OPERATOR SPLITTING FOR WELL-POSED ACTIVE SCALAR EQUATIONS

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ABSTRACT. We analyze operator splitting methods applied to scalar equations with a nonlinear advection operator, and a linear (local or nonlocal) diffusion operator or a linear dispersion operator. The advection velocity is determined from the scalar unknown itself and hence the equations are so-called active scalar equations. Examples are provided by the surface quasi-geostrophic and aggregation equations. In addition, Burgers-type equations with fractional diffusion as well as the KdV and Kawahara equations are covered. Our main result is that the Godunov and Strang splitting methods converge with the expected rates provided the initial data is sufficiently regular.

1. INTRODUCTION AND MAIN RESULTS

We consider operator splitting applied to a class of evolution equations having a nonlinear "transport part", and a linear (local or nonlocal) "diffusion part" or a linear "dispersion part". These equations are posed on $\mathbb{R}^N$ for $N = 1, 2, 3$, and are of the form

$$u_t + \operatorname{div}(u\,\boldsymbol{v}(u)) = A(u), \tag{1.1}$$

where $u(t, x)$ is a scalar function and div is the spatial divergence operator.

The vector-valued operator $\boldsymbol{v}(\,\cdot\,)$ and the real-valued operator $A(\,\cdot\,)$ are linear and satisfy a set of hypotheses (given in Definition 2.3 below). These hypotheses are met, for example, by the popular fractional quasi-geostrophic [3, 4, 6, 7, 8, 10, 19] and aggregation equations [1, 2, 11, 12, 23, 24, 25]:

$$u_t + \operatorname{div}(u\,\boldsymbol{v}(u)) + (-\Delta)^{\alpha/2}u = 0, \quad \boldsymbol{v}(u) = \operatorname{curl}(-\Delta)^{-\beta/2}u, \tag{1.2}$$

$$u_t + \operatorname{div}(u\,\boldsymbol{v}(u)) + (-\Delta)^{\alpha/2}u = 0, \quad \boldsymbol{v}(u) = \nabla\Phi \star u, \tag{1.3}$$

where $\beta, \alpha \geq 1$. Both these equations are paramount examples of so-called *active scalar equations*, that is, equations in which the advection velocity $\boldsymbol{v}(u)$ is determined by the scalar unknown u itself. In three dimensions, the general formulation (1.1) encompasses also the active scalar equation [5]

$$u_t + \operatorname{div}(u\,\boldsymbol{v}(u)) - \Delta u = 0, \quad \boldsymbol{v}(u) = \operatorname{div}\mathbb{T}(u),$$

for which well-posedness was established recently in [14]. Here, $\mathbb{T}$ is a matrix of Calderon–Zygmund operators such that $\operatorname{div}\boldsymbol{v}(u) = \operatorname{div}\operatorname{div}\mathbb{T}(u) = 0$. This equation is a generalized 3D version of the quasi-geostrophic equation (1.2) ($\alpha = 2$).

In one dimension, our linearity requirement on $\boldsymbol{v}(\,\cdot\,)$ limits the type of equations we can consider to those of Burgers type, such as

$$u_t + (u^2)_x = u_{xxx}, \qquad \text{(KdV)},$$

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$$u_t + (u^2)_x = u_{xx}, \qquad \text{(viscous Burgers)},$$
$$u_t + (u^2)_x = -u_{xxx} + u_{xxxxx}, \qquad \text{(Kawahara)}.$$

Since the class of equations studied herein covers a variety of physical models, we will give a proper discussion of applications at the end of the paper.

The main topic of the present paper is analysis of operator splitting methods for constructing approximate solutions to (1.1). The tag "operator splitting" refers to the classical idea of constructing numerical methods for complicated partial differential equations by reducing the original equations to a series of equations with simpler structure, each of which can be handled by some efficient and tailored numerical method. We do not survey the literature on operator splitting here, referring the reader instead to the bibliography in [15].

The purpose of this paper is to prove, in the context of active scalar equations (1.1), the well-posedness and convergence rates for two frequently used operator splitting methods. Both methods are based on applying repeatedly the transport operator and the diffusion/dispersion operator A in separate steps. This splitting is very reasonable since one can then use "hyperbolic" numerical methods in the transport step and "Fourier space" methods in the diffusion/dispersion step. The methods are in the literature referred to as Godunov and Strang splitting and are widely used for both numerical computations and analysis. The reader can consult [15] for a recent survey of theory and applications; see also [22] for analysis of splitting algorithms for the incompressible Navier-Stokes equations.

Let us now discuss our splitting methods in more detail. For this purpose, we first recast (1.1) in the form

$$u_t = C(u), \quad C(u) = A(u) + B(u),$$

where we have introduced the operator $B(u) := -\operatorname{div}(u\boldsymbol{v}(u))$. We can then construct two solution operators Φ_A and Φ_B associated with the abstract ordinary differential equations

$$\partial_t \Phi_A(t, u_0) = A(\Phi_A(t, u_0)), \quad \Phi_A(0, u_0) = u_0,$$
$$\partial_t \Phi_B(t, u_0) = B(\Phi_B(t, u_0)), \quad \Phi_B(0, u_0) = u_0.$$

The first method we will consider, Godunov splitting, is defined as follows. For $\Delta t > 0$ given, construct a sequence $\{u^n, u^{n+1/2}\}_{n=1}^{\lfloor T/\Delta t \rfloor}$ of approximate solutions to (1.1) by the following procedure: Let $u^0 = u_0$ and determine inductively

$$u^{n+1/2} = \Phi_B(\Delta t, u^n), \quad u^{n+1} = \Phi_A(\Delta t, u^{n+1/2}), \quad n = 0, \dots, \lfloor T/\Delta t \rfloor - 1, \tag{1.4}$$

where $\lfloor z \rfloor$ gives the greatest integer less than or equal to z.

For Godunov splitting we prove that it is well-posed and that it convergences linearly in Δt. Specifically, we prove the following theorem.

Theorem 1.1. *Let $T > 0$ be given and assume $u_0 \in H^k$ with $6 \le k \in \mathbb{N}$. Then, for $\Delta t > 0$ sufficiently small we have the following:*

(1) *The Godunov method* (1.4) *is well-defined with*
$$\|u^n\|_{H^k} \le C, \quad n = 1, \dots, \lfloor T/\Delta t \rfloor.$$

(2) *The error satisfies*
$$\|u^n - u(n\Delta t)\|_{H^{k-\max\{\alpha,2\}}} \le C\|u_0\|_{H^k}\Delta t, \tag{1.5}$$
where α is the highest number of derivatives occuring in A.

The other method we consider, Strang splitting, is defined as follows: For $\Delta t > 0$ given, construct a sequence $\{u^n, u^{n+1/4}, u^{n+3/4}\}_{n=1}^{\lfloor T/\Delta t \rfloor}$ of approximate solutions to (1.1) by the following procedure: Let $u^0 = u_0$ and determine inductively, for $n = 0, \dots, \lfloor T/\Delta t \rfloor - 1$,

$$\begin{aligned} u^{n+1/4} &= \Phi_B(\frac{1}{2}\Delta t, u^n), \quad u^{n+3/4} = \Phi_A(\Delta t, u^{n+1/4}), \\ u^{n+1} &= \Phi_B(\frac{1}{2}\Delta t, u^{n+3/4}). \end{aligned} \tag{1.6}$$

For the Strang splitting algorithm we prove well-posedness and second order convergence, provided the initial data are sufficiently regular.

Theorem 1.2. *Let $T > 0$ be given and assume $u_0 \in H^k$ with $6 \le k \in \mathbb{N}$. Then, for $\Delta t > 0$ sufficiently small we have the following:*

(1) *The Strang method* (1.6) *is well-defined with*

$$\|u^n\|_{H^k} \le C, \quad n = 1, \dots, \lfloor T/\Delta t \rfloor.$$

(2) *The error satisfies* (1.5) *and*

$$\|u^n - u(n\Delta t)\|_{H^{k-3\max\{\alpha,1\}}} \le C(\Delta t)^2. \tag{1.7}$$

The α occurring in (1.7) *is the highest number of derivatives in A.*

The approach leading to Theorems 1.1 and 1.2 utilizes the analysis framework put forth in the recent paper [16] for the KdV equation. This framework works with a definition of the Godunov method given in terms of a specific extension of the splitting solution $\{u^n, u^{n+1/2}\}_n$ to all of $[0, T]$. An adaption of this extension that also applies to Strang splitting was provided in [17]. This extension, which is different from the one used in [16] for Strang splitting, will be employed herein. This allows us to treat in a unified manner a rather general class of equations, covering those treated in [16], [17], and [18], as well many additional equations not treated in these papers. At variance with [17], we do not require the divergence of the velocity field $\boldsymbol{v}(\,\cdot\,)$ to be zero, thereby enlarging significantly the class of equations that can be handled.

The remaining part of this paper is organized as follows: Section 2 collects some preliminary results needed later on. The Godunov and Strang splitting methods are analyzed in Sections 3 and 4, respectively.

2. Preliminary existence and regularity results

2.1. **Notation and two technical estimates.** We will use L^p to denote the Lebesgue space of integrable function on $\mathbb{R}^N$ with exponent p. If l denotes a N-dimensional multi-index, i.e., $l = (l_1, \dots, l_N)$, $l_j \in \mathbb{N}_0$, we write

$$D^l f = \nabla^l f = \frac{\partial^{|l|} f}{\partial x_1^{l_1} \cdots \partial x_N{}^{l_N}}, \qquad |l| = l_1 + \cdots + l_N,$$

to denote any derivative of the lth order. If $\ell \in \mathbb{N}$, we let

$$\nabla^\ell f = \{\nabla^l f \mid |l| = \ell\}$$

and

$$\nabla^\ell f : \nabla^\ell g = \sum_{|l| \le \ell} \nabla^l f \, \nabla^l g.$$

We will be working the Sobolev spaces

$$H^k = \{f \in \mathcal{S}' \mid (1 + |\xi|^2)^{k/2} \mathcal{F}(f(\xi)) \in L^2\},$$

(where $\mathcal{S}'$ denotes the set of tempered distributions) and $\mathcal{F}$ denotes Fourier transform. If k is a natural number, H^k is the standard Sobolev space with inner product and norm given by

$$\langle f, g\rangle_{H^k} = \sum_{\ell=0}^{k} \langle \nabla^\ell f, \nabla^\ell g\rangle_{L^2}, \quad \|f\|_{H^k} = \langle f, f\rangle_{H^k}^{1/2},$$

where we have introduced

$$\langle \nabla^\ell f, \nabla^\ell g\rangle_{L^2} = \sum_{\substack{l \\ |l|=\ell}} \langle D^l f, D^l g\rangle_{L^2}.$$

Throughout the paper we will strongly rely on the following two technical lemmas. The validity of these estimates are the primary reasons for the requirements on $\boldsymbol{v}$ given in Definition 2.3. Their proofs are straightforward, but somewhat tedious. For this reason, proofs are deferred to the appendix.

Lemma 2.1. *Let $k \geq 6$. Then*

$$\sum_{s=0}^{k} \left| \int_{\mathbb{R}^N} \nabla^s(\operatorname{div}(f\boldsymbol{v}(f))) : \nabla^s f \; dx \right| \leq C\|f\|_{H^{k-2}}\|f\|_{H^k}^2, \quad f \in H^k.$$

Lemma 2.2. *Let $k \geq 4$. Then the following estimates hold*

$$\sum_{s=0}^{k} \left| \int_{\mathbb{R}^N} \nabla^s \operatorname{div}(f\boldsymbol{v}(g)) : \nabla^s f \; dx \right| \leq C\|g\|_{H^k}\|f\|_{H^k}^2, \quad f, g \in H^k, \tag{2.1}$$

$$\sum_{s=0}^{k} \left| \int_{\mathbb{R}^N} \nabla^s \operatorname{div}(g\boldsymbol{v}(f)) : \nabla^s f \; dx \right| \leq C\|g\|_{H^{k+1}}\|f\|_{H^k}^2, \quad f \in H^k,\, g \in H^{k+1}. \tag{2.2}$$

2.2. **Existence and regularity results.** Our splitting methods are based on alternately solving the equations

$$u_t = A(u), \quad u|_{t=0} = u_0, \tag{2.3}$$

$$u_t + \operatorname{div}(u\,\boldsymbol{v}(u)) = 0, \qquad u|_{t=0} = u_0. \tag{2.4}$$

To analyze the methods we will need to have some control on the behavior of solutions to each of the two equations separately, at least on short time intervals. Before we start discussing available results, let us state the specific properties we will require of the operators $\boldsymbol{v}$ and A.

Definition 2.3. We say that the operators $\boldsymbol{v}, A$ are admissible provided:

(1) A and $\boldsymbol{v}$ are linear operators satisfying the commutative property

$$A(v_i(\,\cdot\,)) = v_i(A(\,\cdot\,)), \quad i = 1, \dots, N,$$

where v_i is the ith component of $\boldsymbol{v}$.

(2) $\boldsymbol{v} : L^p \to L^p$ is bounded for any $p < \infty$, and, if $N \geq 2$,

$$\|\operatorname{div} \boldsymbol{v}(u)\|_{L^p} \leq C\|u\|_{L^p}, \quad N \geq 2.$$

(3) A is a differential operator; specifically, there is a positive integer α such that $A\colon W^{\alpha+k,p} \to W^{k,p}$, is bounded for all $k < \infty$ and $p < \infty$.

(4) A is either conservative or diffusive:

$$\int_{\mathbb{R}^N} A(u)u \; dx \leq 0, \quad u \in W^{\alpha+k,p}.$$

(5) A satisfies the following commutator estimate

$$\|A(fg) - fA(g) - gA(f)\|_{H^k} \leq C\|f\|_{H^{k+\max\{\alpha,2\}-1}}\|g\|_{H^{k+\max\{\alpha,2\}-1}},$$

for all $f,\, g \in H^{k+\max\{\alpha,2\}-1}$ for $k \geq 3$.

(6) The equation (1.1) is well-posed in the sense that for any given finite $T > 0$ and initial data $u_0 \in H^k$, there is a solution of (1.1) satisfying

$$u \in C([0,T]; H^k).$$

The first equation (2.3) is a linear equation with constant coefficients ((1) in Definition 2.3) that preserves (or diffuses) all Sobolev norms over time. From this, the following lemma follows readily.

Lemma 2.4. *If $u_0 \in H^k$ for some $\alpha \le k < \infty$, then there is a unique solution $u \in C([0,T]; H^k)$ of (2.3).*

The nonlinear equation (2.4) is of hyperbolic type. Typically, the best available existence results for this type of equations are local in time existence of smooth solutions.

Lemma 2.5. *Let $u_0 \in H^k$ for some $k \ge 4$. There exists a time $T > 0$ and a function $u \in C([0,T); H^k) \cap C^1([0,T); H^{k-1})$ such that u is a unique solution of (2.4) on $[0,T)$. Moreover, if T is the maximal time of existence for (2.4), then*

$$\lim_{t \to T} \|u(t)\|_{H^k} = \infty.$$

Proof. We will prove the local in time existence by demonstrating compactness of solutions to the approximation scheme:

$$u^m_t + \boldsymbol{v}(u^{m-1}) \cdot \nabla u^m + u^m \operatorname{div} \boldsymbol{v}(u^{m-1}) = 0, \qquad u^m|_{t=0} = u_0. \tag{2.5}$$

For this purpose, we let $u^0 = u_0$ and sequentially determine the sequence $\{u^m\}_{m=1}^{\infty}$ as the solutions to the approximation scheme (2.5). Now, for each given u^{m-1}, the requirements of Definition 2.3 yields $\boldsymbol{v}(u^{m-1})$, $\operatorname{div} \boldsymbol{v}(u^{m-1}) \in C([0,T]; H^k)$. Thus, (2.5) is a linear transport equation with smooth coefficients and consequently admits a smooth solution for all times.

Let us now calculate the H^k norm of u^m. To achieve this, we apply ∇^s to (2.5), multiply with $\nabla^s u^m$, sum over all $s = 0, \dots, k$, and integrate to obtain

$$\begin{aligned} \partial_t \frac{1}{2} \|u^m(t)\|_{H^k}^2 &= -\sum_{s=0}^{k} \int_{\mathbb{R}^N} \left(\nabla^s \operatorname{div}(u^m \boldsymbol{v}(u^{m-1})) : \nabla^s u^m\right)(x,t)\, dx \\ &\le C \|u^{m-1}(t)\|_{H^k} \|u^m(t)\|_{H^k}^2, \end{aligned}$$

where the last inequality is an application of Lemma 2.2. Applying the Gronwall inequality to the previous inequality yields

$$\|u^m\|_{L^\infty([0,T];H^k)} \le e^{CT\|u^{m-1}\|_{L^\infty([0,T];H^k)}} \|u_0\|_{H^k}. \tag{2.6}$$

Next, we fix

$$T \le \frac{\log 2}{2C\|u_0\|_{H^k}},$$

where the constant C is the one appearing in (2.6). Using the bound on T in (2.6) and iterating the resulting inequality (starting from $m = 1$), we obtain

$$\|u^m\|_{L^\infty([0,T];H^k)} \le 2\|u_0\|_{H^k}.$$

Next, let us calculate the H^{k-1} norm of u^m_t. By direct calculation,

$$\begin{aligned} \|u^m_t(t)\|_{H^{k-1}} &= \left| \sum_{s=0}^{k-1} \int_{\mathbb{R}^N} \left(\nabla^s \operatorname{div}(u^m \boldsymbol{v}(u^{m-1})) : \nabla^s u^m_t\right)(x,t)\, dx \right| \\ &\le C \|u^m(t)\|_{H^k} \|u^{m-1}(t)\|_{H^k} \|u^m_t(t)\|_{H^{k-1}}. \end{aligned}$$

At this point, we have proved that $u^m \in C([0,T); H^k) \cap C^1([0,T); H^{k-1})$, independently of m. Hence, we can assert the existence of a function $u \in C([0,T); H^k) \cap C^1([0,T); H^{k-1})$ such that

$$u^m \rightharpoonup u \quad \text{in } C([0,T); H^k) \cap C^1([0,T); H^{k-1}),$$

as $m \to \infty$, where the convergence might take place along a subsequence. Compact Sobolev embedding then tells us that

$$u^m \to u, \quad \text{in } C([0,T); H^{k-1}),$$

again along a subsequence. Note that this is not sufficient to pass to the limit in (2.5). Indeed, we need that the whole sequence converges. To prove this, we let $w^m = u^m - u^{m-1}$ and observe that

$$w_t^m + \operatorname{div}(\boldsymbol{v}(u^{m-1})w^m) + \operatorname{div}(\boldsymbol{v}(w^{m-1})u^{m-1}) = 0.$$

It follows that

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|w^m\|_{L^2}^2 &= -\int \frac{1}{2}|w^m|^2 \operatorname{div} \boldsymbol{v}(u^{m-1}) + \operatorname{div}\left(\boldsymbol{v}(w^{m-1})u^{m-1}\right) w^m \, dx \\ &\leq C\|u^{m-1}\|_{H^k}\left(\|w^m\|_{L^2}^2 + \|w^{m-1}\|_{L^2}^2\right),\end{aligned}$$

where we have used the requirements on $\boldsymbol{v}(\cdot)$ (Definition 2.3), Sobolev embedding, and the Cauchy inequality. An application of the Gronwall inequality, using that $w^m(0) = 0$, we obtain

$$\begin{aligned}\|w^m\|_{L^2}(t) \leq Ce^{CT}\int_0^t \|w^{m-1}(s)\|_{L^2} \, ds &\leq \frac{\left(tCe^{CT}\right)^m}{m!} \sup_{t\in(0,T)} \|w^1\|_{L^2} \\ &\leq \frac{C^m}{m!} \sup_{t\in(0,T)} \|w^1\|_{L^2} \overset{m\to\infty}{\to} 0.\end{aligned}$$

From this we conclude that u^m is a Cauchy sequence in L^2 and hence that the entire sequence converges. We can now pass to the limit in (2.5) to conclude the existence part of the lemma. Uniqueness is an immediate consequence of the regularity of the solution (i.e subtracting one solution from a possibly different solution, adding and subtracting, and using the regularity).

The blow-up at maximal time of existence follows from u being continuous in time. □

3. Godunov splitting (proof of Theorem 1.1)

To prove Theorem 1.1, we will utilize the analysis framework put forth in [16]. To this end, we introduce an extension of the splitting solution $\{u^n, u^{n+1/2}\}_n$ to all of $[0,T]$. The definition is posed on the two-dimensional time domain

$$\Omega_{\Delta t} = \bigcup_{n=0}^{\lfloor T/\Delta t\rfloor - 1} [t_n, t_{n+1}] \times [t_n, t_{n+1}],$$

and goes as follows:

Definition 3.1 (Godunov splitting)**.** For $\Delta t > 0$ given, we say that ϑ is the Godunov splitting approximation to (1.1) provided that ϑ satisfies

$$\begin{aligned}\vartheta(0,0) &= \theta_0, \\ \vartheta_t(t, t_n) &= B(\vartheta(t, t_n)), \quad t \in (t_n, t_{n+1}], \\ \vartheta_\tau(t, \tau) &= A(\vartheta(t,\tau)), \quad (t,\tau) \in [t_n, t_{n+1}] \times (t_n, t_{n+1}].\end{aligned}$$

Observe that

$$\vartheta(t_n, t_n) = u^n, \quad n = 0, \dots, \lfloor T/\Delta t \rfloor - 1.$$

Thus, $\vartheta(t,t)$ is indeed an extension of u^n to all of $[0,T]$.

To measure the error, we will use the function

$$e(t) = \vartheta(t,t) - u(t),$$

where u is the (smooth) solution of (1.1).

Since ϑ is an extension of u^n to all of $[0,T]$, it is clear that Theorem 1.1 is an immediate consequence of the following lemma, which is our main result in this subsection.

Lemma 3.2. *Let $T > 0$ be given and assume $u_0 \in H^k$ with $6 \le k \in \mathbb{N}$. Then, for Δt sufficiently small we have the following:*

(1) *The Godunov method in Definition 3.1 is well-posed with $\vartheta \in C(\Omega_{\Delta t}; H^k)$.*
(2) *The error $e(t)$ satisfies*

$$\|e(t)\|_{H^{k-\max\{\alpha,2\}}} = \|\vartheta(t,t) - u(t)\|_{H^{k-\max\{\alpha,2\}}} \le C\|u_0\|_{H^k}^2 t\,\Delta t.$$

The proof of Lemma 3.2 will be a consequence of the results stated and proved in the ensuing subsections. The closing arguments will be given in Section 3.3.

3.1. **Error evolution equations.** The main benefit of the new definition of the Godunov method is that it allows us to derive continuous-in-time evolution equations for the error e. Though these equations was previously derived in [16], we repeat the derivation here for the convenience of the reader. For this purpose, we need the following Taylor expansion that holds for any smooth operator E

$$E(f+g) = E(f) + dE(f)[g] + \int_0^1 (1-\gamma) d^2E(f+\gamma g)[g]^2 \, d\gamma.$$

Using the definition of ϑ and the above Taylor formula, we deduce

$$\begin{aligned}
e_t - dC(u)[e] &= \vartheta_t + \vartheta_\tau - u_t - dA(u)[e] - dB(u)[e] \\
&= \vartheta_t + A(\vartheta) - (A+B)(u) - dA(u)[e] - dB(u)[e] \\
&= \vartheta_t - B(\vartheta) + (A(\vartheta) - A(u) - dA(u)[e]) \\
&\qquad + (B(\vartheta) - B(u) - dB(u)[e]) \\
&= F(t,t) + \int_0^1 (1-\gamma) d^2C(u+\gamma e)[e]^2 \, d\gamma,
\end{aligned} \tag{3.1}$$

where we have introduced the “forcing” term

$$F(t,\tau) = \vartheta_t(t,\tau) - B(\vartheta(t,\tau)).$$

By direct calculation,

$$\begin{aligned}
F_\tau - dA(\vartheta)[F] &= v_{t\tau} - B(\vartheta)_\tau - dA(\vartheta)[\vartheta_t - B(\vartheta)] \\
&= A(\vartheta)_t - dB(\vartheta)[\vartheta_\tau] - dA(\vartheta)[\vartheta_t] + dA(\vartheta)[B(\vartheta)] \\
&= dA(\vartheta)[\vartheta_t] - dB(\vartheta)[A(\vartheta)] - dA(\vartheta)[\vartheta_t] + dA(\vartheta)[B(\vartheta)] \\
&= [A,B](\vartheta),
\end{aligned} \tag{3.2}$$

where we have defined the commutator

$$[A,B](f) = dA(f)[B(f)] - dB(f)[A(f)].$$

In our case, the operator A is linear and hence

$$dA(f)[g] = A(g), \qquad d^2A(f)[g,h] = 0, \tag{3.3}$$

while the operator B satisfies

$$\begin{aligned} B(f) &= -\operatorname{div}(f\boldsymbol{v}(f)), \\ dB(f)[g] &= -\operatorname{div}(f\boldsymbol{v}(g) + g\boldsymbol{v}(f)), \\ d^2B(f)[g,h] &= -\operatorname{div}(h\boldsymbol{v}(g) + g\boldsymbol{v}(h)). \end{aligned} \tag{3.4}$$

This and the requirement (1) in Definition 2.3 yields

$$\begin{aligned} [A,B](f) &= -A\left(\operatorname{div}(f\boldsymbol{v}(f))\right) + \operatorname{div}\left(f\boldsymbol{v}(A(f)) + A(f)\boldsymbol{v}(f)\right) \\ &= -\operatorname{div}\left(A(f\boldsymbol{v}(f)) - fA(\boldsymbol{v}(f)) - A(f)\boldsymbol{v}(f)\right), \end{aligned}$$

which is exactly the divergence of the commutator appearing in (5) of Definition 2.3. Thus, the following lemma follows directly from this requirement.

Lemma 3.3. *Let $f \in H^k$, where α is given by (3) in Definition 2.3. Then,*

$$\|[A,B](f)\|_{H^{k-\max\{2,\alpha\}}} \leq C\|f\|_{H^k}^2.$$

Using (3.3) and (3.4) in (3.1) and (3.2), we obtain the following evolution equations for the error e

$$e_t + \operatorname{div}\left(e\boldsymbol{v}(e) + u\boldsymbol{v}(e) + e\boldsymbol{v}(u)\right) - A(e) = F, \quad t \in (0,T) \tag{3.5}$$

$$F_\tau - A(F) = [A,B](\vartheta), \quad (t,\tau) \in \Omega_{\Delta t}. \tag{3.6}$$

Upon inspection of these equations, we see that the error e satisfies an equation similar to (1.1), but with an additional source term F.

3.2. **Estimates valid under the assumption of regularity.** In this subsection we state and prove some results that will be needed in order to prove Lemma 3.2. Due to the special configuration of the time domain, it will be convenient to use the following notation for all times prior to a given time $(\sigma,\zeta) \in \Omega_{\Delta t}$:

$$\Omega_{\Delta t}^{\sigma,\zeta} = \left\{(t,\tau) \in \Omega_{\Delta t} \mid \ 0 \leq t \leq \sigma,\ 0 \leq \tau \leq \zeta\right\},$$

The following lemma is the most essential ingredient in the proof of Lemma 3.2.

Lemma 3.4. *Let $\vartheta(0,0) := u_0 \in H^k$. Assume the existence of a time $(\sigma,\zeta) \in [0,T]^2$ and a finite constant $\gamma > 0$ such that*

$$\|\vartheta(t,\tau)\|_{H^{k-\max\{\alpha,2\}}} \leq \gamma, \quad (t,\tau) \in \Omega_{\Delta t}^{\sigma,\zeta}.$$

There is a constant $C(\gamma)$ determined by γ, u_0, and T such that

$$\|\vartheta(t,\tau)\|_{H^k} \leq C(\gamma), \quad (t,\tau) \in \Omega_{\Delta t}^{\sigma,\zeta}.$$

Proof. Fix any $(t,\tau) \in \Omega_{\Delta t}^{\sigma,\tau}$ and let n be such that $t,\tau \in [t_n,t_{n+1}]$. Using the definition of ϑ (Definition 3.1), we see that

$$\partial_\tau \frac{1}{2}\|\vartheta(t,\tau)\|_{H^\ell}^2 = \sum_{s=0}^{\ell}\int_{\mathbb{R}^N} \nabla^s\vartheta_\tau : \nabla^s\vartheta\, dx = \sum_{s=0}^{\ell}\int_{\mathbb{R}^N} A(\nabla^s\vartheta) : \nabla^s\vartheta\, dx \leq 0,$$

for any $\ell = 1,\dots,k$, where the last inequality is requirement (4) in Definition 2.3. The previous inequality yields

$$\|\vartheta(t,\tau)\|_{H^k} \leq \|\vartheta(t,t_n)\|_{H^k}. \tag{3.7}$$

Next, we let $\ell \geq 4$ be an integer and apply Definition 3.1 to obtain

$$\begin{aligned} \partial_t \frac{1}{2}\|\vartheta(t,t_n)\|_{H^\ell}^2 &= -\sum_{s=0}^{\ell}\int_{\mathbb{R}^N} \nabla^s \operatorname{div}(\vartheta\boldsymbol{v}(\vartheta)) : \nabla^s\vartheta\, dx \\ &\leq C\|\vartheta\|_{H^{\ell-2}}\|\vartheta\|_{H^\ell}^2, \end{aligned}$$

where we have applied Lemma 2.1. If ℓ is such that $\ell - 2 \leq k - \max\{2, \alpha\}$, we can apply the Gronwall lemma to the previous inequality to obtain

$$\|\vartheta(t, t_n)\|_{H^\ell} \leq \|\vartheta(t_n, t_n)\|_{H^\ell} e^{C\gamma \Delta t}. \tag{3.8}$$

By combining (3.7) and (3.8), we thus obtain the bound

$$\|\vartheta(t, \tau)\|_{H^\ell} \leq \|u_0\|_{H^\ell} e^{C\gamma T}, \quad 4 \leq \ell \leq k - \max\{2, \alpha\}. \tag{3.9}$$

Now, we can repeat the above arguments with a new $\gamma := C(\gamma) = \|u_0\|_{H^\ell} e^{C\gamma T}$, to obtain

$$\|\vartheta(t, \tau)\|_{H^{\ell+2}} \leq \|u_0\|_{H^{\ell+2}} e^{C(\gamma)T}, \quad 4 \leq \ell \leq k - \max\{2, \alpha\}.$$

Clearly, we can repeat this process n times until $\ell + 2n = k$ (α even) or until $\ell + 2n = k - 1$ (α odd). In the last case, we can conclude that

$$\|\vartheta(t, \tau)\|_{H^{k-2}} \leq \|\vartheta(t, \tau)\|_{H^{k-1}} \leq \|u_0\|_{H^{k-1}} e^{C(\gamma)T},$$

and hence the arguments leading to (3.9) hold for $l = k$. □

In the following lemma, we give our main error estimate. Note that the result is only valid under a strong regularity assumption on the splitting solution ϑ. This assumption will be fully justified when we prove Lemma 3.2 in the next subsection.

Lemma 3.5. *Assume the existence of $(\sigma, \zeta) \in [0, T]^2$ such that the splitting solution*

$$\|\vartheta(t, \tau)\|_{H^k} \leq C(\gamma), \quad (t, \tau) \in \Omega^{\sigma,\zeta}_{\Delta t}.$$

Then,

$$\|e(t)\|_{H^{k-\max\{\alpha,2\}}} \leq \tilde{C}(\gamma)\Delta t, \quad t \leq \sigma.$$

Proof. To shorten the notation in this proof we write $\ell = k - \max\{\alpha, 2\}$. Let us commence by estimating the size of the source term F. For this purpose, we apply ∇^s to (3.6), multiply the result with $\nabla^s F$, integrate by parts, and sum over $s = 0, \dots, k$, to obtain

$$\begin{aligned}
\frac{1}{2}\partial_\tau \|F(t, \tau)\|^2_{H^\ell} &= \sum_{s=0}^{\ell} \int_{\mathbb{R}^N} \left(A(\nabla^s F) : \nabla^s F + \nabla^s [A, B](\vartheta) : \nabla^s F\right) \, dx \\
&\leq \left| \sum_{s=0}^{\ell} \int_{\mathbb{R}^N} \nabla^s [A, B](\vartheta) : \nabla^s F \, dx \right| \leq C\|F\|_{H^\ell} \, \|[A, B](\vartheta)\|_{H^\ell} \, .
\end{aligned}$$

Since $\ell = k - \max\{\alpha, 2\}$, we can apply Lemma 3.3 to obtain

$$\partial_\tau \|F(t, \tau)\|_{H^\ell} \leq \|[A, B](\vartheta)\|_{H^\ell} \leq C\|\vartheta\|^2_{H^k} \leq C(\gamma).$$

By definition $F(t, t_n) = 0$. Hence, integration in the τ direction from t_n to τ yields

$$\|F(t, \tau)\|_{H^\ell} \leq C(\gamma)\Delta t. \tag{3.10}$$

Let us now turn to the evolution equation (3.5) for the error. By applying ∇^s to (3.5), multiplying the result with $\nabla^s e$, integrating by parts, and summing over $s = 0, \dots, k$, we obtain

$$\begin{aligned}
\frac{1}{2}\partial_t \|e(t)\|^2_{H^\ell} &= \sum_{s=0}^{\ell} \Big(\int_{\mathbb{R}^N} A(\nabla^s e) : \nabla^s e \, dx + \int_{\mathbb{R}^N} \nabla^s F : \nabla^s e \, dx \Big) \\
&\qquad - \sum_{s=0}^{\ell} \int_{\mathbb{R}^N} \nabla^s \left(\operatorname{div}\left(e\boldsymbol{v}(e) + u\boldsymbol{v}(e) + e\boldsymbol{v}(u)\right)\right) : \nabla^s e \, dx \\
&\leq C\|F(t, t)\|_{H^\ell} \|e(t)\|_{H^\ell} + C\|u(t)\|_{H^{\ell+1}} \|e(t)\|^2_{H^\ell},
\end{aligned}$$

where the last inequality is an application of Lemma 2.2 and Hölder's inequality. Now, since $u_0 \in H^k$, requirement (6) in Definition 2.3 tells us that the analytical solution $u \in C([0,T];H^k)$. Thus, the previous inequality leads us to the conclusion

$$\partial_t \|e(t)\|_{H^\ell} \leq C\|F(t,t)\|_{H^\ell} + C\|e(t)\|_{H^\ell} \leq C\left(\Delta t + \|e(t)\|_{H^\ell}\right),$$

where the last inequality is (3.10). An application of the Gronwall inequality concludes the proof. □

3.3. **Proof of Lemma 3.2.** We will determine the maximal size of $\Delta t > 0$ during the course of the proof. Specifically, the size of Δt will be determined in accordance to the maximal existence time of solutions to the nonlinear transport equation (2.4). For the convenience of the reader we recall that the maximal existence time T^* of solutions u to (2.4) is characterized by

$$\lim_{t\to T^*} \|u\|_{H^k} = +\infty. \tag{3.11}$$

1. The proof of well-posedness ((1) in Lemma 3.2) will be a direct consequence of the following lemma.

Lemma 3.6. *Let $k \geq 6$ and $u_0 \in H^k$. There exists constants β, γ, depending only on u_0 and T, such that if $\Delta t < \beta$ and*

$$\|\vartheta(t,\tau)\|_{H^{k-\max\{\alpha,2\}}} \leq \gamma, \quad (t,\tau) \in \Omega^{\sigma,\zeta}_{\Delta t},$$

for some $(\sigma,\zeta) \in \Omega_{\Delta t}$, then

$$\|\vartheta(t,\tau)\|_{H^{k-\max\{\alpha,2\}}} \leq \frac{\gamma}{2}, \quad (t,\tau) \in \Omega^{\sigma,\zeta}_{\Delta t}.$$

Let us for the moment take Lemma 3.6 for granted and explain why it concludes our proof of Lemma 3.2. For this purpose, we let $\Delta t < \beta$, where β is dictated by Lemma 3.6. Since the initial data $\vartheta(0,0) = u_0 \in H^k$, our short-time existence result (Lemma 2.5) enables us to start constructing $\vartheta(t,0)$ according to Definition 3.1 up to some time T^*, that is, to start applying the B operator (the nonlinear transport equation (2.4)) in the first time step. Now, on the interval $(0,T^*)$ there is no problem determining γ in accordance to both Lemma 3.6 and such that

$$\|\vartheta(t,0)\|_{H^{k-\max\{\alpha,2\}}} \leq \gamma, \quad 0 \leq t < T^*.$$

Lemma 3.6 can then be applied to conclude that

$$\|\vartheta(t,0)\|_{H^{k-\max\{\alpha,2\}}} \leq \frac{\gamma}{2}, \quad 0 \leq t < T^*.$$

However, in view of (3.11), and since the norm is continuous in time there must exist an $\epsilon > 0$ such that

$$\|\vartheta(t,0)\|_{H^{k-\max\{\alpha,2\}}} \leq \gamma, \quad 0 \leq t \leq T^* + \epsilon.$$

Thus, by repeating these three steps, we can conclude that

$$\|\vartheta(t,0)\|_{H^{k-\max\{\alpha,2\}}} \leq \gamma, \quad 0 \leq t \leq \Delta t.$$

We can now start to determine $\vartheta(t,\tau)$, for $0 < \tau \leq \Delta t$, according to Definition 3.1, that is, to start solving with the A operator (the linear equation (2.3)). By virtue of our existence result for (2.3) (Lemma 2.4), we can repeat the above argument to conclude that $\vartheta(t,\tau)$ satisfies

$$\|\vartheta(t,\tau)\|_{H^{k-\max\{\alpha,2\}}} \leq \gamma, \quad 0 \leq t,\tau \leq \Delta t.$$

Besides well-posedness of the first time step, this result tell us that γ also bounds the $H^{k-\max\{\alpha,2\}}$ norm of $\vartheta(\Delta t,\Delta t)$, which is indeed the initial data in the next time step. Hence, we can repeat the entire process for all time steps recursively to conclude that

$$\|\vartheta(t,\tau)\|_{H^{k-\max\{\alpha,2\}}} \leq \gamma, \quad (t,\tau) \in \Omega_{\Delta t}.$$

Lemma 3.4 can the be applied to obtain

$$\|\vartheta(t,\tau)\|_{H^k} \le C(\gamma), \quad (t,\tau) \in \Omega_{\Delta t},$$

which concludes the proof of well-posedness, i.e., (1) of Lemma 3.2.

2. Since we now know that $\|\vartheta(t,\tau)\|_{H^k} \le C(\gamma)$, $(t,\tau) \in \Omega_{\Delta t}$, the postulates of Lemma 3.5 are satisfied. Thus,

$$\|e(t)\|_{H^{k-\max\{\alpha,2\}}} \le \Delta t C, \quad t \in (0,T),$$

which is precisely (2) in Lemma 3.2.

We conclude this section by proving Lemma 3.6.

Proof of Lemma 3.6. To shorten the notation, we let $\ell = k - \max\{\alpha, 2\}$. Assume that $(\sigma,\zeta) \in \Omega_{\Delta t}$ is such that

$$\|\vartheta(t,\tau)\|_{H^\ell} \le \gamma, \quad (t,\tau) \in \Omega_{\Delta t}^{\sigma,\tau}, \tag{3.12}$$

where the values of γ and Δt are still to be determined.

From Lemma 3.4 we know that (3.12) implies that

$$\|\vartheta(t,\tau)\|_{H^k} \le C(\gamma), \quad (t,\tau) \in \Omega_{\Delta t}^{\sigma,\tau}.$$

We can also apply Lemma 3.5 to obtain

$$\|e(t)\|_{H^\ell} \le \Delta t \tilde{C}(\gamma). \tag{3.13}$$

Now, fix any $(t,\tau) \in \Omega_{\Delta t}^{\sigma,\tau}$. By definition of the Godunov method (Definition 3.1),

$$\begin{aligned}\left|\partial_\tau \frac{1}{2}\|\vartheta(t,\tau)\|_{H^\ell}^2\right| &= \left|\sum_{s=0}^{k}\int_{\mathbb{R}^N} \nabla^s\vartheta_\tau : \nabla^s\vartheta \; dx\right| = \left|\sum_{s=0}^{k}\int_{\mathbb{R}^N} A(\nabla^s\vartheta) : \nabla^s\vartheta \; dx\right| \\ &\le C\|\vartheta\|_{H^\ell}\|A(\vartheta)\|_{H^\ell} \le C\|\vartheta\|_{H^\ell}\|\vartheta\|_{H^{\ell+\alpha}} \le C\|\vartheta\|_{H^\ell}\|\vartheta\|_{H^k},\end{aligned}$$

from which it follows that

$$|\partial_\tau\|\vartheta(t,\tau)\|_{H^\ell}| \le C\|\vartheta\|_{H^k}.$$

Using this inequality and Lemma 3.4, we find

$$\begin{aligned}\|\vartheta(t,\tau)\|_{H^\ell} &\le \|\vartheta(t,t)\|_{H^\ell} + \int_t^\tau |\partial_\tau\|\vartheta(t,\tilde{\tau})\|_{H^k}| \;\; d|tilde\tau \\ &\le \|\vartheta(t,t)\|_{H^\ell} + \Delta t C \sup_{\tilde{\tau}\in[t,\tau]} \|\vartheta(t,\tilde{\tau})\|_{H^k} \\ &\le \|\vartheta(t,t)\|_{H^\ell} + \Delta t C(\gamma) \\ &\le \|e(t)\|_{H^\ell} + \|u(t)\|_{H^\ell} + \Delta t C(\gamma) \\ &\le \Delta t \tilde{C}(\gamma) + C_2,\end{aligned}$$

by applying (3.13), we obtain the estimate

$$\begin{aligned}\|\vartheta(t,\tau)\|_{H^\ell} &\le \|e(t)\|_{H^\ell} + \|u(t)\|_{H^\ell} + \Delta t C(\gamma) \\ &\le \Delta t C(\gamma) + C_2.\end{aligned} \tag{3.14}$$

Finally, by fixing γ and Δt according to

$$\gamma = 4C_2, \qquad \Delta t \le \frac{C_2}{\tilde{C}(\gamma)} := \beta,$$

and applying this information to (3.14), we see that

$$\|\vartheta(t,\tau)\|_{H^\ell} \le \frac{\gamma}{2},$$

which concludes the proof. □

4. Strang splitting (proof of Theorem 1.2)

As for the Godunov method, our convergence analysis will require a continuous definition of the Strang method. In contrast to the Godunov case, we will now introduce three time variables instead of two [17]. We consider the domain:

$$\Omega_{\Delta t} = \bigcup_{n=0}^{\lfloor T/\Delta t\rfloor -1} [\frac{t_n}{2}, \frac{t_{n+1}}{2}] \times [t_n, t_{n+1}] \times [\frac{t_n}{2}, \frac{t_{n+1}}{2}].$$

The continuous Strang method is given by the following definition.

Definition 4.1. For $\Delta t > 0$ given, we say that ϑ is the Strang splitting approximation to (1.1) whenever ϑ solves

$$\begin{aligned}
&\vartheta(0,0,0) = u_0,\\
&\vartheta_t(t, t_n, \frac{t_n}{2}) = B(\vartheta(t, t_n, \frac{t_n}{2})), \quad t \in (\frac{t_n}{2}, \frac{t_{n+1}}{2}],\\
&\vartheta_\tau(t, \tau, \frac{t_n}{2}) = A(\vartheta(t, \tau, \frac{t_n}{2})), \quad (t,\tau) \in [\frac{t_n}{2}, \frac{t_{n+1}}{2}] \times (t_n, t_{n+1}],\\
&\vartheta_\omega(t, \tau, \omega) = B(\vartheta(t,\tau,\omega)), \quad (t,\tau,\omega) \in [\frac{t_n}{2}, \frac{t_{n+1}}{2}] \times [t_n, t_{n+1}] \times (\frac{t_n}{2}, \frac{t_{n+1}}{2}).
\end{aligned} \tag{4.1}$$

In each box $[\frac{t_n}{2}, \frac{t_{n+1}}{2}] \times [t_n, t_{n+1}] \times [\frac{t_n}{2}, \frac{t_{n+1}}{2}]$, we will mainly consider the function ϑ along the diagonal, i.e., the function $\vartheta(\frac{t}{2}, t, \frac{t}{2})$ for $t \in [t_n, t_{n+1}]$. Observe that each point on this diagonal is a Strang splitting solution for a specific time step. More precisely, $\vartheta(\frac{t}{2}, t, \frac{t}{2})$ is a Strang splitting solution with time step $t - t_n$. An easy consequence is that

$$\vartheta(\frac{t_n}{2}, t_n, \frac{t_n}{2}) = u^n, \quad n = 0, \dots, \lfloor T/\Delta t\rfloor,$$

and hence that $\vartheta(\frac{t}{2}, t, \frac{t}{2})$ can be seen as an extension of $\{u^n\}_n$ to all of $[0,T]$.

To measure the error, we will use the function

$$e(t) = \vartheta(\frac{t}{2}, t, \frac{t}{2}) - u(t),$$

where u is the (smooth) solution of (1.1).

Theorem 1.2 is an immediate consequence of the two following lemmas.

Lemma 4.2 (Well-posedness)**.** *Let ϑ the Strang splitting solution of* (1.1) *in the sense of Definition 4.1 and* (4.1). *Let $T > 0$ be given and assume $u_0 \in H^k$ with $6 \leq k \in \mathbb{N}$. Then, for $\Delta t > 0$ sufficiently small*

(1) *The Strang method is well-posed with $\vartheta \in C(\Omega_{\Delta t}; H^k)$.*
(2) *The error $e(t)$ satisfies*

$$\|e(t)\|_{H^{k-\max\{\alpha,2\}}} = \|\vartheta(\frac{t}{2}, t, \frac{t}{2}) - u(t)\|_{H^{k-\max\{\alpha,2\}}} \leq tC\Delta t.$$

Lemma 4.3 (Convergence)**.** *Let ϑ the Strang splitting solution of* (1.1) *in the sense of Definition 4.1 and* (4.1). *Let $T > 0$ be given and assume $u_0 \in H^k$ with $6 \leq k \in \mathbb{N}$. Then, for $\Delta t > 0$ sufficiently small*

$$\|e(t)\|_{H^{k-3\max\{\alpha,1\}}} \leq C(\Delta t)^2.$$

Lemmas 4.2 and 4.3 will be consequences of the results stated and proved in the ensuing subsections.

4.1. **Error evolution equations.** By direct calculation, we see that the error e satisfies the time-evolution

$$\begin{aligned}
e_t - dC(u)[e] &= \frac{\vartheta_t}{2} + \vartheta_\tau + \frac{\vartheta_\omega}{2} - u_t - dA(u)[e] - dB(u)[e] \\
&= \frac{\vartheta_t}{2} + \vartheta_\tau + \frac{1}{2}B(\vartheta) - (A+B)(u) - dA(u)[e] - dB(u)[e] \\
&= \frac{1}{2}\left(\vartheta_t - B(\vartheta)\right) + \vartheta_\tau - A(\vartheta) \\
&\qquad + \left(A(\vartheta) - A(u) - dA(u)[e]\right) + \left(B(\vartheta) - B(u) - dB(u)[e]\right) \\
&= F(t) + \int_0^1 (1-\gamma) d^2C(u+\gamma e)[e]^2 \, d\gamma,
\end{aligned}$$

where $F(t) = F(\frac{t}{2}, t, \frac{t}{2})$ and

$$F(t,\tau,\omega) = \frac{1}{2}\left(\vartheta_t(t,\tau,\omega) - B(\vartheta(t,\tau,\omega))\right) + \vartheta_\tau - A(\vartheta(t,\tau,\omega)).$$

Since $\vartheta_\tau - A(\vartheta(t,\tau,\omega)) = 0$ when $\omega = \frac{t_n}{2}$, $n = 0,\dots,\lfloor T/\Delta t\rfloor - 1$, we can apply the arguments of (3.2) to obtain

$$F_\tau - dA(\vartheta)[F] = \frac{1}{2}[A,B](\vartheta), \quad (t,\sigma,\frac{t_n}{2}) \in \Omega_{\Delta t}$$

where as before

$$[A,B](f) = dA(f)[B(f)] - dB(f)[A(f)].$$

We also derive the following equation for the evolution of F in ω:

$$\begin{aligned}
&F_\omega - dB(\vartheta)[F] \\
&\quad = \frac{1}{2}\vartheta_{t\omega} - \frac{1}{2}B(\vartheta)_\omega - \frac{1}{2}dB(\vartheta)[\vartheta_t - B(\vartheta)] \\
&\qquad + \vartheta_{\tau\omega} - dA(\vartheta)[\vartheta_\omega] - dB(\vartheta)[\vartheta_\tau - A(\vartheta)] \\
&\quad = \frac{1}{2}B(\vartheta)_t - \frac{1}{2}dB(\vartheta)[\vartheta_\omega] - \frac{1}{2}dB(\vartheta)[\vartheta_t] + \frac{1}{2}dB(\vartheta)[B(\vartheta)] \\
&\qquad + dB(\vartheta)[\vartheta_\tau] - dA(\vartheta)[B(\vartheta)] - dB(\vartheta)[\vartheta_\tau] + dB(\vartheta)[A(\vartheta)] \\
&\quad = \frac{1}{2}\left(dB(\vartheta)[\vartheta_t] - dB(\vartheta)[B(\vartheta)] - dB(\vartheta)[\vartheta_t] + dB(\vartheta)[B(\vartheta)]\right) \\
&\qquad + dB(\vartheta)[A(\vartheta)] - dA(\vartheta)[B(\vartheta)].
\end{aligned}$$

Thus, recalling the definition of $[\,\cdot\,,\,\cdot\,]$, we find

$$F_\omega - dB(\vartheta)[F] = [B,A](\vartheta).$$

For the class of equations (1.1), the evolution equations read:

$$e_t + \operatorname{div}\left(e\boldsymbol{v}(e) + u\boldsymbol{v}(e) + e\boldsymbol{v}(u)\right) - A(e) = F, \quad t \in (0,T) \tag{4.2}$$

$$F_\tau - A(F) = \frac{1}{2}[A,B](\vartheta), \quad (t,\tau,\frac{t_n}{2}) \in \Omega_{\Delta t}, \tag{4.3}$$

$$F_\omega + \operatorname{div}\left(\vartheta\boldsymbol{v}(F) + F\boldsymbol{v}(\vartheta)\right) = -[A,B](\vartheta), \quad (t,\tau,\omega) \in \Omega_{\Delta t}. \tag{4.4}$$

As for the Godunov method, it will be convenient to define a notation for all times prior to a given time $(\sigma,\zeta,\nu) \in \Omega_{\Delta t}$:

$$\Omega_{\Delta t}^{\sigma,\tau,\nu} = \left\{(t,\tau,\omega) \in \Omega_{\Delta t} \mid 0 \le t \le \sigma,\ 0 \le \tau \le \zeta,\ 0 \le \omega \le \nu\right\}.$$

To prove Lemma 4.2, we will need Strang versions of Lemmas 3.4 and 3.5. Clearly, there is no problem to extend Lemma 3.4 to conclude the following result.

Lemma 4.4. *Let $\vartheta(0,0) := u_0 \in H^k$. Assume the existence of a time $(\sigma,\tau,\nu) \in [0,T]^3$ and a finite constant $\gamma > 0$ such that*

$$\|\vartheta(t,\tau,\omega)\|_{H^{k-\max\{\alpha,2\}}} \leq \gamma, \quad (t,\tau,\omega) \in \Omega_{\Delta t}^{\sigma,\zeta,\nu}.$$

Then there is a constant $C(\gamma)$ determined by γ, u_0, and T such that

$$\|\vartheta(t,\tau,\omega)\|_{H^k} \leq C(\gamma), \quad (t,\tau,\omega) \in \Omega_{\Delta t}^{\sigma,\zeta,\nu}.$$

Let us now prove the Strang version of Lemma 3.5.

Lemma 4.5. *Assume the existence of $(\sigma,\zeta,\nu) \in \Omega_{\Delta t}$ such that the splitting solution*

$$\|\vartheta(t,\tau,\omega)\|_{H^k} \leq C(\gamma), \quad (t,\tau,\omega) \in \Omega_{\Delta t}^{\sigma,\zeta,\nu}.$$

Then,

$$\|e(t)\|_{H^{k-\max\{\alpha,2\}}} \leq \Delta t \tilde{C}(\gamma), \quad t \leq \sigma.$$

Proof. Let $\ell = k - \max\{\alpha, 2\}$. As in the proof of Lemma 3.5, we first estimate the size of the source term F. For this purpose, fix any $(t,\tau,\omega) \in \Omega_{\Delta t}^{\sigma,\zeta,\nu}$ and let n be such that $(t,\tau,\omega) \in \left[\frac{t_n}{2}, \frac{t_{n+1}}{2}\right] \times [t_n, t_{n+1}] \times \left[\frac{t_n}{2}, \frac{t_{n+1}}{2}\right]$.

Now, in the plane given by $\omega = \frac{t_n}{2}$, we have that F satisfies the equation (4.3). Since this equation is similar to (3.6), we can repeat the arguments in the proof of Lemma 3.5 to conclude that

$$\|F(t,\sigma,\frac{t_n}{2})\|_{H^\ell} \leq \Delta t C(\gamma). \tag{4.5}$$

Hence, to estimate F at the point (t,τ,ω), we can integrate from (t,τ,ω) in the ω direction to the plane given by $\omega = \frac{t_n}{2}$ and apply (4.5). To achieve this, we first apply ∇^s to (4.4), multiply with $\nabla^s F$, integrate, and sum over $s = 0, \dots, k$, to obtain

$$\begin{aligned}\partial_\omega \frac{1}{2}\|F(t,\tau,\omega)\|_{H^\ell}^2 &= -\sum_{s=0}^{\ell}\int_{\mathbb{R}^N} \nabla^s \operatorname{div}\left(\vartheta \boldsymbol{v}(F) + F\boldsymbol{v}(\vartheta)\right) : \nabla^s F \, dx \\ &\quad - \sum_{s=0}^{\ell}\int_{\mathbb{R}^N} \nabla^s [A,B](\vartheta) : \nabla^s F \, dx \\ &\leq C\left(\|\vartheta\|_{H^{\ell+1}}\|F\|_{H^\ell}^2 + \|\vartheta\|_{H^\ell}\|F\|_{H^\ell}^2\right) + C\|\vartheta\|_{H^k}^2\|F\|_{H^\ell}.\end{aligned}$$

In the last inequality we have applied Lemmas 2.2 and 3.3. Since $\vartheta \in H^k$, we can integrate the last inequality to obtain

$$\|F(t,\tau,\omega)\|_{H^\ell} \leq \|F(t,\sigma,\frac{t_n}{2})\|_{H^\ell} + C(\gamma)\Delta t \leq \Delta t C(\gamma),$$

where the last inequality is (4.5).

Since (4.2) is identical to (3.5), we can repeat the arguments in the proof of Lemma 3.5 to conclude the proof. □

4.2. **Proof of well-posedness (Lemma 4.2).** We can apply similar arguments as those of Section 3.3 (for the Godnov method) to prove Lemma 4.2. In particular, upon inspection of the arguments in Section 3.3, it is clear that Lemma 4.2 is a consequence of the following result.

Lemma 4.6. *Let $k \geq 6$ and $\vartheta(0,0,0) := u_0 \in H^k$. There exists constants β and γ, depending only on T and u_0, such that if $\Delta t < \beta$ and*

$$\|\vartheta(t,\tau,\omega)\|_{H^{k-\max\{\alpha,2\}}} \leq \gamma, \quad (t,\tau,\omega) \in \Omega_{\Delta t}^{\sigma,\zeta,\nu},$$

for some $(\sigma,\zeta,\nu) \in \Omega_{\Delta t}$, then

$$\|\vartheta(t,\tau,\omega)\|_{H^{k-\max\{\alpha,2\}}} \leq \frac{\gamma}{2}, \quad (t,\tau,\omega) \in \Omega_{\Delta t}^{\sigma,\zeta,\nu}.$$

Proof. Let $\ell = k - \max\{\alpha, 2\}$ and assume that $(t, \tau, \omega) \in \Omega_{\Delta t}^{\sigma,\zeta,\nu}$ is such that

$$\|\vartheta(t,\tau,\omega)\|_{H^\ell} \leq \gamma, \quad (t,\tau,\omega) \in \Omega_{\Delta t}^{\sigma,\zeta,\nu},$$

where the values of γ and Δt are to be determined. Lemma 4.4 can then be applied and yields

$$\|\vartheta(t,\tau,\omega)\|_{H^k} \leq C(\gamma), \quad \forall (t,\tau,\omega) \in \Omega_{\Delta t}^{\sigma,\zeta,o}.$$

Furthermore, we can apply Lemma 4.5 to conclude the estimate

$$\|e(t)\|_{H^\ell} \leq \Delta t \tilde{C}(\gamma).$$

Now, to proceed we fix any $(t,\tau,\omega) \in \Omega_{\Delta t}^{\sigma,\zeta,\nu}$. Using the definition of the Strang method (Definition 4.1) and Lemma 2.1, we see that

$$\begin{aligned}
\left|\partial_\omega \frac{1}{2}\|\vartheta(t,\tau,\omega)\|_{H^\ell}^2\right| &= \left|\sum_{s=0}^{\ell} \int_{\mathbb{R}^N} \nabla^s \vartheta_\omega : \nabla^s \vartheta \; dx\right| \\
&= \left|-\sum_{s=0}^{\ell} \int_{\mathbb{R}^N} \nabla^s \operatorname{div}(\vartheta \boldsymbol{v}(\vartheta)) : \nabla^s \vartheta \; dx\right| \\
&\leq C\|\vartheta\|_{H^{\ell-2}}\|\vartheta\|_{H^\ell}^2 \leq C\|\vartheta\|_{H^\ell}^3 \leq C\gamma^3.
\end{aligned}$$

From this it easily follows that

$$\|\vartheta(t,\tau,\omega)\|_{H^\ell} \leq \|\vartheta(t,\tau,t)\|_{H^\ell} + \Delta t C(\gamma). \tag{4.6}$$

By the same calculations as those in the proof of Lemma 3.6, we also deduce

$$\begin{aligned}
\|\vartheta(t,\tau,t)\|_{H^\ell} &\leq \|\vartheta(t,2t,t)\|_{H^\ell} + \Delta t C(\gamma) \\
&\leq \|e(2t)\|_{H^\ell} + \|u(2t)\|_{H^\ell} + \Delta t C(\gamma) \\
&\leq C(\gamma)\Delta t + C_2.
\end{aligned}$$

Using this in (4.6) allow us to conclude

$$\|\vartheta(t,\tau,\omega)\|_{H^\ell} \leq C(\gamma)\Delta t + C_2. \tag{4.7}$$

Since (t,τ,ω) was arbitrary, we can conclude that (4.7) holds for all $(t,\tau,\omega) \in \Omega_{\Delta t}^{\sigma,\zeta,\nu}$.

Finally, we fix γ and Δt according to

$$\gamma = 4C_2, \qquad \Delta t \leq \frac{C_2}{C(\gamma)} := \beta,$$

in (4.7) to obtain

$$\|\vartheta(t,\tau,\omega)\|_{H^\ell} \leq \frac{\gamma}{2}, \quad (t,\tau,\omega) \in \Omega_{\Delta t}^{\sigma,\zeta,\nu}.$$

□

Equipped with the previous lemma, Lemma 4.2 can be proved as we did with Lemma 3.2 in the Godunov case.

4.3. **Temporal regularity.** To prove second-order convergence (Lemma 4.3), we will need regularity in time of our splitting solution. Since we are working with three time variables, it will be convenient to recall the notation

$$\nabla_t^l f = \frac{\partial^{|l|} f}{\partial t^{l_1} \partial \tau^{l_2} \partial \omega^{l_3}}, \qquad |l| = l_1 + l_2 + l_3,$$

for a multi-index $l = (l_1, l_2, l_3)$. We will also use the notation

$$\nabla_t^k f = \left\{ \frac{\partial^{|l|} f}{\partial t^{l_1} \partial \tau^{l_2} \partial \omega^{l_3}} \mid |l| = k \right\}$$

for any natural number k

Lemma 4.7. *Let ϑ be the Strang splitting solution in the sense of Definition 4.1. If $u_0 \in H^k$, then*

$$\|\nabla_t^l \vartheta(t,\tau,\omega)\|_{H^{k-|l|\max\{\alpha,1\}}} \leq C\,, \quad (t,\tau,\omega) \in \Omega_{\Delta t}.$$

Proof. We will argue by induction on $|l|$. For $|l| = 0$, the result follows from Lemma 4.2. To proceed we assume that the result holds for $|l| = 0,\dots,q$. To close the induction argument it remains to prove the result for $|l| = q+1$.

To simplify notation, we let

$$\lambda = k - |l|\max\{\alpha,1\} = k - (q+1)\max\{\alpha,1\}.$$

Now, let $(t',\tau',\omega') \in \Omega_{\Delta t}$ be arbitrary and fix n such that $(t',\tau',\omega') \in [\frac{t_n}{2}, \frac{t_{n+1}}{2}] \times [t_n, t_{n+1}] \times [\frac{t_n}{2}, \frac{t_{n+1}}{2}]$. An arbitrary component of $\nabla_t^l \vartheta$ can be written in the form

$$\Theta_{i,j,\ell}^{q+1} = \frac{\partial^{q+1}}{\partial t^i \partial \tau^j \partial \omega^\ell}\vartheta, \quad 0 \leq i,j,\ell \leq q+1, \quad i+j+\ell = q+1, \quad i,j,\ell \in \mathbb{N}.$$

To estimate the arbitrary component at the point (t',τ',ω'), we apply the fundamental theorem of calculus to obtain

$$\begin{aligned}
&\frac{1}{2}\|\Theta_{i,j,\ell}^{q+1}(t',\tau',\omega')\|_{H^\lambda}^2 \\
&\quad = \frac{1}{2}\|\Theta_{i,j,\ell}^{q+1}(t',\tau',\frac{t_n}{2})\|_{H^\lambda}^2 \\
&\qquad + \sum_{s=0}^{\lambda} \int_{\frac{t_n}{2}}^{\omega'} \int_{\mathbb{R}^N} \nabla^s \partial_\omega \Theta_{i,j,\ell}^{q+1}(t',\tau',\tilde{\omega}) : \nabla^s \Theta_{i,j,\ell}^{q+1}(t',\tau',\tilde{\omega})\, dx d\tilde{\omega} \\
&\quad = \frac{1}{2}\|\Theta_{i,j,\ell}^{q+1}(t',\tau',\frac{t_n}{2})\|_{H^\lambda}^2 \\
&\qquad\quad + \sum_{s=0}^{\lambda}\sum_{r=0}^{i}\sum_{n=0}^{j}\sum_{m=0}^{\ell} \binom{i}{r}\binom{j}{n}\binom{\ell}{m} \\
&\qquad\qquad \times \int_{\frac{t_n}{2}}^{\omega'} \int_{\mathbb{R}^N} \nabla^s \operatorname{div}\left(\Theta_{r,n,m}^{m+n+r}(t',\tau',\tilde{\omega})\boldsymbol{v}\left(\Theta_{i-r,j-n,\ell-m}^{|l|-r-n-m}(t',\tau',\tilde{\omega})\right)\right) \\
&\qquad\qquad\qquad : \nabla^s \Theta_{i,j,\ell}^{q+1}(t',\tau',\tilde{\omega})\, dx d\tilde{\omega},
\end{aligned} \tag{4.8}$$

where we used the definition of $\Theta_{i,j,\ell}^{q+1}$ and ϑ_ω (cf. (4.1)) to conclude the last equality. Let us consider three separate cases of $m+n+r$ in the quadruple sum above.
(i) If $m+n+r = q+1$, the corresponding term in the above reads

$$\begin{aligned}
&\sum_{s=0}^{\lambda} \int_{\frac{t_n}{2}}^{\omega'} \int_{\mathbb{R}^N} \nabla^s \operatorname{div}\left(\Theta_{i,j,\ell}^{q+1}(t',\tau',\tilde{\omega})\boldsymbol{v}\left(\vartheta(t',\tau',\tilde{\omega})\right)\right) : \nabla^s \Theta_{i,j,\ell}^{q+1}(t',\tau',\tilde{\omega})\, dx d\tilde{\omega} \\
&\qquad \leq C \int_{\frac{t_n}{2}}^{\omega'} \|\boldsymbol{v}(\vartheta(t',\tau',\tilde{\omega}))\|_{H^\lambda} \left\|\Theta_{i,j,\ell}^{q+1}(t',\tau',\tilde{\omega})\right\|_{H^\lambda}^2 \, d\tilde{\omega} \\
&\qquad \leq C \int_{\frac{t_n}{2}}^{\omega'} \left\|\Theta_{i,j,\ell}^{q+1}(t',\tau',\tilde{\omega})\right\|_{H^\lambda}^2 \, d\tilde{\omega},
\end{aligned} \tag{4.9}$$

where we have applied Lemmas 2.2 and 4.4.
(ii) If $m+n+r = 0$, we can also apply Lemmas 2.2 and 4.4 to conclude

$$\sum_{s=0}^{\lambda} \int_{\frac{t_n}{2}}^{\omega'} \int_{\mathbb{R}^N} \nabla^s \operatorname{div}\left(\vartheta(t',\tau',\tilde{\omega})\boldsymbol{v}\left(\Theta_{i,j,\ell}^{q+1}(t',\tau',\tilde{\omega})\right)\right) : \nabla^s \Theta_{i,j,\ell}^{q+1}(t',\tau',\tilde{\omega})\, dx d\tilde{\omega}$$

$$
\begin{aligned}
&\le C\int_{\frac{t_n}{2}}^{\omega'} \|\vartheta(t',\tau',\tilde\omega)\|_{H^{\lambda+1}}\|\Theta^{q+1}_{i,j,\ell}(t',\tau',\tilde\omega)\|^2_{H^\lambda}\ d\tilde\omega\\
&\le C\int_{\frac{t_n}{2}}^{\omega'} \|\Theta^{q+1}_{i,j,\ell}(t',\tau',\tilde\omega)\|^2_{H^\lambda}\ d\tilde\omega.
\end{aligned}
$$

(iii) For the remaining cases ($1 \le m+n+r \le q$), we apply the Hölder inequality to obtain

$$
\begin{aligned}
&\sum_{s=0}^{\lambda}\int_{\frac{t_n}{2}}^{\omega'}\int_{\mathbb{R}^N} \nabla^s \operatorname{div}\left(\Theta^{m+n+r}_{r,n,m}(t',\tau',\tilde\omega)\boldsymbol{v}\left(\Theta^{q+1-r-n-m}_{i-r,j-n,\ell-m}(t',\tau',\tilde\omega)\right)\right)\\
&\qquad\qquad : \nabla^s\Theta^{q+1}_{i,j,\ell}(t',\tau',\tilde\omega)\ dx d\tilde\omega\\
&\le C\int_{\frac{t_n}{2}}^{\omega'} \|\Theta^{q+1}_{i,j,\ell}(t',\tau',\tilde\omega)\|_{H^\lambda}\|\Theta^{m+n+r}_{r,n,m}(t',\tau',\tilde\omega)\|_{H^{\lambda+1}} \qquad (4.10)\\
&\qquad\qquad \times\|\Theta^{q+1-r-n-m}_{i-r,j-n,\ell-m}(t',\tau',\tilde\omega)\|_{H^{\lambda+1}}\ d\tilde\omega\\
&\le C\int_{\frac{t_n}{2}}^{\omega'} \|\Theta^{q+1}_{i,j,\ell}(t',\tau',\tilde\omega)\|_{H^\lambda}\ d\tilde\omega,
\end{aligned}
$$

where we have used that our induction hypothesis yields

$$
\|\nabla_t^q\vartheta\|_{H^{\lambda+1}} = \|\nabla_t^q\vartheta\|_{H^{k+1-(q+1)\max\{\alpha,1\}}} \le \|\nabla_t^q\vartheta\|_{H^{k-q\max\{\alpha,1\}}} \le C.
$$

By applying (4.9)–(**??**) to (4.8), we gather

$$
\begin{aligned}
&\frac{1}{2}\|\Theta^{q+1}_{i,j,\ell}(t',\tau',\omega')\|^2_{H^\lambda}\\
&\quad\le \frac{1}{2}\|\Theta^{q+1}_{i,j,\ell}(t',\tau',\frac{t_n}{2})\|^2_{H^\lambda} + C\int_{\frac{t_n}{2}}^{\omega'}\|\Theta^{q+1}_{i,j,\ell}(t',\tau',\tilde\omega)\|_{H^\lambda}\ d\tilde\omega\\
&\qquad + C\int_{\frac{t_n}{2}}^{\omega'}\|\Theta^{q+1}_{i,j,\ell}(t',\tau',\tilde\omega)\|^2_{H^\lambda}\ d\tilde\omega\\
&\quad\le \frac{1}{2}\|\Theta^{q+1}_{i,j,\ell}(t',\tau',\frac{t_n}{2})\|^2_{H^\lambda} + C\Delta t + C\int_{\frac{t_n}{2}}^{\omega'}\|\Theta^{q+1}_{i,j,\ell}(t',\tau',\tilde\omega)\|^2_{H^\lambda}\ d\tilde\omega,
\end{aligned}
$$

where the last inequality as an application of the Hölder inequality to the second term. Now, by applying the Gronwall inequality to the previous inequality, we conclude

$$
\|\Theta^{q+1}_{i,j,\ell}(t',\tau',\omega')\|^2_{H^\lambda} \le \left(\|\Theta^{q+1}_{i,j,\ell}(t',\tau',\frac{t_n}{2})\|^2_{H^\lambda} + C\Delta t\right)e^{C\Delta t}. \qquad (4.11)
$$

We have now derived a bound on $\Theta^{q+1}_{i,j,\ell}(t',\tau',\omega')$ in terms of $\Theta^{q+1}_{i,j,\ell}(t',\tau',\frac{t_n}{2})$. Next, we derive a bound on $\Theta^{q+1}_{i,j,\ell}(t',\tau',\frac{t_n}{2})$ in terms of $\Theta^{q+1}_{i,j,\ell}(t',t_n,\frac{t_n}{2})$. For this purpose, we once more apply the fundamental theorem of calculus to obtain

$$
\begin{aligned}
&\frac{1}{2}\|\Theta^{q+1}_{i,j,\ell}(t',\tau',\frac{t_n}{2})\|^2_{H^\lambda} \\
&\quad= \frac{1}{2}\|\Theta^{q+1}_{i,j,\ell}(t',t_n,\frac{t_n}{2})\|^2_{H^\lambda} \\
&\qquad+ \sum_{s=0}^{\lambda}\int_{t_n}^{\tau'}\int_{\mathbb{R}^N} \nabla^s \partial_\tau \Theta^{q+1}_{i,j,\ell}(t',\tilde{s},\frac{t_n}{2}) : \nabla^s \Theta^{q+1}_{i,j,\ell}(t',\tilde{s},\frac{t_n}{2})\, dxd\tilde{s} \\
&\quad= \frac{1}{2}\|\Theta^{q+1}_{i,j,\ell}(t',t_n,\frac{t_n}{2})\|^2_{H^\lambda} \\
&\qquad+ \sum_{s=0}^{\lambda}\int_{t_n}^{\tau'}\int_{\mathbb{R}^N} \nabla^s A\left(\Theta^{q+1}_{i,j,\ell}(t',\tilde{s},\frac{t_n}{2})\right) : \nabla^s \Theta^{q+1}_{i,j,\ell}(t',\tilde{s},\frac{t_n}{2})\, dxd\tilde{s} \\
&\quad\leq \frac{1}{2}\|\Theta^{q+1}_{i,j,\ell}(t',t_n,\frac{t_n}{2})\|^2_{H^\lambda},
\end{aligned}
$$

where we have used that $\vartheta_\tau = A(\vartheta)$ for $\omega = \frac{t_n}{2}$ and (4) of Definition 2.3. It follows that

$$
\|\Theta^{q+1}_{i,j,\ell}(t',\tau',\frac{t_n}{2})\|^2_{H^k} \leq \|\Theta^{q+1}_{i,j,\ell}(t',t_n,\frac{t_n}{2})\|^2_{H^k}. \tag{4.12}
$$

Finally, we perform our last application of the fundamental theorem to obtain

$$
\begin{aligned}
&\frac{1}{2}\|\Theta^{q+1}_{i,j,\ell}(t',t_n,\frac{t_n}{2})\|^2_{H^\lambda} - \frac{1}{2}\|\Theta^{q+1}_{i,j,\ell}(\frac{t_n}{2},t_n,\frac{t_n}{2})\|^2_{H^\lambda} \\
&\qquad= \sum_{s=0}^{\lambda}\int_{\frac{t_n}{2}}^{\omega'}\int_{\mathbb{R}^N} \nabla^s \partial_t \Theta^{q+1}_{i,j,\ell}(\tilde{s},t_n,\frac{t_n}{2}) : \nabla^s \Theta^{q+1}_{i,j,\ell}(\tilde{s},t_n,\frac{t_n}{2})\, dxd\tilde{s}.
\end{aligned}
$$

By applying the same calculations to the previous equations as those leading to (4.11), we find that

$$
\|\Theta^{q+1}_{i,j,\ell}(t',t_n,\frac{t_n}{2})\|^2_{H^\lambda} \leq \left(\|\Theta^{q+1}_{i,j,\ell}(\frac{t_n}{2},t_n,\frac{t_n}{2})\|^2_{H^\lambda} + C\Delta t\right)e^{C\Delta t}. \tag{4.13}
$$

Combining (4.11), (4.12), and (4.13) gives

$$
\|\Theta^{q+1}_{i,j,\ell}(t',\tau',\omega')\|_{H^\lambda} \leq \|\Theta^{q+1}_{i,j,\ell}(\frac{t_n}{2},t_n,\frac{t_n}{2})\|_{H^\lambda} e^{2C\Delta t} + C\Delta t\left(e^{2C\Delta t} + e^{C\Delta t}\right),
$$

which immediately leads to the conclusion

$$
\begin{aligned}
\|\Theta^{q+1}_{i,j,\ell}(t',\tau',\omega')\|_{H^\lambda} &\leq \|\Theta^{q+1}_{i,j,\ell}(0,0,0)\|_{H^\lambda} e^{nC\Delta t} + C\Delta t \sum_{\varrho=1}^{N} e^{\varrho C\Delta t} \\
&\leq \|\Theta^{q+1}_{i,j,\ell}(0,0,0)\|_{H^\lambda} e^{Ct'} + Ct' e^{Ct'}.
\end{aligned} \tag{4.14}
$$

Finally, let us estimate $\|\Theta^{q+1}_{i,j,\ell}(0,0,0)\|_{H^\lambda}$. By definition, we have that

$$
\Theta^{q+1}_{i,j,\ell}(0,0,0) = \frac{\partial^{q+1}}{\partial t^i \partial \tau^j \partial \omega^\ell}\vartheta(0,0,0).
$$

At this point, we can apply each of the time-derivatives to $\vartheta(0,0,0)$ and use the definition (4.1) to translate time-derivatives into spatial derivatives. The maximal number of spatial derivatives for each time derivative is $\max\{\alpha,1\}$ and consequently $q+1$ time derivatives translates into at most $(q+1)\max\{\alpha,1\}$ spatial derivatives. As a consequence, we can conclude that

$$
\|\Theta^{q+1}_{i,j,\ell}(0,0,0)\|_{H^\lambda} = \|\Theta^{q+1}_{i,j,\ell}(0,0,0)\|_{H^{k-(q+1)\max\{\alpha,1\}}} \leq C\|u_0\|_{H^k}.
$$

From this, and (4.14), we conclude that the lemma holds also for $|l| = q+1$. □

Using the previous lemma, we now deduce time regularity of the source term F.

Lemma 4.8. *Let ϑ be the Strang splitting solution in the sense of Definition 4.1 and* (4.1)*. Then, for any $3\max\{\alpha,1\} \leq k \in \mathbb{N}$ such that $\|u_0\|_{H^k} \leq C$, we have*

$$\|\nabla_t^2 F(t,\tau,\omega)\|_{H^{k-3\max\{\alpha,1\}}} \leq C, \quad (t,\tau,\omega) \in \Omega_{\Delta t}.$$

Proof. Let $\ell = k - 3\max\{\alpha,1\}$. Let i and j denote any one of t, τ, or ω. An arbitrary component of $\nabla_t^2 F$ can then be written $F_{ij} := \partial_i\partial_j F$. By definition, we have that

$$\begin{aligned} F_{ij} &= \partial_i\partial_j \left[\frac{1}{2}\left(\vartheta_t + \operatorname{div}(\vartheta \boldsymbol{v}(\vartheta))\right) + \vartheta_\tau - A(\vartheta)\right] \\ &= \frac{1}{2}\partial_t \vartheta_{ij} + \partial_\tau \vartheta_{ij} - A(\vartheta_{ij}) + \frac{1}{2}\operatorname{div}\left(\vartheta_{ij}\boldsymbol{v}(\vartheta) + \vartheta_i \boldsymbol{v}(\vartheta_j) + \vartheta_j \boldsymbol{v}(\vartheta_i) + \vartheta \boldsymbol{v}(\vartheta_{ij})\right). \end{aligned}$$

By applying the Hölder inequality and Sobolev embedding, we make the gross overestimation (note that we can have $\ell \leq 2$):

$$\begin{aligned} \|F_{ij}\|_{H^\ell} \leq{}& \frac{3}{2}\|\nabla_t^3\vartheta\|_{H^\ell} + \|\nabla_t^2\vartheta\|_{H^{\ell+\alpha}} + \|\nabla\vartheta_{ij}\cdot\boldsymbol{v}(\vartheta)\|_{H^\ell} + \|\vartheta_{ij}\operatorname{div}\boldsymbol{v}(\vartheta)\|_{H^\ell} \\ &+ \|\nabla\vartheta_i\cdot\boldsymbol{v}(\vartheta_j)\|_{H^\ell} + \|\vartheta_i\operatorname{div}\boldsymbol{v}(\vartheta_j)\|_{H^\ell} + \|\nabla\vartheta_j\cdot\boldsymbol{v}(\vartheta_i)\|_{H^\ell} \\ &+ \|\vartheta_j\operatorname{div}\boldsymbol{v}(\vartheta_i)\|_{H^\ell} + \|\nabla\vartheta\cdot\boldsymbol{v}(\vartheta_{ij})\|_{H^\ell} + \|\vartheta\operatorname{div}\boldsymbol{v}(\vartheta_{ij})\|_{H^\ell} \\ \leq{}& \frac{3}{2}\|\nabla_t^3\vartheta\|_{H^\ell} + \|\nabla_t^2\vartheta\|_{H^{\ell+\alpha}} + C\|\nabla_t^2\vartheta\|_{H^{\ell+1}}\left(\|\boldsymbol{v}(\vartheta)\|_{H^{\ell+2}} + \|\operatorname{div}\boldsymbol{v}(\vartheta)\|_{H^{\ell+2}}\right) \\ &+ 2\|\nabla_t\vartheta\|_{H^{\ell+1}}\|\boldsymbol{v}(\nabla_t\vartheta)\|_{H^{\ell+2}} + 2\|\nabla_t\vartheta\|_{H^{l+2}}\|\boldsymbol{v}(\nabla_t\vartheta)\|_{H^{l+1}} \\ &+ C\|\vartheta\|_{H^{\ell+3}}\left(\|\boldsymbol{v}(\nabla_t^2\vartheta)\|_{H^\ell} + \|\operatorname{div}\boldsymbol{v}(\nabla_t^2\vartheta)\|_{H^\ell}\right) \\ \leq{}& C\left(\|\nabla_t^3\vartheta\|_{H^\ell} + \|\nabla_t^2\vartheta\|_{H^{\ell+\max\{\alpha,1\}}} + \|\nabla_t\vartheta\|_{H^{\ell+2}}\right). \end{aligned}$$

Since $\vartheta(0,0) := u_0 \in H^k$, Lemma 4.7 tell us that the right-hand side is bounded and hence our proof is complete. □

The next lemma is the main property giving second order convergence. Observe that the result does not depend on our specific choice of operators A and B.

Lemma 4.9. *There holds*

$$\nabla_t F(\frac{t_n}{2}, t_n, \frac{t_n}{2}) \cdot \begin{pmatrix}1\\2\\1\end{pmatrix} = 0.$$

Proof. We will prove Lemma 4.9 by direct calculation. Let us begin by estimating $F_\omega(\frac{t_n}{2}, t_n, \frac{t_n}{2})$. Since $F(t, t_n, \frac{t_n}{2}) = 0$, we have that

$$F_\omega(\frac{t_n}{2}, t_n, \frac{t_n}{2}) = -[A,B](\vartheta). \tag{4.15}$$

Similarly, we see that (4.3) yields

$$F_\tau = \frac{1}{2}[A,B](\vartheta). \tag{4.16}$$

It only remains to estimate $F_t(\frac{t_n}{2}, t_n, \frac{t_n}{2})$. However, as $F(t, t_n, \frac{t_n}{2}) = 0$, we must have that $F_t(\frac{t_n}{2}, t_n, \frac{t_n}{2}) = 0$. This, together with (4.15) and (4.16), concludes the proof. □

Using the previous lemma, we can now prove that the error produced along the diagonal $(t/2, t, t/2)$ is second order in Δt.

Lemma 4.10. *Let ϑ be the Strang splitting solution in the sense of Definition 4.1 and* (4.1)*. Then, if $u_0 \in H^k$,*

$$\left\| F\left(\frac{t}{2}, t, \frac{t}{2}\right)\right\|_{H^{k-3\max\{\alpha,1\}}} \leq C(\Delta t)^2.$$

Proof. Since $F(\frac{t_n}{2}, t_n, \frac{t_n}{2}) = 0$, a Taylor expansion provides the identity

$$\begin{aligned} F\left(\frac{t}{2}, t, \frac{t}{2}\right) &= \nabla_t F(\frac{t_n}{2}, t_n, \frac{t_n}{2}) \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} (\frac{t}{2} - \frac{t_n}{2}) \\ &\quad + \frac{1}{2} \int_{t_n/2}^{t/2} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}^T \nabla_t^2 F\left(\frac{s}{2}, s, \frac{s}{2}\right) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} (\frac{s}{2} - \frac{t_n}{2})\, ds \qquad (4.17) \\ &= \frac{1}{2} \int_{t_n/2}^{t/2} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}^T \nabla_t^2 F\left(\frac{s}{2}, s, \frac{s}{2}\right) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} (\frac{s}{2} - \frac{t_n}{2})\, ds, \end{aligned}$$

where the last equality is an application of Lemma 4.9. By taking the $H^{k-3\max\{\alpha,1\}}$ norm on both sides of (4.17) and applying the previous lemma, we gather

$$\left\| F(\frac{t}{2}, t, \frac{t}{2}) \right\|_{H^k} \leq C(\Delta t)^2,$$

which concludes the proof. □

Proof of Lemma 4.3. By performing the same calculations as those in the proof of Lemma 3.5 with the new estimate on F given by Lemma 4.10, we obtain the estimate

$$\frac{1}{2}\partial_t \|e(t)\|_{H^{k-3\max\{\alpha,1\}}} \leq C(T)\left((\Delta t)^2 + \|e(t)\|_{H^{k-3\max\{\alpha,1\}}}\right), \qquad t \in (0,T).$$

Since $e(0) = 0$, an application of the Gronwall inequality to the previous inequality gives

$$\|e(t)\|_{H^{k-3\max\{\alpha,1\}}} \leq t\,(\Delta t)^2 C, \quad t \in [0,T],$$

which concludes the proof of Lemma 4.3 and consequently also Theorem 1.2. □

5. Applications

In the previous sections we have established well-posedness and convergence rates for both Godunov and Strang splitting applied to (1.1). In this section we will examine a range of different equations that are all of the form (1.1) with A and $\boldsymbol{v}$ being admissible (in the sense of Definition 2.3). Our primary goal is to equip the reader with some relevant applications of our framework. Let us for the convenience of the reader repeat our equation and assumptions here. We are working with the equation

$$u_t + \operatorname{div}(u\,\boldsymbol{v}(u)) = A(u), \qquad u|_{t=0} = u_0,$$

where the operators are required to be admissible in the sense of Definition 2.3.

Given a specific equation, the only non-trivial conditions in Definition 2.3 are the well-posedness (6) and the commutator estimate (5). The following lemma is of help to determine the latter.

Lemma 5.1. *Let $\alpha \in (0,2)$ and l be any multi-index. The standard differential operator $A(u) = D^l u$ and the fractional Laplacian $A(u) = (-\Delta)^{\alpha/2}u$ both satisfy (3)–(5) in Definition 2.3.*

Proof. Conditions (3) and (4) are trivially satisfied. It remains to prove that A satisfies the commutator estimate (5). In the case of the fractional Laplacian ($A(u) = (-\Delta)^{\alpha/2}u$), the commutator estimate was proved in Corollary 2.5 in [17]. See also the lecture notes by Constantin [4]. For the standard differential operator, the result is immediate from the standard Leibniz rule. □

Remark 5.2. By combining the usual Leibniz rule with the corresponding rule for the fractional Laplacian, the previous lemma can be extended to also include operators on the form $A(u) = (-\Delta)^{\alpha/2} D^l u$, that is, any mix of fractional and standard derivatives.

5.1. **Burgers type equations.** If we restrict to one spatial dimension ($N = 1$), the only valid velocity operator $\boldsymbol{v}$ satisfying requirement (1) of Definition 2.3 is

$$\boldsymbol{v}(u) = au, \qquad a \in \mathbb{R}.$$

Thus, the type of equations we can consider consists of a Burgers term and a linear differential term. In the literature one can find several equations of this type that are well-posed in the sense of (6) in Definition 2.3. Some examples are:

$$\begin{aligned} &u_t + (u^2)_x = u_{xxx}, && \text{(KdV)},\\ &u_t + (u^2)_x = -(-\partial_x^2)^{\alpha/2} u, && \text{(viscous Burgers)},\\ &u_t + (u^2)_x = -u_{xxx} + u_{xxxxx}, && \text{(Kawahara)}. \end{aligned}$$

For the viscous Burgers equation, $\alpha \geq 1$ is required to ensure well-posedness [20].

5.2. **Quasi-geostrophic flow.** The following equation has been proposed as a toy model for strongly rotating atmospheric flow

$$u_t + \operatorname{div}(u\, \boldsymbol{v}(u)) = 0, \quad \boldsymbol{v}(u) = \operatorname{curl}(-\Delta)^{-1/2} u, \tag{5.1}$$

where u is the potential temperature, $\boldsymbol{v}(u)$ is the fluid velocity, and the equation is valid in two dimensions ($N = 2$). The reader can consult [4, 7] and the references therein for more on the physical aspects of the model. The equation (5.1) is in the literature referred to as the *quasi-geostrophic equation* and has in the recent years been the subject of numerous analytical studies. This recent interest was probably sparked by the Constantin, Majda, and Tabak, paper [7] in which they give numerical evidence for a connection between the blow-up of solutions to (5.1) and the three dimensional Euler equations. Though the precise type of blow-up has been later dismissed by D. Cordoba [9], there remains hope that understanding the behavior of solutions to (5.1) can aid in characterizing blow-up of solutions to the incompressible Euler equations, a long standing open problem. As a consequence, most of the recent studies concerns regularity of solutions to equations of the form (5.1). A particularly well-studied case is the dissipative quasi-geostrophic equation

$$u_t + \operatorname{div}(u\, \boldsymbol{v}(u)) = A(u), \quad \boldsymbol{v}(u) = \operatorname{curl}(-\Delta)^{-\beta/2} u, \tag{5.2}$$

where $A(u) = -(-\Delta)^{\alpha/2} u$. Through the papers [3, 6, 8, 19], it is has been shown that (5.2) is well-posed in the sense of (6) in Definition 2.3 for α, $\beta \in [1, 2]$. Since in addition $\operatorname{div} \boldsymbol{v}(u) = 0$, (2) in Definition 2.3 is also satisfied. Hence, (5.2) with α, $\beta \in [1, 2]$ is admissible in our framework.

5.3. **Aggregation equations.** The following active scalar equation has been proposed [23, 24, 25] as a model for the long-range attraction between individuals in flocks, schools, or swarms:

$$u_t + \operatorname{div}(u\, \boldsymbol{v}(u)) = 0. \tag{5.3}$$

The velocity is determined as the convolution of u with an interaction potential:

$$\boldsymbol{v}(u) = \nabla\Phi \star u.$$

In these models, u is the density of individuals and $\boldsymbol{v}(u)$ incorporates the pairwise attractive forces between individuals in the flock. Two relevant examples of potentials Φ in applications are the radially symmetric $\Phi = 1 - e^{-|x|}$ and $\Phi = 1 - e^{-|x|^2}$.

In the papers [1, 2], sharp conditions are derived on the potential Φ under which solutions of (5.3) blows up in finite time. In particular, whenever

$$\int_0^1 \frac{1}{\Phi'(r)}\, dr < \infty,$$

solutions of (5.3) collapse to a point at the center of mass in finite time. Thus, the relevant case $\Phi = 1 - e^{-|x|}$ leads to blow up in finite time. This clearly non-realistic behavior can be attributed to the lack of any effect incorporating collision avoidance in the model. Since collision avoidance is a short-range phenomena, a simple way to incorporate it is to add diffusion to the model

$$u_t + \operatorname{div}(u\, \boldsymbol{v}(u)) = A(u), \qquad A(u) = -(-\Delta)^{\alpha/2} u. \tag{5.4}$$

The equation (5.4) has been the subject of the recent studies [11, 12, 13] and well-posedness in the sense of (6) in Definition 2.3 has been established for $\alpha \in (1, 2]$. In view of this well-posedness result and Lemma 5.1, (5.4) is included in our framework provided we can verify condition (2) in Definition 2.3.

By applying the Hölder inequality, we see that

$$\int_{\mathbb{R}^N} \operatorname{div} \boldsymbol{v}(f)\phi\, dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Delta\Phi(y) f(x-y)\phi(x)\, dxdy \leq \|f\|_{L^p}\|\phi\|_{L^{p'}}\|\Delta\Phi\|_{L^1}.$$

Hence, $\operatorname{div} \boldsymbol{v}(f)$ satisfies (2) in Definition 2.3 for any potential Φ satisfying $\Delta\Phi \in L^1$. Comparing this condition with the results in the paper [21], we see that we can include all the potentials for which well-posedness is known. For instance, the case $\Phi = 1 - e^{-|x|}$, since

$$\Delta\Phi = \operatorname{div}\left(\frac{x}{|x|}e^{-|x|}\right) = -e^{-|x|} + \frac{N-1}{|x|}e^{-|x|}, \quad x \neq 0,$$

and

$$\int_{\{|x|\leq 1\}} \frac{N-1}{|x|}\, dx \leq C, \qquad N \geq 2.$$

5.4. **Magneto geostrophic dynamics.** In the recent paper [14], well-posedness was established for the following class of active scalar equations

$$u_t + \operatorname{div}(u\, \boldsymbol{v}(u)) = A(u), \quad \boldsymbol{v}(u) = \operatorname{div} \mathbb{T}(u), \tag{5.5}$$

where $\mathbb{T}$ is a matrix of Calderon–Zygmund operators satisfying

$$\operatorname{div} \boldsymbol{v}(u) = \operatorname{div}\operatorname{div} \mathbb{T}(u) = 0,$$

and A is the Laplace operator $A(u) = \Delta u$. The equation (5.5) can be seen as a generalization of the quasi-geostrophic equation (5.2) (with $\alpha = 2$) since the latter can be obtained from (5.5) with a particular choice of $\mathbb{T}$ in 2D. The equation (5.5) is also included in the framework considered by Constantin in [5].

The physical motivation for the study [14] was that (5.5) with a particular choice of $\mathbb{T}$ has been proposed as a model for magnetostrophic turbulence in the Earth's fluid core. See the paper [14] and the references therein for more on this application. Since (5.5) is well-posed in the sense of (6) in Definition 2.3, our framework does indeed include this class of active scalar equations.

Appendix A. Proof of Lemmas 2.1 and 2.2

In this appendix we have gathered the proofs of Lemmas 2.1 and 2.2. Both lemmas have been used in an essential fashion throughout the convergence analysis.

Lemma 2.1. *Let $k \geq 6$ and $\boldsymbol{v}$ be an operator that satisfies Definition 2.3. Then,*

$$\sum_{s=0}^{k} \left| \int_{\mathbb{R}^N} \nabla^s (\operatorname{div}(f\boldsymbol{v}(f))) : \nabla^s f \; dx \right| \leq C \|f\|_{H^{k-2}} \|f\|_{H^k}^2, \quad f \in H^k.$$

Proof. Let us confine to the three dimensional case ($N = 3$) as the other cases are almost identical. The product rule provides us with the identity

$$\begin{aligned}
&\sum_{s=0}^{k} \left| \int_{\mathbb{R}^N} \nabla^s \operatorname{div}\left(f\boldsymbol{v}(f)\right) : \nabla^s f \; dx \right| \\
&\quad \leq \sum_{s=0}^{k} \left(\left| \int_{\mathbb{R}^N} \nabla^s \left(\nabla f \cdot \boldsymbol{v}(f)\right) : \nabla^s f \; dx \right| + \left| \int_{\mathbb{R}^N} \nabla^s (f \operatorname{div} \boldsymbol{v}(f)) : \nabla^s f \; dx \right| \right) \\
&\quad := \sum_{s=0}^{k} |I_1^s| + |I_2^s| \,.
\end{aligned}$$

In the remaining parts of the proof, our strategy is to bound each of I_1^s and I_2^s, $s = 0, \dots, k$, separately. We begin with I_1^s.

1. By applying the Leibniz rule to I_1^s (with multi-index notation $\alpha = (\alpha_1, \alpha_2, \alpha_3)$), we obtain the following expression

$$\begin{aligned}
I_1^s &= \sum_{|\alpha|=s} \int_{\mathbb{R}^N} \nabla^\alpha (\nabla f \cdot \boldsymbol{v}(f)) \nabla^\alpha f \; dx \\
&= \sum_{|\alpha|=s} \sum_{i_1=0}^{\alpha_1} \sum_{i_2=0}^{\alpha_2} \sum_{i_3=0}^{\alpha_3} \binom{\alpha_1}{i_1} \binom{\alpha_2}{i_2} \binom{\alpha_3}{i_3} \\
&\quad \times \int_{\mathbb{R}^N} \left(\nabla \frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}} \cdot \boldsymbol{v} \left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1 - i_1} \partial y^{\alpha_2 - i_2} \partial z^{\alpha_3 - i_3}} \right) \right) \frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \; dx.
\end{aligned} \tag{2.1}$$

Let us now consider four separate cases of $i_1 + i_2 + i_3$ in the above quadruple sum.

(i) If $i_1 + i_2 + i_3 = k = s$, i.e., $(\alpha_1, \alpha_2, \alpha_3) = (i_1, i_2, i_3)$, the above term can be rewritten as follows

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\nabla \frac{\partial^k f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \cdot \boldsymbol{v}(f) \right) \frac{\partial^k f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \; dx \\
&= \int_{\mathbb{R}^N} \frac{1}{2} \nabla \left| \frac{\partial^k f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \right|^2 \cdot \boldsymbol{v}(f) \; dx = - \int_{\mathbb{R}^N} \frac{1}{2} \left| \frac{\partial^k f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \right|^2 \operatorname{div} \boldsymbol{v}(f) \; dx \\
&\leq \frac{1}{2} \left\| \frac{\partial^k f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \right\|_{L^2}^2 \| \operatorname{div} \boldsymbol{v}(f) \|_{L^\infty} \leq C \|f\|_{H^k}^2 \|\boldsymbol{v}(f)\|_{H^3} \leq C \|f\|_{H^k}^2 \|f\|_{H^{k-2}},
\end{aligned}$$

where we have used the Sobolev embedding $H^2 \subset L^\infty$ and $k \geq 6$.

(ii) If $2 \leq i_1 + i_2 + i_3 \leq k - 3$, we find

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\nabla \frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}} \cdot \boldsymbol{v} \left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1 - i_1} \partial y^{\alpha_2 - i_2} \partial z^{\alpha_3 - i_3}} \right) \right) \frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \; dx \\
&\leq \left\| \nabla \frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}} \right\|_{L^\infty} \left\| \boldsymbol{v} \left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1 - i_1} \partial y^{\alpha_2 - i_2} \partial z^{\alpha_3 - i_3}} \right) \right\|_{L^2} \left\| \frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \right\|_{L^2} \\
&\leq C \|f\|_{H^k} \|f\|_{H^{k-2}} \|f\|_{H^k}.
\end{aligned}$$

(iii) If $k-2 \leq i_1+i_2+i_3 \leq k-1$, we find

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\nabla \frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}} \cdot \boldsymbol{v}\left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1-i_1} \partial y^{\alpha_2-i_2} \partial z^{\alpha_3-i_3}}\right)\right) \frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}\, dx \\
&\leq \left\|\nabla \frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}}\right\|_{L^2} \left\|\boldsymbol{v}\left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1-i_1} \partial y^{\alpha_2-i_2} \partial z^{\alpha_3-i_3}}\right)\right\|_{L^\infty} \left\|\frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}\right\|_{L^2} \\
&\leq C\|f\|_{H^k}\|f\|_{H^4}\|f\|_{H^k} \leq C\|f\|_{H^{k-2}}\|f\|_{H^k}^2.
\end{aligned}$$

To conclude the second last inequality, we have used that $k-2 \leq s \leq k$ and hence that $s-i_1-i_2-i_3 \leq 2$. The inequality then follows from the embedding $H^2 \subset L^\infty$.

(iv) If $0 \leq i_1+i_2+i_3 \leq 1$, we find

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\nabla \frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}} \cdot \boldsymbol{v}\left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1-i_1} \partial y^{\alpha_2-i_2} \partial z^{\alpha_3-i_3}}\right)\right) \frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}\, dx \\
&\leq \left\|\nabla \frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}}\right\|_{L^\infty} \left\|\boldsymbol{v}\left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1-i_1} \partial y^{\alpha_2-i_2} \partial z^{\alpha_3-i_3}}\right)\right\|_{L^2} \left\|\frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}\right\|_{L^2} \\
&\leq C\|f\|_{H^4}\|f\|_{H^k}^2 \leq C\|f\|_{H^{k-2}}\|f\|_{H^k}^2.
\end{aligned}$$

Hence, by applying (i)–(iv) in (2.1), we see that

$$\sum_{s=0}^{k} |I_1^s| \leq C\|f\|_{H^{k-2}}\|f\|_{H^k}^2. \tag{2.2}$$

2. We now bound the I_2^s terms. The standard Leibniz rule provides the identity

$$\begin{aligned}
I_2^s &= \sum_{|\alpha|=s} \int_{\mathbb{R}^N} \nabla^\alpha \left(f \operatorname{div} \boldsymbol{v}(f)\right) \nabla^\alpha f\, dx \\
&= \sum_{|\alpha|=s} \sum_{i_1=0}^{\alpha_1} \sum_{i_2=0}^{\alpha_2} \sum_{i_3=0}^{\alpha_3} \binom{\alpha_1}{i_1}\binom{\alpha_2}{i_2}\binom{\alpha_3}{i_3} \\
&\quad \times \int_{\mathbb{R}^N} \left(\frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}} \operatorname{div} \boldsymbol{v}\left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1-i_1} \partial y^{\alpha_2-i_2} \partial z^{\alpha_3-i_3}}\right)\right) \frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}\, dx.
\end{aligned} \tag{2.3}$$

Let us again consider four separate cases of $i_1+i_2+i_3$.

(i) If $i_1+i_2+i_3 = 0$, we apply the Hölder inequality and (3) in Definition 2.3, which yields to deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(f \operatorname{div} \boldsymbol{v}\left(\frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}\right)\right) \frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}\, dx \\
&\leq \|f\|_{L^\infty} \left\|\operatorname{div} \boldsymbol{v}\left(\frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}\right)\right\|_{L^2} \|f\|_{H^s} \leq C\|f\|_{H^{k-2}}\|f\|_{H^k}^2.
\end{aligned}$$

(ii) If $3 \leq i_1+i_2+i_3 \leq k-2$, we find

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}} \operatorname{div} \boldsymbol{v}\left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1-i_1} \partial y^{\alpha_2-i_2} \partial z^{\alpha_3-i_3}}\right)\right) \frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}\, dx \\
&\leq \left\|\frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}}\right\|_{L^2} \left\|\operatorname{div} \boldsymbol{v}\left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1-i_1} \partial y^{\alpha_2-i_2} \partial z^{\alpha_3-i_3}}\right)\right\|_{L^\infty} \left\|\frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}\right\|_{L^2} \\
&\leq C\|f\|_{H^{k-2}}\|\boldsymbol{v}(f)\|_{H^s}\|f\|_{H^s} \leq C\|f\|_{H^{k-2}}\|f\|_{H^k}^2.
\end{aligned}$$

(iii) If $1 \le i_1 + i_2 + i_3 \le 2$, we find

$$\int_{\mathbb{R}^N} \left(\frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}} \operatorname{div} \boldsymbol{v} \left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1 - i_1} \partial y^{\alpha_2 - i_2} \partial z^{\alpha_3 - i_3}} \right) \right) \frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \, dx$$
$$\le \left\| \frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}} \right\|_{L^\infty} \left\| \operatorname{div} \boldsymbol{v} \left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1 - i_1} \partial y^{\alpha_2 - i_2} \partial z^{\alpha_3 - i_3}} \right) \right\|_{L^2} \left\| \frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \right\|_{L^2}$$
$$\le C \|f\|_{H^4} \|\boldsymbol{v}(f)\|_{H^s} \|f\|_{H^s} \le C \|f\|_{H^{k-2}} \|f\|_{H^k}^2.$$

(iv) If $k - 1 \le i_1 + i_2 + i_3 \le k$, we find

$$\int_{\mathbb{R}^N} \left(\frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}} \operatorname{div} \boldsymbol{v} \left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1 - i_1} \partial y^{\alpha_2 - i_2} \partial z^{\alpha_3 - i_3}} \right) \right) \frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \, dx$$
$$\le \left\| \frac{\partial^{i_1+i_2+i_3} f}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3}} \right\|_{L^2} \left\| \operatorname{div} \boldsymbol{v} \left(\frac{\partial^{s-i_1-i_2-i_3} f}{\partial x^{\alpha_1 - i_1} \partial y^{\alpha_2 - i_2} \partial z^{\alpha_3 - i_3}} \right) \right\|_{L^\infty} \left\| \frac{\partial^s f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \right\|_{L^2}$$
$$\le C \|f\|_{H^k} \|\boldsymbol{v}(f)\|_{H^4} \|f\|_{H^s} \le C \|f\|_{H^{k-2}} \|f\|_{H^k}^2.$$

Applying (i)–(iv) in (2.3) gives

$$\sum_{s=0}^{k} |I_2^s| \le C \|f\|_{H^{k-2}} \|f\|_{H^k}^2.$$

Together with (2.2), this brings our proof to an end. □

Lemma 2.2. *Let $k \ge 4$. Then the following estimates hold*

$$\sum_{s=0}^{k} \left| \int_{\mathbb{R}^N} \nabla^s \operatorname{div} (f \boldsymbol{v}(g)) : \nabla^s f \, dx \right| \le C \|g\|_{H^k} \|f\|_{H^k}^2, \tag{2.4}$$

$$\sum_{s=0}^{k} \left| \int_{\mathbb{R}^N} \nabla^s \operatorname{div} (g \boldsymbol{v}(f)) : \nabla^s f \, dx \right| \le C \|g\|_{H^{k+1}} \|f\|_{H^k}^2. \tag{2.5}$$

Proof. The proof of (2.4) is easily obtained by the calculations of the previous proof. To prove (2.5), it is only step (i) in part 1 of the previous proof that does not go through. Instead, we now make the calculation

$$\int_{\mathbb{R}^N} \left(\nabla \frac{\partial^k g}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \cdot \boldsymbol{v}(f) \right) \frac{\partial^k f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \, dx$$
$$\le \left\| \nabla \frac{\partial^k g}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} \right\|_{L^2} \|\boldsymbol{v}(f)\|_{L^\infty} \|f\|_{H^k} \le C \|f\|_{H^{k+1}}^2 \|f\|_{H^2},$$

and the proof is complete. □

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