} - \max\{3\frac{4-\gamma}{4\gamma}, 0\} - \frac{\delta}{2(2-\delta)}$ and $\delta > 0$ is a small number. To achieve the previous inequality, we have applied the inverse inequality in time (Lemma 2.9).

Now, from Lemma 5.3, we know what

$$
\varrho_h \widehat{u}_h(t) - \varrho_h \widehat{u}_h(t - \Delta t) \to 0 \quad \text{a.e on } (0,T) \times \Omega.
$$

Hence, since $\varrho_h \widehat{u}_h \in L^\infty(0,T;L^{\frac{3}{2}}(\Omega)) \cap L^2(0,T;L^3(\Omega))$,

$$
\varrho_h \widehat{u}_h(t) - \varrho_h \widehat{u}_h(t - \Delta t) \to 0 \quad \text{in } L^{p_1}(0,T;L^{q_1}(\Omega)) \cap L^{p_2}(0,T;L^{q_2}(\Omega)),
$$

for any $p_1 < \infty$, $q_1 < \frac{2\gamma}{\gamma+1}$, $p_2 < 2$, and $q_2 < \frac{6\gamma}{3+\gamma}$. Moreover, since $\gamma > 3$, we can choose δ small such that $\alpha(\gamma, \delta) > 0$. As a consequence, we can pass to the limit in (9.26) to obtain

$$
\lim_{h \to 0} |K_3| = 0.
\tag{9.27}
$$

The K_4 term is directly bounded by Lemma 5.5;

$$
\begin{aligned}
|K_4| &= \left| h^{1-\epsilon} \sum_\Gamma \int_{\Delta t}^T \int_\Gamma [\![\varrho_h]\!]_\Gamma \left[\!\left[\Pi_h^Q \left[\phi \mathcal{A}^{\mathrm{div}}[\varrho_h \widehat{u}_h \psi(\cdot - \Delta t)] \right] \right]\!\right]_\Gamma \; dS(x) dt \right| \\
&\leq h^{\frac{11-4\epsilon}{8}} C \left\| \nabla \left(\phi \mathcal{A}^{\mathrm{div}}[\varrho_h \widehat{u}_h \psi(\cdot - \Delta t)] \right) \right\|_{L^2(\Delta t, T; L^2(\Omega))} \\
&\leq h^{\frac{11-4\epsilon}{8}} C \|\varrho_h \widehat{u}_h\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}
\tag{9.28}
$$

where the norm is bounded by Corollary 4.3.

Next, since

$$\varrho_h \widehat{u}_h \rightharpoonup \varrho u \quad \text{in } L^2(0,T;L^{m_2}(\Omega)), \, m_2 > 3,$$

and $\mathcal{A}^\nabla[\phi\varrho_h] \to \mathcal{A}^\nabla[\phi\varrho]$ in $C(0,T;L^p(\Omega))$, for any $p < \infty$, we see that

$$
\begin{aligned}
&\lim_{h\to 0} (K_5 + K_6) \\
&= -\lim_{h\to 0} \int_{\Delta t}^T \int_\Omega (\varrho_h \widehat{u}_h)(\cdot - \Delta t)\, \mathcal{A}^\nabla[\phi\varrho_h]\, D_t^h \psi \; dxdt \\
&\quad - \lim_{h\to 0} \int_\Omega \varrho_h^0 \widehat{u}_h^0 v(\Delta t) \; dx \\
&= -\int_0^T \int_\Omega \varrho u \mathcal{A}^\nabla[\phi\varrho]\psi_t \; dxdt - \int_\Omega m_0 w_0 \; dx.
\end{aligned}
\tag{9.29}
$$

To bound K_7, we apply Lemma 5.3 and the interpolation error for Π_h^V and obtain

$$
\begin{aligned}
|K_7| = \left| \int_0^T \int_\Omega D_t^h(\varrho_h \widehat{u}_h)(v_h - v) \; dxdt \right| &\le h^{\frac{1}{\gamma}} C \|\nabla v\|_{L^\infty(0,T;L^3(\Omega))} \\
&\le h^{\frac{1}{\gamma}} C \|\varrho_h\|_{L^\infty(0,T;L^\gamma(\Omega))},
\end{aligned}
\tag{9.30}
$$

which is bounded by Corollary 4.3.

At this stage, we can send $h \to 0$ in (9.24), using (9.25), (9.27), (9.28), (9.29), and (9.30), to obtain

$$
\begin{aligned}
\lim_{h\to 0} J_3^h = \int_0^T \int_\Omega \varrho u \nabla \left(\phi \mathcal{A}^{\text{div}}[\psi\varrho u] \right) - \varrho u \otimes u : \nabla(\psi \mathcal{A}^\nabla[\phi\varrho]) \; dxdt \\
- \int_0^T \int_\Omega \varrho u \mathcal{A}^\nabla[\phi\varrho]\psi_t \; dxdt - \int_\Omega m_0 w_0 \; dx.
\end{aligned}
\tag{9.31}
$$

Finally, since the limit (ϱ, u) is a solution to the continuity equation (Lemma 7.2), we have that (recall that ϕ does not depend on time)

$$
\begin{aligned}
&\int_0^T \int_\Omega \varrho u \nabla \left(\phi \mathcal{A}^{\text{div}}[\psi\varrho u] \right) \\
&= -\int_0^T \int_\Omega \phi\varrho \left(\mathcal{A}^{\text{div}}[\psi\varrho u] \right)_t \; dxdt - \int_\Omega \phi\varrho_0 \mathcal{A}^{\text{div}}[\psi(0,\cdot)m_0] \; dx \\
&= -\int_0^T \int_\Omega \left(\mathcal{A}^\nabla[\varrho\phi] \right)_t \psi\varrho u \; dxdt \\
&= -\int_0^T \int_\Omega \varrho u w_t \; dxdt + \int_0^T \int_\Omega \varrho u \, \mathcal{A}^\nabla[\phi\varrho]\, \psi_t \; dxdt.
\end{aligned}
$$

Thus, by using this identity in (9.31), we obtain

$$
\begin{aligned}
\lim_{h\to 0} J_3^h &= -\int_0^T \int_\Omega \varrho u w_t + \varrho u \otimes u : \nabla w \; dxdt - \int_\Omega m_0 w(0) \; dx \\
&= J_3,
\end{aligned}
$$

which was the desired result. $\qquad\square$

The proof of Proposition 9.1 is now complete.

10. Strong convergence and proof of Theorem 3.5

Equipped with Proposition 9.1 we can now obtain strong convergence of the density approximation. The following argument uses only the continuity approximation and is very similar to the corresponding argument in [14, 15, 16]. We include it here for the sake of completeness.

Lemma 10.1. *Let (ϱ_h, u_h) be the numerical solution obtain through Definition 3.1 and (3.7). The density approximation converges a.e*

$$\varrho_h \to \varrho \quad a.e \ on \ (0,T) \times \Omega.$$

Proof. From Lemma 4.1 with $B(\varrho) = \varrho \log \varrho$, we have the inequality

$$\int_\Omega \varrho_h \log \varrho_h \ dx \ (t) - \int_\Omega \varrho_h^0 \log \varrho_h^0 \ dx \le - \int_0^t \int_\Omega \varrho_h \operatorname{div} u_h \ dxdt.$$

From Lemma 2.3, we know that the limit (ϱ, u) is a renormalized solution to the continuity equation. In particular, the following identity holds

$$\int_\Omega \varrho \log \varrho \ dx \ (t) - \int_\Omega \varrho_0 \log \varrho_0 \ dx = - \int_0^t \int_\Omega \varrho \operatorname{div} u \ dxdt.$$

Note that there is no problems with integrability of $\varrho \operatorname{div} u$. Indeed, since $\varrho \in L^\infty(0,T; L^\gamma(\Omega))$ with $\gamma > 3$ and $\operatorname{div} u \in L^2(0,T; L^2(\Omega))$, Hölders inequality provides $\varrho \operatorname{div} u \in L^2(0,T; L^{\frac{6}{5}}(\Omega))$.

By subtracting the two previous identities, we obtain

$$\int_\Omega \varrho_h \log \varrho_h - \varrho \log \varrho \ dx \ (t)$$

$$\le \int_0^t \int_\Omega \varrho \operatorname{div} u - \varrho_h \operatorname{div}_h u_h \ dxdt + \int_\Omega \varrho_h^0 \log \varrho_h^0 - \varrho_0 \log \varrho_0 \ dx$$

$$= \int_0^{t^k} \int_\Omega \phi^2 (\varrho \operatorname{div} u - \varrho_h \operatorname{div}_h u_h) \ dxdt$$

$$+ \int_0^{t^k} \int_\Omega (1 - \phi^2)(\varrho \operatorname{div} u - \varrho_h \operatorname{div}_h u_h) \ dxdt$$

$$+ \int_\Omega \varrho_h^0 \log \varrho_h^0 - \varrho_0 \log \varrho_0 \ dx.$$

From Proposition 9.1, we have that

$$\lim_{h \to 0} \int_0^{t^k} \int_\Omega \phi^2 (\varrho \operatorname{div} u - \varrho_h \operatorname{div} u_h) \ dxdt$$

$$= \lim_{h \to 0} \int_0^{t^k} \int_\Omega \phi^2 (\varrho p(\varrho) - \varrho_h p(\varrho_h)) \ dxdt \le 0,$$

where the last inequality follows from the convexity of $p(\varrho)$. Hence, this, together with the strong convergence of the initial density, allow us to conclude

$$\int_\Omega \overline{\varrho \log \varrho} - \varrho \log \varrho \ dx \ (t) \le \int_0^t \int_\Omega (1 - \phi^2) Q \ dxdt,$$

where $Q = \varrho \operatorname{div} u - \overline{\varrho \operatorname{div} u} \in L^2(0,T; L^{\frac{6}{5}}(\Omega))$. Hence, we see that

$$0 \leq \int_\Omega \overline{\varrho \log \varrho} - \varrho \log \varrho \; dx \; (t) \leq C \left\| 1 - \phi^2 \right\|_{L^6(\Omega)},$$

where the first inequality follows by convexity of $z \mapsto z \log z$. Now, for any $\epsilon > 0$, we can choose $\phi \in W^{1,\infty}(\Omega)$ such that

$$0 \leq \int_\Omega \overline{\varrho \log \varrho} - \varrho \log \varrho \; dx \; (t) \leq \epsilon.$$

As a consequence, $\overline{\varrho \log \varrho} - \varrho \log \varrho = 0$ a.e and hence $\varrho_h \to \varrho$ a.e in $(0,T) \times \Omega$. $\qquad \square$

10.1. **Proof of the main result (Theorem 3.5).** To conclude the main result, the only remaining part is to prove that (ϱ, u) is a weak solution of the momentum equation (1.2) and that it satisfies the energy inequality (2.1). The other statements in Theorem 3.5 are all covered by (7.1), Lemma 7.1, Lemma 7.2, and Lemma 10.1. Since we now know that $\varrho_h \to \varrho$ a.e, we have in particular

$$\overline{p(\varrho)} = p(\varrho) \quad \text{a.e in } (0,T) \times \Omega.$$

By applying this information in (7.7), we immediately see that (ϱ, u) is a weak-solution of the momentum equation.

By passing to the limit $h \to 0$ in the numerical energy inequality (4.1) (Lemma 4.2), using convexity, we discover that the limit (ϱ, u) satisfies the energy inequality (2.1).

Appendix A. Existence of a numerical solution

Since the numerical method in Definition 3.1 is nonlinear and implicit it is not trivial that it is actually well-defined (i.e admits a solution). In addition, the discretization of the momentum transport is posed using element averages of the velocity. As we will see the only part of the discretization that provides sufficient number of equations to determine all the degrees of freedom of u_h is the discretization of the diffusion operator. Hence, in it's present form, our discretization is not suitable for the Euler equations.

The purpose of this section, is to prove the following the existence result which we have relied on in our analysis.

Proposition 3.3. *For each fixed $h > 0$, there exists a solution*

$$(\varrho_h^k, u_h^k) \in Q_h(\Omega) \times V_h(\Omega), \quad \varrho_h^k(\cdot) > 0, \quad k = 1, \dots, M,$$

to the numerical method posed in Definition 3.1.

To prove this result, we shall use a topological degree argument. The argument is strongly inspired by a very similar argument in the paper [12]. We will argue the existence of solutions to the following finite element map.

Definition A.1. Let the finite element map
$H : Q_h^+(\Omega) \times V_h(\Omega) \times [0,1] \mapsto Q_h(\Omega) \times V_h(\Omega)$ be given by

$$H(\varrho_h, u_h, \alpha) = (f_h(\alpha), g_h(\alpha)),$$

where $(f_h(\alpha), g_h(\alpha))$ are obtained through the mappings:

$$
\begin{aligned}
\int_\Omega f_h(\alpha) q_h \; dx = \int_\Omega \frac{\varrho_h - \varrho_h^{k-1}}{\Delta t} q_h \; dx \\
- \alpha \sum_\Gamma \int_\Gamma \mathrm{Up}(\varrho u) \, [\![q_h]\!]_\Gamma \; dS(x) \\
+ \alpha h^{1-\epsilon} \sum_\Gamma \int_\Gamma [\![\varrho_h]\!]_\Gamma \, [\![q_h]\!]_\Gamma \; dS(x),
\end{aligned}
\tag{A.1}
$$

for all $q_h \in Q_h(\Omega)$ and

$$
\begin{aligned}
\int_\Omega g_h(\alpha) v_h \; dx = \int_\Omega \frac{\varrho_h \widehat{u}_h - \varrho_h^{k-1} \widehat{u}_h^{k-1}}{\Delta t} v_h \; dx + \int_\Omega \nabla_h u_h \nabla_h v_h \; dx \\
- \alpha \sum_\Gamma \int_\Gamma \mathrm{Up}(\varrho u \otimes \widehat{u}) \, [\![\widehat{v}]\!]_\Gamma \; dS(x) \\
- \alpha \int_\Omega p(\varrho_h) \operatorname{div} v_h \; dx \\
- \alpha h^{1-\epsilon} \sum_E \int_{\partial E} \left(\frac{\widehat{u}_- + \widehat{u}_+}{2} \right) [\![\varrho_h]\!] \, \widehat{v}_h \; dS(x),
\end{aligned}
\tag{A.2}
$$

for all $v_h \in V_h(\Omega)$.

Observe that a solution of $H(\varrho_h, u_h, 1) = (0,0)$ is a solution to our numerical method as posed in Definition 3.1.

Before proceeding, let us make clear what we mean by topological degree in the present finite element context and denote by $d(F, \Omega, y)$ the $\mathbb{Z}$–valued (Brouwer) degree of a continuous function $F : \bar{O} \to \mathbb{R}^M$ at a point $y \in \mathbb{R}^M \backslash F(\partial O)$ relative to an open and bounded set $O \subset \mathbb{R}^M$.

Definition A.2. Let S_h be a finite element space, $\| \cdot \|$ be a norm on this space, and introduce the bounded set

$$
\tilde{S}_h = \left\{ q_h \in S_h; \| q_h \| \leq C \right\},
$$

where $C > 0$ is a constant. Let $\{\sigma_i\}_{i=1}^M$ be a basis such that $\operatorname{span}\{\sigma_i\}_{i=1}^M = S_h$ and define the operator $\Pi_\mathcal{B} : S_h \to \mathbb{R}^M$ by

$$
\Pi_\mathcal{B} q_h = (q_1, q_2, \ldots, q_M), \qquad q_h = \sum_{i=1}^M q_i \sigma_i.
$$

The degree $d_{S_h}(F, \tilde{S}_h, q_h)$ of a continuous mapping $F : \tilde{S}_h \to S_h$ at $q_h \in S_h \backslash F(\partial \tilde{S}_h)$ relative to $\tilde{S}_h$ is defined as

$$
d_{S_h}(F, \tilde{S}_h, q_h) = d\left(\Pi_\mathcal{B} F(\Pi_\mathcal{B}^{-1}), \Pi_\mathcal{B} \tilde{S}_h, \Pi_\mathcal{B} q_h \right).
$$

The next lemma is a consequence of some basic properties of the degree, cf. [7].

Lemma A.3. *Fix a finite element space S_h, and let $d_{S_h}(F, \tilde{S}_h, q_h)$ be the associated degree of Definition A.2. The following properties hold:*

(1) *$d_{S_h}(F, \tilde{S}_h, q_h)$ does not depend on the choice of basis for S_h.*
(2) *$d_{S_h}(\mathrm{Id}, \tilde{S}_h, q_h) = 1$.*

(3) $d_{S_h}(H(\cdot, \alpha), \tilde{S}_h, q_h(\alpha))$ *is independent of* $\alpha \in J := [0,1]$ *for* $H : \tilde{S}_h \times J \to S_h$ *continuous,* $q_h : J \to S_h$ *continuous, and* $q_h(\alpha) \notin H(\partial \tilde{S}_h, \alpha)$ $\forall \alpha \in [0,1]$.

(4) $d_{S_h}(F, \tilde{S}_h, q_h) \neq 0 \Rightarrow F^{-1}(q_h) \neq \emptyset$.

To prove Proposition 3.3, we shall apply Lemma A.3 with $q_h = 0$ and mapping H given by Definition A.1. Let us first prove that our mapping H satisfies (3) in Lemma A.3.

Lemma A.4. *Let* $H : Q_h^+(\Omega) \times V_h(\Omega) \times [0,1] \mapsto Q_h(\Omega) \times V_h(\Omega)$ *be the finite element mapping of Definition A.1. There is a subset* $\tilde{S}_h \subset Q_h^+(\Omega) \times V_h(\Omega)$ *for which* $H : \tilde{S}_h \times J \to S_h$ *is continuous and the zero solution* $(0,0) \notin H(\partial \tilde{S}, \alpha)$ *for all* $\alpha \in [0,1]$.

Proof. For any subset $\tilde{S} \subset Q_h^+(\Omega) \times V_h(\Omega)$ bounded independently of α, the corresponding mapping $H(\tilde{S}_h, \alpha, 0)$ is clearly continuous. This follows directly from (A.1) and (A.2) using the equivalence of finite dimensional norms. The more involved part is to determine a subset $\tilde{S}_h$ for which $(0,0) \notin H(\partial \tilde{S}, \alpha)$ independently of α.

Now, let us for the moment assert the existence of of (ϱ, u) satisfying

$$H(\varrho_h, u_h, \alpha) = (0,0).$$

Then, from Lemma 3.4, we have that $\varrho_h > 0$ and moreover (A.1) yields

$$\int_\Omega \varrho_h \, dx = \int_\Omega \varrho_h^{k-1} \, dx.$$

Consequently, we can conclude that

$$\|\varrho_h\|_{L^\infty(\Omega)} \leq C_\dagger, \tag{A.3}$$

independently of α.

To derive a bound on the velocity u_h, we can repeat the steps in the proof of Proposition 4.2 (the energy estimate) while keeping track of α to obtain

$$\int_\Omega \frac{\varrho_h |\hat{u}_h|^2}{2} + \frac{\alpha}{\gamma - 1} p(\varrho_h) \, dx + \Delta t \int_\Omega |\nabla u_h|^2 \, dx$$
$$\leq \int_\Omega \frac{\varrho_h^{k-1} |\hat{u}_h^{k-1}|^2}{2} + \frac{\alpha}{\gamma - 1} p(\varrho_h^{k-1}) \, dx$$
$$\leq \int_\Omega \frac{\varrho_h^{k-1} |\hat{u}_h^{k-1}|^2}{2} + \frac{1}{\gamma - 1} p(\varrho_h^{k-1}) \, dx \leq C,$$

where C is independent of α. Together with (A.3), this allow us to conclude

$$\|\varrho_h\|_{L^\infty(\Omega)} + \|u_h\|_{L^\infty(\Omega)} \leq C_\dagger.$$

We can now define the subspace

$$\tilde{S}_h = \left\{ (\varrho_h, u_h) \in Q_h^+ \times V_h; \|\varrho_h\|_{L^\infty(\Omega)} + \|u_h\|_{L^\infty(\Omega)} \leq C_\dagger \right\},$$

which by definition has the property that $(0,0) \notin H(\partial \tilde{S}, \alpha)$ for all $\alpha \in [0,1]$. This concludes the proof. $\qquad \square$

Lemma A.5. *Let $\tilde{S}$ be the subspace obtained by the previous lemma. Then, the topological degree of $H(\tilde{S}_h, 0)$ at $q_h = 0$ is non-zero:*

$$d_{S_h}(H(\cdot, 0), \tilde{S}_h, 0) \neq 0. \tag{A.4}$$

As a consequence, there exists $(\varrho_h, u_h) \in \tilde{S}_h$ such that

$$H(\varrho_h, u_h, 1) = (0, 0),$$

and hence Proposition 3.3 holds true.

Proof. First, we note that proving (A.4) is equivalent to proving the existence of $(\varrho_h, u_h) \in Q_h^+ \times V_h$ satisfying, for all $(q_h, v_h) \in Q_h \times V_h$,

$$
\begin{aligned}
\int_\Omega \varrho_h q_h \; dx &= \int_\Omega \varrho_h^{k-1} q_h \; dx, \\
\int_\Omega \varrho_h \widehat{u}_h v_h \; dx + \Delta t \int_\Omega \nabla_h u_h \nabla_h v_h \; dx &= \int_\Omega \varrho_h^{k-1} \widehat{u}_h^{k-1} v_h \; dx.
\end{aligned}
\tag{A.5}
$$

The first equation has the solution $\varrho_h = \varrho_h^{k-1}$. Setting this into the second equation in (A.5), we see that the resulting linear system is a sum of a positive matrix $\varrho_h^{k-1} \widehat{u}_h v_h$ and a symmetric positive definite matrix $\Delta t \nabla_h u_h \nabla_h v_h$. Since the Laplace problem with the Crouzeix-Raviart element space and dirichlet conditions is well-defined, there is no problems with concluding the existence of u_h satisfying the second equation in (A.5). $\qquad\square$

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